THE F_N METHOD APPLIED TO MULTIGROUP TRANSPORT THEORY IN PLANE GEOMETRY

by
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ABSTRACT

GARCIA, ROBERTO DAVID MARTINEZ. The F_N Method Applied to Multigroup Transport Theory in Plane Geometry. (Under the direction of CHARLES EDWARD SIEWERT.)

A study of multigroup transport theory for the special case of slowing-down in plane geometry is reported. Anisotropic scattering effects are included by using an L^{th} order Legendre expansion of the transfer cross section. The theory reduces the calculation of the reflected and transmitted angular fluxes to a sequence of one-group problems involving only angular fluxes at the boundaries. The theory is then extended to yield the angular flux at any location within a slab. The F_N method is used to establish particularly accurate solutions for several test problems. The computational aspects of the method are studied, and numerical results are given, accurate to five significant figures, for all considered problems.

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BIOGRAPHY

Roberto David Martinez Garcia was born in Vigo, Spain on June 5, 1950 and became a Brazilian citizen in 1977. He received his high-school education in São Paulo, Brazil at the Instituto Estadual de Educação Professor Alberto Conte, entered Universidade de São Paulo, São Paulo, Brazil in 1969, and received a Bachelor of Science degree in Physics in 1974.

In 1975 the author accepted employment with the Instituto de Pesquisas Energéticas e Nucleares, São Paulo, Brazil as a Researcher at the Reactor Physics Division. In the same year he initiated his graduate studies at the Escola Politécnica da Universidade de São Paulo, São Paulo, Brazil and received a Master of Science degree in Nuclear Engineering in 1977 with a thesis in neutron transport theory.

The author entered North Carolina State University in August 78 and since then has been working towards the Doctor of Philosophy degree in Nuclear Engineering.

The author is married to the former Iara Maria Pinheiro Simões and has a daughter, Juliana.

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1. INTRODUCTION

The task of solving the steady-state, energy-dependent, linear transport (Boltzmann) equation in plane geometry without azimuthal dependence

$$\mu \frac{\partial}{\partial z} \psi(z,\mu,E) + \sigma(z,E)\,\psi(z,\mu,E)$$
$$= \iint \sigma(z,E')\, f(z;\mu',E'\to\mu,E)\,\psi(z,\mu',E')\,d\mu' dE'$$
$$+ Q(z,\mu,E) \qquad (1.1)$$

is frequently encountered in transport problems. It is well known that the Boltzmann equation, formulated in early studies on kinetic theory of gases (Boltzmann, 1872), plays also an important role in the fields of radiative transfer (Chandrasekhar, 1950), neutron transport theory (Davison, 1957), radiation shielding (Goldstein, 1959), and rarefied gas dynamics (Cercignani, 1969). In the context of neutron transport theory $\psi(z,\mu,E)$, the fundamental quantity in equation (1.1), is known as the energy-dependent angular flux, a function of the space variable (z), the direction cosine of the neutron velocity with respect to the z-axis (μ), and the neutron energy (E). In addition, $\sigma(z,E)$ denotes the macroscopic total cross section, $f(z;\mu',E'\to\mu,E)$ the transfer probability (both are assumed to be known experimentally) and $Q(z,\mu,E)$ represents extraneous sources that may be present throughout the host medium.

In this work the multigroup method (Davison, 1957; Bell and Glasstone, 1970) is employed to discretize the energy dependence of equation (1.1) and the resulting set of multigroup equations for a class of radiation transport problems is solved by an extension of the F_N method initially introduced for monoenergetic neutron transport theory (Siewert and Benoist, 1979).

In Chapter 2, a review of previous work in multigroup transport theory is presented and applications of the F_N method since its introduction as well as its main characteristics are summarized.

In Chapter 3 a derivation of the multigroup equations from equation (1.1) is presented in an abbreviated manner and in Chapters 4, 5, and 6 basic multigroup problems are solved and accurate numerical results are reported.

2. REVIEW OF LITERATURE

2.1 Multigroup Methods

Historically the multigroup approach to the Boltzmann equation originated from attempts to describe the phenomenon of neutron transport in a more realistic manner than that provided by one-speed theory.

Early methods for solving the neutron transport equation (Davison, 1957) include the spherical harmonics method, generalized for multigroup theory (Mandl, 1953) some years after its introduction (Mark, 1944, 1945; Marshak, 1947) and the discrete ordinates method, based on replacing the integral term in the transport equation by a numerical quadrature (Wick, 1943; Chandrasekhar, 1944). The S_N method (Carlson, 1953), a different version of the original discrete ordinates method, proved to be very important from a practical point of view due to its capability of handling problems with a high degree of complexity. In today's computer codes this method, including a series of improvements incorporated since its introduction, is the most widely employed technique for solving the one-dimensional multigroup transport equation. Although its utilization in practical calculations is evidence of its merits, the S_N method is known to have difficulties in dealing with deep-penetration problems and strong absorbing media (e.g., reactor control rods).

Analytical methods were also developed for studying the multigroup transport equation. Because these methods have limitations impeding their application to practical calculations they are mainly used to obtain highly accurate solutions to basic problems which can be used as standards for accuracy assessment of numerical methods. Another important aspect of analytical methods is the fact that solutions can be carried out in a systematic and rigorous way and thus a greater insight into the mathematical structure of the transport equation is gained. In the past twenty years the method of singular eigenfunction expansions has been employed much more than any other analytical method, especially in multigroup theory, and for this reason a summary of the most important achievements of this method seems appropriate here. Some basic proofs were carried out initially by Davison (1945) but several aspects remained obscure until Case (1960) demonstrated clearly the adequacy of the method for one-speed neutron transport theory. A large number of papers devoted to two-group neutron transport theory with isotropic or linearly anisotropic scattering is available in the literature (McCormick and Kuščer, 1973). The reader is referred, for example, to the works of Żelazny and Kuszell (1961), Siewert and Shieh (1967), Metcalf and Zweifel (1968a, 1968b), Siewert and Ishiguro (1972), Reith and Siewert (1972), Siewert, Burniston, and Kriese (1972) and Burniston, Mullikin, and Siewert (1972). Several papers with numerical results for various basic problems in two-group theory are also available and some studies not restricted to the two-group case have been reported, for example, by Yoshimura and Katsuragi (1968) and Pahor and Shultis (1969a, 1969b) for isotropic scattering and Leuthäuser (1971) for linearly anisotropic scattering. Some works based on modal expansions for the energy variable in the transport equation that yield essentially the same multigroup equations have been reported in the literature. Among others, reference can be made to the works of Bednarz and Mika (1963), Zumbrunn (1965), Leonard and Ferziger (1966a, 1966b), Shultis (1969), and Silvennoinen and Zweifel (1972). The method of Bowden and co-workers, based on the analytic continuation of the μ-variable into the complex plane (Bowden, McCrosson, and Rhodes, 1968; Bowden and Bullard, 1969), has also been extended to multigroup transport theory with isotropic scattering (Bowden and McCrosson, 1971). It is apparent from all these works, however, that an extraordinary computational effort would be required to solve multigroup problems with very general scattering laws by purely analytical methods.

Several additional methods applicable to multigroup theory are available in the literature (Williams, 1971); a review of numerical methods in neutron transport theory has been recently completed by Sanchez and McCormick (1981).

2.2 The F_N Method

The F_N method was introduced initially in one-speed neutron transport theory by Siewert and Benoist (1979). Some similarities with the C_N method (Benoist and Kavenoky, 1968) were apparent but soon it was realized that the method could be derived in an independent and simpler way (Grandjean and Siewert, 1979) by using the full-range orthogonality properties of the singular eigenfunctions (Case and Zweifel, 1967). The method was extended for solving problems with L^{th} order anisotropic in the transport equation that yield essentially the same multigroup equations have been reported in the literature. Among others, reference can be made to the works of Bednarz and Mika (1963), Zumbrunn (1965), Leonard and Ferziger (1966a, 1966b), Shultis (1969), and Silvennoinen and Zweifel (1972). The method of Bowden and co-workers, based on the analytic continuation of the μ-variable into the complex plane (Bowden, McCrosson, and Rhodes, 1968; Bowden and Bullard, 1969), has also been extended to multigroup transport theory with isotropic scattering (Bowden and McCrosson, 1971). It is apparent from all these works, however, that an extraordinary computational effort would be required to solve multigroup problems with very general scattering laws by purely analytical methods.

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Basically, in plane geometry, the method starts by deriving a set of singular integral equations and constraints for the surface angular fluxes. By approximating the unknown surface fluxes in terms of appropriate basis functions and using collocation, a set of linear algebraic equations that can be solved by standard techniques is obtained for the coefficients in the approximation. Once the linear system is solved and the surface fluxes are established, similar ideas can be used to derive the angular flux at any position inside the slab (Devaux, Siewert, and Yuan, 1982; Garcia and Siewert, 1982b). Computationally, the method is easy to use and can yield accurate numerical results.

With regard to multigroup transport theory, Siewert and Benoist (1981) and Garcia and Siewert (1981b) generalized the F_N method for slowing-down problems with isotropic scattering. An extension of the method to multigroup problems with L^{th} order anisotropic scattering was also developed (Garcia and Siewert, 1982c).

3. THE MULTIGROUP EQUATIONS

In this chapter we review briefly how the multigroup equations can be obtained from equation (1.1). Further details can be found in standard references on neutron transport theory (Davison, 1957; Bell and Glasstone, 1970).

Here we assume rotational symmetry for scattering events, i.e., the transfer probability is a function of the cosine of the scattering angle as viewed in the laboratory system (μ_0). In addition, we assume the following Legendre representation to be valid:

$$\sigma(z,E')f(z;E'\to E,\mu_0) = \frac{1}{2}\sum_{\ell=0}^{L}(2\ell+1)\sigma_\ell(z;E'\to E)P_\ell(\mu_0) \tag{3.1}$$

where

$$\sigma_\ell(z;E'\to E) = \int_{-1}^{1}\sigma(z,E')f(z;E'\to E,\mu_0)P_\ell(\mu_0)d\mu_0 \quad . \tag{3.2}$$

If equation (3.1) is substituted into equation (1.1) and the addition theorem for the Legendre polynomials is used, we obtain

$$\mu\frac{\partial}{\partial z}\psi(z,\mu,E) + \sigma(z,E)\psi(z,\mu,E)$$

$$= \frac{1}{2}\sum_{\ell=0}^{L}(2\ell+1)P_\ell(\mu)\int\sigma_\ell(z;E'\to E)\int_{-1}^{1}\psi(z,\mu',E')P_\ell(\mu')d\mu'dE'$$

$$+ Q(z,\mu,E) \quad . \tag{3.3}$$

At this point the energy interval of interest is divided in M subintervals (groups) and equation (3.3) is integrated over the energy range for each subinterval. In general, the groups are numbered such that the i^{th} group contains all energies $E_i \le E \le E_{i-1}$. We obtain, for $i = 1,2,\ldots,M$, in the absence of upscattering:

$$\mu \frac{\partial}{\partial z} \psi_i(z,\mu) + \sigma_i(z)\psi_i(z,\mu) = \frac{1}{2} \sum_{j=1}^{i} \sum_{\ell=0}^{L} \sigma_{ij}(\ell,z)P_\ell(\mu)\phi_{j,\ell}(z) + Q_i(z,\mu) \qquad (3.4)$$

where

$$\psi_i(z,\mu) = \int_{E_i}^{E_{i-1}} \psi(z,\mu,E)dE \quad , \qquad (3.5)$$

$$\sigma_i(z) = \psi_i^{-1}(z,\mu) \int_{E_i}^{E_{i-1}} \sigma(z,E)\psi(z,\mu,E)dE \quad , \qquad (3.6)$$

$$\phi_{j,\ell}(z) = \int_{-1}^{1} \psi_j(z,\mu)P_\ell(\mu)d\mu \quad , \qquad (3.7)$$

and

$$\sigma_{ij}(\ell,z) = (2\ell+1)\phi_{j,\ell}^{-1}(z) \int_{E_j}^{E_{j-1}} \phi_\ell(z,E') \int_{E_i}^{E_{i-1}} \sigma_\ell(z;E'\rightarrow E)dE\, dE' \qquad (3.8)$$

with

$$\phi_\ell(z,E) = \int_{-1}^{1} \psi(z,\mu,E)P_\ell(\mu)d\mu \quad . \qquad (3.9)$$

In addition,

$$Q_i(z,\mu) = \int_{E_i}^{E_{i-1}} Q(z,\mu,E)dE \quad . \tag{3.10}$$

Strictly speaking, equations (3.4) through (3.9) define a system of coupled nonlinear equations and one could argue that very little was accomplished in this step. Furthermore the right-hand-side of equation (3.6) depends on μ and thus it would be logical to expect $\sigma_i(z)$ also to be dependent on μ. One simple way to avoid this unpleasant situation is to postulate that the energy-dependent angular flux can be represented by a separable function inside each energy group $j = 1,2,\ldots,i$:

$$\psi(z,\mu,E) = F_j(z,\mu)G_j(E), \; E\epsilon[E_j,E_{j-1}] \quad , \tag{3.11}$$

with the arbitrary normalization

$$\int_{E_j}^{E_{j-1}} G_j(E)\, dE = 1 \quad . \tag{3.12}$$

With these assumptions the nonlinear system reduces to equation (3.4) plus

$$\sigma_i(z) = \int_{E_i}^{E_{i-1}} \sigma(z,E)G_i(E)\, dE \tag{3.13}$$

and

$$\sigma_{ij}(\ell,z) = (2\ell+1) \int_{E_j}^{E_{j-1}} G_j(E') \int_{E_i}^{E_{i-1}} \sigma_\ell(z;E'\to E)dE\, dE' \quad . \tag{3.14}$$

We cannot expect to carry out an exact approach any further. A reasonable procedure is to choose explicit representations for $G_j(E)$, $j = 1,2,...,i$--infinite medium solutions are usually the best choice--and use $\sigma_i(z)$ and $\sigma_{ij}(\ell,z)$ from equations (3.13) and (3.14) as given inputs for solving equation (3.4). Once $\psi_i(z,\mu)$ is available for $i = 1,2,...,M$, a better representation for $\psi(z,\mu,E)$ in terms of energy may be available and improved group-averaged cross sections may be computed from equations (3.13) and (3.14). If any significant difference is found in the solution of equation (3.4) with this improved cross-section set an iterative procedure may be used until the agreement is satisfactory. Alternative approaches to the problem have been reported in the literature (Bell and Glasstone, 1970).

We note that so far no specification concerning the nature of the interacting particles has been made. Since we are interested in applications for radiation shielding, equation (3.4) can be considered specifically for neutrons and gamma rays. Of course the appropriate slowing-down mechanisms must be considered for each case when defining the group-averaged cross sections by equations (3.13) and (3.14). Thus, for neutrons in the absence of fissionable material, absorption and scattering are the dominant interactions (Davison, 1957) while, for gamma rays, photoelectric effect, Compton scattering and pair production (for energies above 1.022 MeV) must be considered (Goldstein, 1959).

4. A SINGLE SLAB WITH ISOTROPIC SCATTERING[1]

4.1 Introduction

In this chapter we consider the problem of solving equation (3.4) for the case of a homogeneous, source-free, non-multiplying slab, $z\epsilon[L,R]$, in the isotropic scattering model. We thus write, for $i = 1,2,\ldots,M$,

$$\mu \frac{\partial}{\partial z} \psi_i(z,\mu) + \sigma_i \psi_i(z,\mu) = \frac{1}{2} \sum_{j=1}^{i} \sigma_{ij} \phi_j(z) \qquad (4.1)$$

subject to the boundary conditions

$$\psi_i(L,\mu) = L_i(\mu) \quad , \mu > 0 \quad , \qquad (4.2a)$$

and

$$\psi_i(R,-\mu) = R_i(\mu) \quad , \mu > 0 \quad , \qquad (4.2b)$$

where $L_i(\mu)$ and $R_i(\mu)$ are considered specified. Here we follow the analysis of Siewert and Benoist (1981) to reduce the problem of finding the emerging angular fluxes relevant to equations (4.1) and (4.2) to a sequence of one-group problems involving only the boundary data and established emerging fluxes for preceding groups.

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[1] This chapter is partially based on a paper published in Nuclear Science and Engineering (Garcia and Siewert, 1981b).

4.2 Analysis

We note that a Wiener-Hopf factorization for the dispersion matrix in the case of a triangular transfer matrix has been investigated by Larsen and Zweifel (1976). Our approach, however, is based on solving a sequence of one-group problems. We thus begin by expressing the solution for the first group in terms of the elementary solutions (Case and Zweifel, 1967):

$$\psi_1(z,\mu) = A(\nu_1)\phi_1(\nu_1,\mu)\, e^{-\sigma_1 z/\nu_1} + A(-\nu_1)\phi_1(-\nu_1,\mu)\, e^{\sigma_1 z/\nu_1} + \int_{-1}^{1} A_1(\nu)\phi_1(\nu,\mu)\, e^{-\sigma_1 z/\nu} d\nu \qquad (4.3)$$

where, in general,

$$\phi_i(\nu_i,\mu) = \frac{1}{2} c_i \nu_i \left(\frac{1}{\nu_i - \mu}\right) \qquad (4.4a)$$

and

$$\phi_i(\nu,\mu) = \frac{1}{2} c_i\nu \, P\nu\left(\frac{1}{\nu - \mu}\right) + (1 - c_i\nu \tanh^{-1} \nu)\delta(\nu - \mu) \quad . \qquad (4.4b)$$

Here $c_i = \sigma_{ii}/\sigma_i$ and ν_i is the positive zero of the dispersion function

$$\Lambda_i(z) = 1 + \frac{1}{2} c_i z \int_{-1}^{1} \frac{d\mu}{\mu - z} \quad . \qquad (4.5)$$

If we use the full-range orthogonality condition (Case and Zweifel, 1967),

$$(\xi - \xi') \int_{-1}^{1} \mu \, \phi_i(\xi,\mu) \, \phi_i(\xi',\mu) \, d\mu = 0 \quad , \qquad (4.6)$$

where $\pm\xi$, $\pm\xi' \epsilon P_i = \nu_i \cup [0,1]$, we can deduce from equation (4.3) the following singular integral equations and constraints for the emerging fluxes $\psi_1(L,-\mu)$ and $\psi_1(R,\mu)$, $\mu > 0$:

$$\int_{-1}^{1} \mu \, \phi_1(\pm\xi,\mu) \, [\psi_1(L,\mu) - e^{\pm\Delta_1/\xi} \psi_1(R,\mu)] \, d\mu = 0 \quad , \quad \xi \epsilon P_1 \quad , \qquad (4.7)$$

where, in general, $\Delta_i = \sigma_i(R-L)$. We can rewrite equation (4.7) as

$$\int_{0}^{1} \mu \, \phi_1(\xi,\mu) \, \psi_1(L,-\mu) \, d\mu + e^{-\Delta_1/\xi} \int_{0}^{1} \mu \, \phi_1(-\xi,\mu) \, \psi_1(R,\mu) \, d\mu = U_1(\xi) \qquad (4.8a)$$

and

$$\int_{0}^{1} \mu \, \phi_1(\xi,\mu) \psi_1(R,\mu) \, d\mu + e^{-\Delta_1/\xi} \int_{0}^{1} \mu \, \phi_1(-\xi,\mu) \, \psi_1(L,-\mu) \, d\mu = V_1(\xi) \qquad (4.8b)$$

where $\xi \epsilon P_1$ and the known inhomogeneous terms are, in general,

$$U_i(\xi) = \int_{0}^{1} \mu \, \phi_i(-\xi,\mu) \, L_i(\mu) \, d\mu + e^{-\Delta_i/\xi} \int_{0}^{1} \mu \, \phi_i(\xi,\mu) \, R_i(\mu) \, d\mu \qquad (4.9a)$$

and

$$V_i(\xi) = \int_0^1 \mu\phi_i(-\xi,\mu)R_i(\mu)d\mu + e^{-\Delta_i/\xi} \int_0^1 \mu\phi_i(\xi,\mu)L_i(\mu)d\mu \quad . \quad (4.9b)$$

Equations (4.8) can be solved, for example, by the F_N method (Siewert and Benoist, 1979), and thus we now consider the second group. We write

$$\psi_2(z,\mu) = A(\nu_2)\phi_2(\nu_2,\mu)\, e^{-\sigma_2 z/\nu_2} + A(-\nu_2)\phi_2(-\nu_2,\mu)\, e^{\sigma_2 z/\nu_2}$$

$$+ \int_{-1}^{1} A_2(\nu)\phi_2(\nu,\mu)e^{-\sigma_2 z/\nu}\, d\nu + \frac{1}{2}\,\sigma_{21}\,\psi_{21}^{\dagger}(z,\mu) \qquad (4.10)$$

where $\psi_{21}^{\dagger}(z,\mu)$ denotes a particular solution of

$$\mu \frac{\partial}{\partial z}\,\psi(z,\mu) + \sigma_2\,\psi(z,\mu) = \frac{1}{2}\,\sigma_{22} \int_{-1}^{1} \psi(z,\mu')\, d\mu' + \phi_1(z) \quad . \qquad (4.11)$$

We can use equations (4.6) and (4.10) to deduce, for $\xi\varepsilon P_2$,

$$\int_{-1}^{1} \mu\, \phi_2(\pm\xi,\mu)\, [\psi_2(L,\mu) - e^{\pm\Delta_2/\xi}\, \psi_2(R,\mu)]d\mu$$

$$= \frac{1}{2}\,\sigma_{21}\, W_{21}(\pm\xi) \qquad (4.12)$$

where

$$W_{21}(\xi) = \int_{-1}^{1} \mu\, \phi_2(\xi,\mu)\, [\psi_{21}^{\dagger}(L,\mu) - e^{\Delta_2/\xi}\, \psi_{21}^{\dagger}(R,\mu)]\, d\mu \quad . \qquad (4.13)$$

Noting that $\psi_{21}^{\dagger}(z,\mu)$ can be expressed in terms of the infinite-medium Green's function (Case and Zweifel, 1967) basic to equation (4.11), we write

$$\psi_{21}^{\dagger}(z,\mu) = \int_L^R G_2(z_0 \to z;\mu)\,\phi_1(z_0)\,dz_0 \qquad (4.14)$$

where, in general, for $z > z_0$,

$$G_i(z_0 \to z;\mu) = \frac{1}{N_i(\nu_i)}\,\phi_i(\nu_i,\mu)\,e^{-\sigma_i(z-z_0)/\nu_i} + \int_0^1 \frac{1}{N_i(\nu)}\,\phi_i(\nu,\mu)\,e^{-\sigma_i(z-z_0)/\nu}\,d\nu \qquad (4.15a)$$

and, for $z < z_0$,

$$G_i(z_0 \to z;\mu) = -\frac{1}{N_i(-\nu_i)}\,\phi_i(-\nu_i,\mu)\,e^{\sigma_i(z-z_0)/\nu_i} - \int_0^1 \frac{1}{N_i(-\nu)}\,\phi_i(-\nu,\mu)\,e^{\sigma_i(z-z_0)/\nu}\,d\nu \;. \qquad (4.15b)$$

In addition,

$$N_i(\nu_i) = \frac{1}{2}\,c_i\nu_i^3\left(\frac{c_i}{\nu_i^2 - 1} - \frac{1}{\nu_i^2}\right) \qquad (4.16a)$$

and

$$N_i(\nu) = \nu[(1 - c_i\,\nu\,\tanh^{-1}\nu)^2 + \frac{1}{4}\pi^2\nu^2c_i^2] \;. \qquad (4.16b)$$

After using equations (4.14) and (4.15) in equation (4.13) we find

$$W_{21}(\xi) = -e^{-\sigma_2 L/\xi} \int_L^R \phi_1(z)\, e^{\sigma_2 z/\xi}\, dz \tag{4.17}$$

and thus we can write equation (4.12), for $\xi \varepsilon P_2$, as

$$\int_0^1 \mu\, \phi_2(\xi,\mu)\, \psi_2(L,-\mu)\, d\mu$$

$$+ e^{-\Delta_2/\xi} \int_0^1 \mu\, \phi_2(-\xi,\mu)\psi_2(R,\mu)\, d\mu = U_2(\xi) + \frac{1}{2}\sigma_{21}\xi I_{21}(\xi) \tag{4.18a}$$

and

$$\int_0^1 \mu\, \phi_2(\xi,\mu)\psi_2(R,\mu)\, d\mu + e^{-\Delta_2/\xi} \int_0^1 \mu\, \phi_2(-\xi,\mu)\psi_2(L,-\mu)\, d\mu$$

$$= V_2(\xi) + \frac{1}{2}\sigma_{21}\xi J_{21}(\xi) \tag{4.18b}$$

where

$$\xi I_{21}(\xi) = e^{\sigma_2 L/\xi} \int_L^R \phi_1(z)\, e^{-\sigma_2 z/\xi}\, dz \tag{4.19a}$$

and

$$\xi J_{21}(\xi) = e^{-\sigma_2 R/\xi} \int_L^R \phi_1(z)\, e^{\sigma_2 z/\xi}\, dz \quad . \tag{4.19b}$$

Because $I_{21}(\xi)$ and $J_{21}(\xi)$ are given in terms of $\phi_1(z)$ it is clear that the right-hand sides of equations (4.18) can be determined once the solution for the first group is available for all z. However, if the primary interest is in the solution at the boundaries, a theory that deals exclusively with the solution at the boundaries can be developed. Clearly, to accomplish this we need to express $I_{21}(\xi)$ and $J_{21}(\xi)$ in terms of $\psi_1(L,\mu)$ and $\psi_1(R,\mu)$, $\mu\epsilon[-1,1]$. We can write equation (4.1) for the first group as

$$\mu \frac{\partial}{\partial z} [\psi_1(z,\mu)\, e^{\sigma_1 z/\mu}] = \frac{1}{2}\, \sigma_{11}\, e^{\sigma_1 z/\mu}\, \phi_1(z) \qquad (4.20)$$

and integrate equation (4.20) to find, for $\xi\epsilon[0,\sigma_2/\sigma_1]$,

$$I_{21}(\xi) = \frac{2}{\sigma_{11}}\, \frac{\sigma_1}{\sigma_2}\, [\psi_1(L,-\sigma_1\xi/\sigma_2) - R_1(\sigma_1\xi/\sigma_2)\, e^{-\Delta_2/\xi}] \qquad (4.21a)$$

and

$$J_{21}(\xi) = \frac{2}{\sigma_{11}}\, \frac{\sigma_1}{\sigma_2}\, [\psi_1(R,\sigma_1\xi/\sigma_2) - L_1(\sigma_1\xi/\sigma_2)\, e^{-\Delta_2/\xi}] \quad . \qquad (4.21b)$$

It is clear that equations (4.21) express $I_{21}(\xi)$ and $J_{21}(\xi)$ in terms of the boundary fluxes; however these equations are not sufficient since $I_{21}(\xi)$ and $J_{21}(\xi)$ are required for all $\xi\epsilon P_2$ in equations (4.18). In addition, equations (4.21) cannot be used if $\sigma_{11} = 0$ and alternative expressions needed for this special case will be provided later. We can formally solve equation (4.20) and integrate the resulting equation to find

$$\phi_1(z) = K_1(z) + \frac{1}{2}\, \sigma_{11} \int_L^R \phi_1(z')\, E_1(\sigma_1|z - z'|)\, dz' \quad , \qquad (4.22)$$

where $E_1(x)$ is one of the exponential integral functions and, in general,

$$K_j(z) = \int_0^1 [L_j(\mu)e^{-\sigma_j(z-L)/\mu} + R_j(\mu)e^{-\sigma_j(R-z)/\mu}]d\mu \quad . \qquad (4.23)$$

If we multiply equation (4.22) by exp(-z/s), integrate over z and use equations (4.19) and (4.21), we find that

$$\Lambda_1(\sigma_1\xi/\sigma_2)\ I_{21}(\xi)$$
$$= \int_{-1}^{1} \mu[\psi_1(L,\mu) - \psi_1(R,\mu)\ e^{-\Delta_2/\xi}]\ \frac{d\mu}{\sigma_2\ \mu + \sigma_1\ \xi} \qquad (4.24a)$$

and

$$\Lambda_1(\sigma_1\xi/\sigma_2)\ J_{21}(\xi)$$
$$= \int_{-1}^{1} \mu[\psi_1(R,\mu) - \psi_1(L,\mu)\ e^{-\Delta_2/\xi}]\ \frac{d\mu}{\sigma_2\ \mu - \sigma_1\ \xi} \qquad (4.24b)$$

for $\xi \notin [-\sigma_2/\sigma_1,\ \sigma_2/\sigma_1]$. Equations (4.24) can therefore be used to compute the $I_{21}(\xi)$ and $J_{21}(\xi)$ required in equations (4.18) for $\xi \notin [0,\ \sigma_2/\sigma_1]$, except if $\sigma_1\xi/\sigma_2 = \nu_1$. This special case will be discussed later.

We now consider the extension of the foregoing analysis to the i^{th} group. As before, we write

$$\psi_i(z,\mu) = A(\nu_i)\ \phi_i(\nu_i,\mu)\ e^{-\sigma_i z/\nu_i} + A(-\nu_i)\ \phi_i(-\nu_i,\mu)\ e^{\sigma_i z/\nu_i}$$
$$+ \int_{-1}^{1} A_i(\nu)\ \phi_i(\nu,\mu)\ e^{-\sigma_i z/\nu}\ d\nu + \frac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij}\ \psi^{\dagger}_{ij}(z,\mu) \qquad (4.25)$$

where

$$\psi^{\dagger}_{ij}(z,\mu) = \int_L^R G_i(z_0 + z;\ \mu)\ \phi_j(z_0)dz_0. \qquad (4.26)$$

The appropriate generalizations of equations (4.18) are, for $\xi \varepsilon P_i$,

$$\int_0^1 \mu\, \phi_i(\xi,\mu)\, \psi_i(L,-\mu)\, d\mu + e^{-\Delta_i/\xi} \int_0^1 \mu\, \phi_i(-\xi,\mu)\, \psi_i(R,\mu)\, d\mu$$

$$= U_i(\xi) + \frac{1}{2}\, \xi \sum_{j=1}^{i-1} \sigma_{ij}\, I_{ij}(\xi) \qquad (4.27a)$$

and

$$\int_0^1 \mu\, \phi_i(\xi,\mu)\, \psi_i(R,\mu)\, d\mu + e^{-\Delta_i/\xi} \int_0^1 \mu\, \phi_i(-\xi,\mu)\, \psi_i(L,-\mu)\, d\mu$$

$$= V_i(\xi) + \frac{1}{2}\, \xi \sum_{j=1}^{i-1} \sigma_{ij}\, J_{ij}(\xi) \qquad (4.27b)$$

where

$$\xi I_{ij}(\xi) = e^{\sigma_i L/\xi} \int_L^R \phi_j(z)\, e^{-\sigma_i z/\xi}\, dz \qquad (4.28a)$$

and

$$\xi J_{ij}(\xi) = e^{-\sigma_i R/\xi} \int_L^R \phi_j(z)\, e^{\sigma_i z/\xi}\, dz \quad . \qquad (4.28b)$$

We now write equation (4.1) for the j^{th} group as

$$\mu \frac{\partial}{\partial z} [\psi_j(z,\mu)\, e^{\sigma_j z/\mu}] = \frac{1}{2}\, e^{\sigma_j z/\mu} \sum_{k=1}^{j} \sigma_{jk}\, \phi_k(z) \qquad (4.29)$$

and integrate to find the generalizations of equations (4.21) to be, for $\xi \varepsilon [0,1/s_{ij}]$,

$$I_{ij}(\xi) = \frac{1}{\sigma_{jj}} [X_{ij}(\xi) - \sum_{k=1}^{j-1} \sigma_{jk}\, I_{ik}(\xi)] \qquad (4.30a)$$

and

$$J_{ij}(\xi) = \frac{1}{\sigma_{jj}} [Y_{ij}(\xi) - \sum_{k=1}^{j-1} \sigma_{jk} J_{ik}(\xi)] \tag{4.30b}$$

where $s_{ij} = \sigma_j/\sigma_i$,

$$X_{ij}(\xi) = 2 s_{ij} [\psi_j(L,-s_{ij}\xi) - R_j(s_{ij}\xi) e^{-\Delta_i/\xi}] \tag{4.31a}$$

and

$$Y_{ij}(\xi) = 2 s_{ij} [\psi_j(R,s_{ij}\xi) - L_j(s_{ij}\xi) e^{-\Delta_i/\xi}] \ . \tag{4.31b}$$

We can multiply the generalization of equation (4.22),

$$\phi_j(z) = K_j(z) + \frac{1}{2} \sum_{k=1}^{j} \sigma_{jk} \int_L^R \phi_k(z') E_1(\sigma_j|z - z'|) dz' \ , \tag{4.32}$$

by exp(-z/s) and integrate over z to find, for $\xi \notin [-1/s_{ij}, 1/s_{ij}]$,

$$\Lambda_j(s_{ij}\xi)I_{ij}(\xi) = \int_{-1}^{1} \mu [\psi_j(L,\mu) - \psi_j(R,\mu) e^{-\Delta_i/\xi}] \frac{d\mu}{\sigma_i\mu+\sigma_j\xi} - \frac{1}{\sigma_j} \Delta(s_{ij}\xi) \sum_{k=1}^{j-1} \sigma_{jk} I_{ik}(\xi) \tag{4.33a}$$

and

$$\Lambda_j(s_{ij}\xi)J_{ij}(\xi) = \int_{-1}^{1} \mu [\psi_j(R,\mu) - \psi_j(L,\mu) e^{-\Delta_i/\xi}] \frac{d\mu}{\sigma_i\mu-\sigma_j\xi} - \frac{1}{\sigma_j} \Delta(s_{ij}\xi) \sum_{k=1}^{j-1} \sigma_{jk} J_{ik}(\xi) \tag{4.33b}$$

where

$$\Delta(z) = \frac{1}{2} z \int_{-1}^{1} \frac{d\mu}{\mu - z} \quad . \tag{4.34}$$

Equations (4.30) and (4.33) can be used to compute the $I_{ij}(\xi)$ and $J_{ij}(\xi)$ required in equations (4.27), provided that $\sigma_{jj} \neq 0$ in equations (4.30) and $s_{ij}\xi \neq \nu_j$ in equations (4.33). The following alternatives to equations (4.30) for $\sigma_{jj} = 0$ can be derived by letting ξ approach the cut in equations (4.33), using the Plemelj formulas (Muskhelishvili, 1953), and considering equations (4.30):

$$\sigma_i I_{ij}(\xi) = P \int_{-1}^{1} \mu[\psi_j(L,\mu) - \psi_j(R,\mu)\, e^{-\Delta_i/\xi}] \frac{d\mu}{\mu + s_{ij}\xi}$$
$$- \frac{1}{2} \xi X_{ij}(\xi)\, P \int_{-1}^{1} \frac{d\mu}{\mu - s_{ij}\xi} \quad , \sigma_{jj} = 0 \quad , \tag{4.35a}$$

and

$$\sigma_i J_{ij}(\xi) = P \int_{-1}^{1} \mu[\psi_j(R,\mu) - \psi_j(L,\mu)\, e^{-\Delta_i/\xi}] \frac{d\mu}{\mu - s_{ij}\xi}$$
$$- \frac{1}{2} \xi Y_{ij}(\xi)\, P \int_{-1}^{1} \frac{d\mu}{\mu - s_{ij}\xi} \quad , \sigma_{jj} = 0 \quad . \tag{4.35b}$$

In the case that $s_{ij}\xi = \nu_j$ a limiting procedure may be used as shown in the following section to find alternative expressions to equations (4.33).

4.3 The F_N Method

In the previous section, exact analysis was used to reduce the problem of finding the surface angular distributions relevant to equations (4.1) and (4.2) to a system of one-group problems, each of which is based only on surface results. We now wish to demonstrate that the F_N method (Siewert and Benoist, 1979) can be used to establish accurate numerical results for the considered multigroup model and especially for the challenging deep-penetration problem.

It is evident that the H-function (Chandrasekhar, 1950), for example, can be used to convert equations (4.8) to a system of Fredholm integral equations which can, of course, be solved by an iterative numerical method to yield $\psi_1(L,-\mu)$ and $\psi_1(R,\mu)$, $\mu > 0$. It follows that equations (4.27) can, in a like manner, be solved for i = 2, then i = 3 and so on. Rather than pursue this exact analysis, we prefer here to use the F_N method to develop a concise approximate solution. We therefore introduce, for the i^{th} group and $\mu > 0$,

$$\psi_i(L,-\mu) = R_i(\mu)\exp(-\Delta_i/\mu) + \sum_{\alpha=0}^{N} a_{i,\alpha}P_\alpha(2\mu - 1) \qquad (4.36a)$$

and

$$\psi_i(R,\mu) = L_i(\mu)\exp(-\Delta_i/\mu) + \sum_{\alpha=0}^{N} b_{i,\alpha}P_\alpha(2\mu - 1) \qquad (4.36b)$$

into equations (4.27) to find

$$\sum_{\alpha=0}^{N} \{a_{i,\alpha}B_{i,\alpha}(\xi) + c_i\exp(-\Delta_i/\xi)b_{i,\alpha}A_\alpha(\xi)\} = c_iI_i(\xi) + \sum_{j=1}^{i-1} \sigma_{ij}I_{ij}(\xi) \qquad (4.37a)$$

and

$$\sum_{\alpha=0}^{N} \{b_{i,\alpha}B_{i,\alpha}(\xi) + c_i \exp(-\Delta_i/\xi) a_{i,\alpha} A_\alpha(\xi)\} = c_i J_i(\xi) + \sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(\xi) \quad (4.37b)$$

for all $\xi \epsilon P_i$. Here, for $\alpha > 1$,

$$B_{i,\alpha}(\xi) = \left(\frac{2\alpha-1}{\alpha}\right)(2\xi - 1)B_{i,\alpha-1}(\xi) - \left(\frac{\alpha-1}{\alpha}\right)B_{i,\alpha-2}(\xi) - \frac{1}{2}c_i\delta_{\alpha,2} - c_i\delta_{\alpha,1} \quad (4.38a)$$

with

$$B_{i,o}(\xi) = 2 - c_i\,[1 + \xi \ln(1 + 1/\xi)] \quad (4.38b)$$

and

$$A_\alpha(\xi) = -\left(\frac{2\alpha-1}{\alpha}\right)(2\xi + 1)A_{\alpha-1}(\xi) - \left(\frac{\alpha-1}{\alpha}\right)A_{\alpha-2}(\xi) + \frac{1}{2}\delta_{\alpha,2} + \delta_{\alpha,1} \quad (4.39a)$$

with

$$A_o(\xi) = 1 - \xi \ln|1 + 1/\xi| \quad . \quad (4.39b)$$

In equations (4.36) we use a Legendre basis $P_\alpha(2\mu-1)$ that is orthogonal on the half range $\mu \epsilon [0,1]$ in order to avoid, in subsequent systems of linear algebraic equations, the inversion of ill-conditioned matrices that have been encountered, for large N, with the use (Siewert and Benoist, 1979; Grandjean and Siewert, 1979) of the simple basis functions μ^α. The functions $I_i(\xi)$ and $J_i(\xi)$ required in equations (4.37) are given in terms of the boundary data for the i^{th} group, i.e.,

$$I_i(\xi) = \int_0^1 \mu[L_i(\mu)S_i(\mu,\xi) + R_i(\mu)C_i(\mu,\xi)]d\mu \qquad (4.40a)$$

and

$$J_i(\xi) = \int_0^1 \mu[L_i(\mu)C_i(\mu,\xi) + R_i(\mu)S_i(\mu,\xi)]d\mu \qquad (4.40b)$$

where

$$S_i(\mu,\xi) = \frac{1 - \exp(-\Delta_i/\mu)\ \exp(-\Delta_i/\xi)}{\mu + \xi} \qquad (4.41a)$$

and

$$C_i(\mu,\xi) = \frac{\exp(-\Delta_i/\mu) - \exp(-\Delta_i/\xi)}{\mu - \xi} \quad . \qquad (4.41b)$$

The additional known terms in equations (4.37) represent "slowing-down" contributions to the i^{th} group. Thus for $\xi\epsilon[0,1/s_{ij}]$ we write equations (4.30) as

$$\sigma_{jj}I_{ij}(\xi) = 2s_{ij} \sum_{\alpha=0}^{N} a_{j,\alpha}P_\alpha(2s_{ij}\xi-1) - \sum_{k=1}^{j-1} \sigma_{jk}I_{ik}(\xi) \qquad (4.42a)$$

and

$$\sigma_{jj}J_{ij}(\xi) = 2s_{ij} \sum_{\alpha=0}^{N} b_{j,\alpha}P_\alpha(2s_{ij}\xi-1) - \sum_{k=1}^{j-1} \sigma_{jk}J_{ik}(\xi) \quad . \qquad (4.42b)$$

For $\xi\notin[0,1/s_{ij}]$ we deduce from equations (4.33) that

$$\Lambda_j(s_{ij}\xi)\sigma_i I_{ij}(\xi) = T_j(s_{ij}\xi) + \xi\ \tanh^{-1}\left(\frac{1}{s_{ij}\xi}\right)\sum_{k=1}^{j-1} \sigma_{jk}I_{ik}(\xi) \qquad (4.43a)$$

and

$$\Lambda_j(s_{ij}\xi)\sigma_i J_{ij}(\xi) = \Xi_j(s_{ij}\xi) + \xi \tanh^{-1}\left(\frac{1}{s_{ij}\xi}\right)\sum_{k=1}^{j-1} \sigma_{jk} J_{ik}(\xi) \qquad (4.43b)$$

where

$$\Upsilon_j(\xi) = I_j(\xi) + \sum_{\alpha=0}^{N} \{a_{j,\alpha}A_\alpha(-\xi) - \exp(-\Delta_j/\xi)b_{j,\alpha}A_\alpha(\xi)\} \qquad (4.44a)$$

and

$$\Xi_j(\xi) = J_j(\xi) + \sum_{\alpha=0}^{N} \{b_{j,\alpha}A_\alpha(-\xi) - \exp(-\Delta_j/\xi)a_{j,\alpha}A_\alpha(\xi)\} \quad . \qquad (4.44b)$$

It is clear that equations (4.42) and (4.43) establish the required $I_{ij}(\xi)$ and $J_{ij}(\xi)$ except when $\sigma_{jj} = 0$ and thus alternatives to equations (4.42) are required. For this case, we find from equations (4.35), for $\xi\epsilon[0,1/s_{ij}]$,

$$\sigma_i I_{ij}(\xi) = I_j(s_{ij}\xi) + [1+s_{ij}\xi \ln(1+1/s_{ij}\xi)] \sum_{\alpha=0}^{N} a_{j,\alpha}P_\alpha(2s_{ij}\xi-1) + \\ + \sum_{\alpha=0}^{N} \{a_{j,\alpha}G_\alpha(s_{ij}\xi) - \exp(-\Delta_j/s_{ij}\xi)b_{j,\alpha}A_\alpha(s_{ij}\xi)\} \qquad (4.45a)$$

and

$$\sigma_i J_{ij}(\xi) = J_j(s_{ij}\xi) + [1+s_{ij}\xi \ln(1+1/s_{ij}\xi)] \sum_{\alpha=0}^{N} b_{j,\alpha}P_\alpha(2s_{ij}\xi-1) + \\ + \sum_{\alpha=0}^{N} \{b_{j,\alpha}G_\alpha(s_{ij}\xi) - \exp(-\Delta_j/s_{ij}\xi)a_{j,\alpha}A_\alpha(s_{ij}\xi)\} \quad , \qquad (4.45b)$$

where $G_0(\xi) = 0$ and, for $\alpha > 1$,

$$G_\alpha(\xi) = \left(\frac{2\alpha-1}{\alpha}\right)(2\xi - 1)G_{\alpha-1}(\xi) - \left(\frac{\alpha-1}{\alpha}\right)G_{\alpha-2}(\xi) + \frac{1}{2}\,\delta_{\alpha,2} + \delta_{\alpha,1} \quad . \qquad (4.46)$$

We also note that alternatives to equations (4.43) are required if $s_{ij}\xi = \nu_j$. Clearly if $\xi = \nu_j/s_{ij}$ happens to be a collocation point in the continuum [0,1] we can avoid the difficulty simply by choosing a different point. However, if $\xi = \nu_i = \nu_j/s_{ij}$, such a simple remedy is not possible. We note that, from the point-of-view of a matrix formulation of this multi-group model, the phenomenon $\nu_i = \nu_j/s_{ij}$ appears as a degenerate eigenvalue, i.e., a double zero of the dispersion function. On the other hand, if we view the problem as a sequence of one-group problems, this phenomenon is clearly equivalent to seeking a particular solution corresponding to an inhomogeneous source of the form $\exp(-x/n_0)$ where n_0 is the eigenvalue for the homogeneous equation. To find the desired particular solution requires, as noted previously, special attention (Siewert, 1975). For our purpose here we find we can use l'Hospital's rule to obtain the following alternatives to equations (4.43) for $\xi = \nu_j/s_{ij}$:

$$\sigma_i I_{ij}(\xi) = \frac{1}{2}\, c_j(s_{ij}\xi)^2\, N_j^{-1}(s_{ij}\xi)[\Gamma_j(s_{ij}\xi) + \frac{\sigma_i}{s_{ij}\sigma_{jj}} \sum_{k=1}^{j-1} \sigma_{jk} \frac{d}{d\xi} I_{ik}(\xi)] - \frac{\sigma_i}{\sigma_{jj}} \sum_{k=1}^{j-1} \sigma_{jk} I_{ik}(\xi) \qquad (4.47a)$$

and

$$\sigma_i J_{ij}(\xi) = \frac{1}{2} c_j (s_{ij}\xi)^2 N_j^{-1}(s_{ij}\xi)[\Delta_j(s_{ij}\xi)$$

$$+ \frac{\sigma_i}{s_{ij}\sigma_{jj}} \sum_{k=1}^{j-1} \sigma_{jk} \frac{d}{d\xi} J_{ik}(\xi)] - \frac{\sigma_i}{\sigma_{jj}} \sum_{k=1}^{j-1} \sigma_{jk} J_{ik}(\xi) \quad . \qquad (4.47b)$$

Here

$$\Gamma_j(\xi) = \frac{d}{d\xi} I_j(\xi) + \sum_{\alpha=0}^{N} a_{j,\alpha} F_\alpha(-\xi)$$

$$+ \exp(-\Delta_j/\xi) \sum_{\alpha=0}^{N} b_{j,\alpha} \left[F_\alpha(\xi) - \frac{\Delta_j}{\xi^2} A_\alpha(\xi)\right] \qquad (4.48a)$$

and

$$\Delta_j(\xi) = \frac{d}{d\xi} J_j(\xi) + \sum_{\alpha=0}^{N} b_{j,\alpha} F_\alpha(-\xi)$$

$$+ \exp(-\Delta_j/\xi) \sum_{\alpha=0}^{N} a_{j,\alpha} \left[F_\alpha(\xi) - \frac{\Delta_j}{\xi^2} A_\alpha(\xi)\right] \quad . \qquad (4.48b)$$

We have found that the functions $F_\alpha(\xi) = -\frac{d}{d\xi} A_\alpha(\xi)$ appearing in equations (4.48) can be expressed as

$$F_\alpha(\xi) = \frac{1}{2\xi(\xi+1)} \{-[2(\xi+1) + \alpha(2\xi+1)]A_\alpha(\xi) - \alpha A_{\alpha-1}(\xi)$$

$$+ 2\delta_{\alpha,0} + \delta_{\alpha,1}\} \qquad (4.49)$$

and thus no additional recursive relations are required to deduce $F_\alpha(\xi)$. We note that equations (4.47) require, for $j > 1$, the derivatives of $I_{jk}(\xi)$ and $J_{jk}(\xi)$ for all $k < j$.

Clearly there exist possibilities for higher-order degeneracies; however, from a practical point of view the possibility of even a first-order degeneracy is slight. Nevertheless, a given data set should be reviewed with regard to this matter before an F_N calculation is initiated.

If the constants $\{a_{j,\alpha}\}$ and $\{b_{j,\alpha}\}$ have been established for all $j < i$, then clearly the right-hand sides of equations (4.37) are known. Thus on considering equations (4.37) at (N + 1) values of $\xi \epsilon P_i$, say $\xi_{i,\beta}$, we generate 2(N + 1) linear algebraic equations to be solved for the 2(N + 1) unknowns $a_{i,\alpha}$ and $b_{i,\alpha}$, $\alpha = 0,1,2,\ldots N$.

4.4 Computational Aspects and Numerical Results

In order to establish the constants $\{a_{i,\alpha}\}$ and $\{b_{i,\alpha}\}$ required in equations (4.36) we first must define a strategy for selecting the collocation points $\xi_{i,\beta}$. We then must compute the known matrix elements and inhomogeneous terms in the system of linear algebraic equations

$$\sum_{\alpha=0}^{N} \{a_{i,\alpha}B_{i,\alpha}(\xi_{i,\beta}) + c_i \exp(-\Delta_i/\xi_{i,\beta}) b_{i,\alpha}A_\alpha(\xi_{i,\beta})\} = c_i I_i(\xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(\xi_{i,\beta}) \quad (4.50a)$$

and

$$\sum_{\alpha=0}^{N} \{b_{i,\alpha}B_{i,\alpha}(\xi_{i,\beta}) + c_i \exp(-\Delta_i/\xi_{i,\beta}) a_{i,\alpha}A_\alpha(\xi_{i,\beta})\} = c_i J_i(\xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(\xi_{i,\beta}) \quad . \quad (4.50b)$$

In this work we have used, for various orders of the F_N approximation, the collocation scheme given by

$$\xi_{i,0} = \nu_i \text{, all } N, \tag{4.51a}$$

and

$$\xi_{i,\beta} = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\beta-1}{2N}\pi\right), \; \beta = 1, 2, \ldots, N, \; N \neq 0 \; . \tag{4.51b}$$

The points given by equation (4.51b) are the zeros of the Chebyshev polynomial of the first kind $T_N(2x - 1)$. The use of these points to define a collocation strategy was suggested by the work of Sloan and Burn (1979). We now discuss the methods we have used to compute the functions $A_\alpha(\xi)$, $B_{i,\alpha}(\xi)$ and $G_\alpha(\xi)$ required to define equations (4.50). The functions $A_\alpha(\xi)$ and $B_{i,\alpha}(\xi)$ are defined by

$$A_\alpha(\xi) = \frac{2}{c_i \xi} \int_0^1 \mu P_\alpha(2\mu - 1)\phi_i(-\xi,\mu)d\mu \tag{4.52a}$$

and

$$B_{i,\alpha}(\xi) = \frac{2}{\xi} \int_0^1 \mu P_\alpha(2\mu - 1)\phi_i(\xi,\mu)d\mu \; . \tag{4.52b}$$

We note from the previous section that the functions $A_\alpha(\xi)$ are required for all $\xi \notin [-1,0)$ and $B_{i,\alpha}(\xi)$ only for $\xi \epsilon P_i$. Equations (4.38) and (4.39) clearly are recursive relations that are easier to use, from a computational standpoint, than the definitions given by equations (4.52). However, some care must be taken to avoid a loss of accuracy when using the recursive equations (4.38) and (4.39). Here we note the strategy used to compute (working in double precision with an IBM 370/165 machine) the required functions accurate to at least thirteen significant figures for α up to 40. For the functions $A_\alpha(\xi)$ we have found that forward recursion is stable only for $\xi\varepsilon(-1,0]$ and that backward recursion is stable for all $\xi\notin[-1,0]$. For the functions $B_{i,\alpha}(\xi)$ forward recursion is stable for $\xi\varepsilon[0,1]$, and thus backward recursion must be used when $\xi = \nu_i$. In practice the use of backward recursion, in the manner of Miller (1952), can be very time-consuming for ξ close to the transition points, i.e., points that define the regions of forward and backward stability. For this reason we do not always use the defined regions of stability; e.g., in computing $A_\alpha(\xi)$ we have actually used forward recursion for $\xi\varepsilon[0,0.001]$ without losing too many significant figures.

The polynomials $G_\alpha(\xi)$ introduced in equation (4.46) and required for $\xi\varepsilon[0,1]$ are defined by

$$G_\alpha(\xi) = \int_0^1 \mu[P_\alpha(2\mu-1) - P_\alpha(2\xi-1)]\,\frac{d\mu}{\mu-\xi} \quad . \tag{4.53}$$

These polynomials satisfy the same recursive relation as the Legendre polynomials and we can use forward recursion to obtain accurate results.

In order to demonstrate the computational merit of the foregoing solution we now consider a special 16-group albedo problem. A slab of 1 cm thickness has an isotropic incident distribution of neutrons only in the first group and only on the surface at $z = L$, i.e., for $\mu > 0$

$$L_i(\mu) = \delta_{i,1} \tag{4.54a}$$

and

$$R_i(\mu) = 0 \quad . \tag{4.54b}$$

The macroscopic total cross-sections (cm^{-1}) are given by $\sigma_1 = 11$, $\sigma_i = 10 + i/14$, $i = 2,3,\ldots,14$, $\sigma_{15} = 10^4$ and $\sigma_{16} = 20$. In addition, the macroscopic transfer cross-sections are given by $\sigma_{i+k,i} = 3/(k+1)$, $i = 1,2,\ldots,14$ and $k = 0,1,\ldots,(16-i)$; $\sigma_{15,15} = 0$, $\sigma_{16,15} = 10^{-4}$ and $\sigma_{16,16} = 5$. This sample problem was designed to be a severe test of the established solution. Note, for example, the very strong absorption in group 15; in fact, the in-group scattering is zero for this group. For this data set we also have a degenerate case, in other words, $\sigma_{14}\nu_1 = \sigma_1\nu_{14}$, so that we must use the alternatives to equations (4.43) given by equations (4.47). We seek the exit distributions $\psi_i(L,-\mu)$ and $\psi_i(R,\mu)$, $\mu > 0$, the group albedos

$$A_i^* = 2\int_0^1 \mu\psi_i(L,-\mu)d\mu \tag{4.55a}$$

and the group transmission factors

$$B_i^* = 2\int_0^1 \mu\psi_i(R,\mu)d\mu \quad . \tag{4.55b}$$

If we use the solutions given by equations (4.36) in equations (4.55) we find

$$A_i^* = a_{i,o} + \frac{1}{3}a_{i,1} \tag{4.56a}$$

and

$$B_i^* = 2\delta_{i,1}E_3(\Delta_i) + b_{i,0} + \frac{1}{3} b_{i,1} \tag{4.56b}$$

where $E_3(x)$ denotes one of the exponential integral functions. In Tables 4.1 to 4.4 we list converged results for the exit distributions. In Table 4.5 we list final results for the group albedos and transmission factors. Also, in Table 4.5 we compare our F_N results to a calculation of Renken (1981) who used DTF69, a discrete ordinates code (Renken and Adams, 1969), with 100 space points and 8 directions for each of the half ranges of μ. The F_N results given in Tables 4.1 to 4.5 are correct to within ±1 in the fifth significant figure. We note that, in general, the albedos computed by Renken agree to four or five significant figures with our results. The transmission factors, however, agree only to two or three significant figures. An exception is found in group 15 where the DTF69 results clearly show a loss of accuracy. Finally, we note that the collocation strategy defined by equations (4.51) yielded results that represented a significant improvement over the ones initially deduced from either of the equally-spaced schemes used previously (Grandjean and Siewert, 1979; Siewert, Maiorino, and Özişik, 1980).

We now consider a second problem suggested and solved by Renken (1981). Here an iron ($N = 8.466 \times 10^{22}$ atoms/cm^3) slab, of 10 cm thickness, has an isotropic source of gamma rays incident at $z = L$ in the first of 19 groups that span the energy interval 50 kev to 1 Mev. The cross-section reported by Garcia and Siewert (1981b) was generated by Renken (1981), based on the photoelectric effect and the P_0 component of

Table 4.1 The exit angular fluxes $\psi_i(L,-\mu)$ for i = 1 to 8

μ	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8
0.05	1.2813(-1)[a]	7.7362(-2)	5.6265(-2)	4.5317(-2)	3.8517(-2)	3.3837(-2)	3.0396(-2)	2.7746(-2)
0.1	1.1660(-1)	7.1154(-2)	5.2400(-2)	4.2624(-2)	3.6527(-2)	3.2315(-2)	2.9207(-2)	2.6806(-2)
0.2	1.0041(-1)	6.2123(-2)	4.6539(-2)	3.8381(-2)	3.3271(-2)	2.9729(-2)	2.7105(-2)	2.5072(-2)
0.3	8.8866(-2)	5.5490(-2)	4.2071(-2)	3.5035(-2)	3.0621(-2)	2.7556(-2)	2.5284(-2)	2.3521(-2)
0.4	7.9986(-2)	5.0281(-2)	3.8471(-2)	3.2276(-2)	2.8389(-2)	2.5689(-2)	2.3687(-2)	2.2134(-2)
0.5	7.2858(-2)	4.6037(-2)	3.5481(-2)	2.9946(-2)	2.6473(-2)	2.4062(-2)	2.2276(-2)	2.0891(-2)
0.6	6.6973(-2)	4.2493(-2)	3.2946(-2)	2.7943(-2)	2.4806(-2)	2.2631(-2)	2.1021(-2)	1.9773(-2)
0.7	6.2013(-2)	3.9480(-2)	3.0763(-2)	2.6199(-2)	2.3341(-2)	2.1361(-2)	1.9897(-2)	1.8764(-2)
0.8	5.7767(-2)	3.6881(-2)	2.8861(-2)	2.4666(-2)	2.2042(-2)	2.0226(-2)	1.8886(-2)	1.7850(-2)
0.9	5.4084(-2)	3.4613(-2)	2.7186(-2)	2.3306(-2)	2.0881(-2)	1.9206(-2)	1.7971(-2)	1.7019(-2)
1	5.0857(-2)	3.2615(-2)	2.5699(-2)	2.2090(-2)	1.9838(-2)	1.8284(-2)	1.7140(-2)	1.6260(-2)

[a] Read as 1.2813×10^{-1}.

Table 4.2 The exit angular fluxes $\psi_i(L,-\mu)$ for i = 9 to 16

μ	i = 9	i = 10	i = 11	i = 12	i = 13	i = 14	i = 15	i = 16
0.05	2.5634(-2)	2.3904(-2)	2.2458(-2)	2.1228(-2)	2.0166(-2)	1.9238(-2)	1.7897(-5)	9.1192(-3)
0.1	2.4886(-2)	2.3310(-2)	2.1988(-2)	2.0861(-2)	1.9885(-2)	1.9031(-2)	1.7897(-5)	9.1021(-3)
0.2	2.3442(-2)	2.2099(-2)	2.0970(-2)	2.0004(-2)	1.9166(-2)	1.8430(-2)	1.7897(-5)	8.9941(-3)
0.3	2.2106(-2)	2.0939(-2)	1.9957(-2)	1.9116(-2)	1.8385(-2)	1.7743(-2)	1.7896(-5)	8.8398(-3)
0.4	2.0887(-2)	1.9859(-2)	1.8994(-2)	1.8253(-2)	1.7609(-2)	1.7042(-2)	1.7896(-5)	8.6638(-3)
0.5	1.9779(-2)	1.8864(-2)	1.8094(-2)	1.7435(-2)	1.6862(-2)	1.6359(-2)	1.7895(-5)	8.4777(-3)
0.6	1.8773(-2)	1.7951(-2)	1.7260(-2)	1.6669(-2)	1.6156(-2)	1.5705(-2)	1.7895(-5)	8.2880(-3)
0.7	1.7858(-2)	1.7113(-2)	1.6488(-2)	1.5954(-2)	1.5492(-2)	1.5086(-2)	1.7895(-5)	8.0982(-3)
0.8	1.7023(-2)	1.6344(-2)	1.5775(-2)	1.5290(-2)	1.4871(-2)	1.4503(-2)	1.7894(-5)	7.9110(-3)
0.9	1.6259(-2)	1.5637(-2)	1.5116(-2)	1.4673(-2)	1.4291(-2)	1.3956(-2)	1.7894(-5)	7.7276(-3)
1	1.5558(-2)	1.4985(-2)	1.4506(-2)	1.4100(-2)	1.3749(-2)	1.3442(-2)	1.7893(-5)	7.5490(-3)

Table 4.3 The exit angular fluxes $\psi_i(R,\mu)$ for i = 1 to 8

μ	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8
0.05	4.5701(-7)	7.3831(-7)	9.6077(-7)	1.1809(-6)	1.4023(-6)	1.6263(-6)	1.8529(-6)	2.0822(-6)
0.1	4.8998(-7)	7.9750(-7)	1.0392(-6)	1.2788(-6)	1.5200(-6)	1.7641(-6)	2.0112(-6)	2.2612(-6)
0.2	5.6471(-7)	9.2983(-7)	1.2115(-6)	1.4910(-6)	1.7726(-6)	2.0575(-6)	2.3460(-6)	2.6377(-6)
0.3	6.5894(-7)	1.0958(-6)	1.4230(-6)	1.7477(-6)	2.0744(-6)	2.4046(-6)	2.7385(-6)	3.0759(-6)
0.4	7.8516(-7)	1.3185(-6)	1.7003(-6)	2.0786(-6)	2.4585(-6)	2.8416(-6)	3.2281(-6)	3.6177(-6)
0.5	9.6666(-7)	1.6409(-6)	2.0893(-6)	2.5330(-6)	2.9771(-6)	3.4235(-6)	3.8724(-6)	4.3236(-6)
0.6	1.2644(-6)	2.1567(-6)	2.6827(-6)	3.2046(-6)	3.7257(-6)	4.2475(-6)	4.7704(-6)	5.2938(-6)
0.7	1.9188(-6)	3.0745(-6)	3.6719(-6)	4.2769(-6)	4.8834(-6)	5.4902(-6)	6.0965(-6)	6.7014(-6)
0.8	3.8488(-6)	4.7969(-6)	5.4072(-6)	6.0732(-6)	6.7563(-6)	7.4451(-6)	8.1350(-6)	8.8230(-6)
0.9	9.6935(-6)	7.9539(-6)	8.4198(-6)	9.0719(-6)	9.7887(-6)	1.0532(-5)	1.1287(-5)	1.2045(-5)
1	2.5162(-5)	1.3354(-5)	1.3377(-5)	1.3866(-5)	1.4525(-5)	1.5262(-5)	1.6036(-5)	1.6828(-5)

Table 4.4 The exit angular fluxes $\psi_i(R,\mu)$ for $i = 9$ to 16

μ	i = 9	i = 10	i = 11	i = 12	i = 13	i = 14	i = 15	i = 16
0.05	2.3135(-6)	2.5465(-6)	2.7806(-6)	3.0152(-6)	3.2497(-6)	3.4835(-6)	2.3397(-9)	1.2655(-6)
0.1	2.5137(-6)	2.7679(-6)	3.0234(-6)	3.2794(-6)	3.5354(-6)	3.7908(-6)	2.3401(-9)	1.3423(-6)
0.2	2.9322(-6)	3.2287(-6)	3.5265(-6)	3.8250(-6)	4.1232(-6)	4.4206(-6)	2.3409(-9)	1.4865(-6)
0.3	3.4160(-6)	3.7581(-6)	4.1013(-6)	4.4449(-6)	4.7880(-6)	5.1297(-6)	2.3417(-9)	1.6302(-6)
0.4	4.0098(-6)	4.4034(-6)	4.7976(-6)	5.1916(-6)	5.5843(-6)	5.9749(-6)	2.3425(-9)	1.7799(-6)
0.5	4.7762(-6)	5.2293(-6)	5.6820(-6)	6.1333(-6)	6.5821(-6)	7.0275(-6)	2.3433(-9)	1.9399(-6)
0.6	5.8170(-6)	6.3390(-6)	6.8589(-6)	7.3754(-6)	7.8876(-6)	8.3944(-6)	2.3441(-9)	2.1142(-6)
0.7	7.3039(-6)	7.9030(-6)	8.4974(-6)	9.0859(-6)	9.6676(-6)	1.0241(-5)	2.3449(-9)	2.3070(-6)
0.8	9.5070(-6)	1.0185(-5)	1.0857(-5)	1.1519(-5)	1.2172(-5)	1.2813(-5)	2.3457(-9)	2.5235(-6)
0.9	1.2799(-5)	1.3548(-5)	1.4288(-5)	1.5018(-5)	1.5734(-5)	1.6437(-5)	2.3465(-9)	2.7700(-6)
1	1.7625(-5)	1.8421(-5)	1.9209(-5)	1.9986(-5)	2.0749(-5)	2.1495(-5)	2.3473(-9)	3.0545(-6)

Table 4.5 A_i^* and Bi^* for the 16-group problem

	Present Work		DTF69	
i	A_i^*	B_i^*	A_i^*	B_i^*
1	6.6351(-2)	5.1058(-6)	6.6339(-2)	5.0413(-6)
2	4.2002(-2)	4.4781(-6)	4.1996(-2)	4.4362(-6)
3	3.2483(-2)	4.9308(-6)	3.2480(-2)	4.8877(-6)
4	2.7501(-2)	5.4649(-6)	2.7499(-2)	5.4197(-6)
5	2.4382(-2)	6.0283(-6)	2.4380(-2)	5.9806(-6)
6	2.2221(-2)	6.6050(-6)	2.2220(-2)	6.5547(-6)
7	2.0624(-2)	7.1879(-6)	2.0623(-2)	7.1349(-6)
8	1.9388(-2)	7.7730(-6)	1.9387(-2)	7.7175(-6)
9	1.8399(-2)	8.3577(-6)	1.8398(-2)	8.2996(-6)
10	1.7586(-2)	8.9399(-6)	1.7586(-2)	8.8792(-6)
11	1.6904(-2)	9.5178(-6)	1.6903(-2)	9.4547(-6)
12	1.6321(-2)	1.0090(-5)	1.6321(-2)	1.0025(-5)
13	1.5816(-2)	1.0655(-5)	1.5816(-2)	1.0587(-5)
14	1.5373(-2)	1.1212(-5)	1.5373(-2)	1.1140(-5)
15	1.7895(-5)	2.3446(-9)	1.7702(-5)	0.0
16	8.1585(-3)	2.2968(-6)	8.1578(-3)	2.2805(-6)

the Klein-Nishina differential scattering cross-section for photons (Klein and Nishina, 1929). In Table 4.6 we report our final results for A_i^* and B_i^* along with those found by Renken with DTF69 (again with 100 space points and 8 discrete directions for each half range of μ). Here our F_N results are also correct to within ±1 in the fifth significant figure. Further, the degree of agreement with the results of Renken is essentially the same as for the 16-group problem. Finally, we note what we believe to be a slight deterioration in the DTF69 results when there is strong absorption.

Table 4.6 A_i^* and B_i^* for the 19-group problem

	Present Work		DTF69	
i	A_i^*	B_i^*	A_i^*	B_i^*
1	1.3060(-2)	2.4188(-3)	1.3059(-2)	2.4154(-3)
2	2.6476(-2)	3.9163(-4)	2.6475(-2)	3.9124(-4)
3	2.0013(-2)	2.9446(-4)	2.0013(-2)	2.9415(-4)
4	2.0420(-2)	3.0057(-4)	2.0420(-2)	3.0026(-4)
5	2.1216(-2)	3.1339(-4)	2.1216(-2)	3.1306(-4)
6	2.2650(-2)	3.3596(-4)	2.2650(-2)	3.3560(-4)
7	1.6399(-2)	2.4287(-4)	1.6398(-2)	2.4260(-4)
8	1.8059(-2)	2.6601(-4)	1.8059(-2)	2.6572(-4)
9	2.0613(-2)	2.9930(-4)	2.0612(-2)	2.9896(-4)
10	2.4717(-2)	3.4873(-4)	2.4717(-2)	3.4834(-4)
11	3.1745(-2)	4.2569(-4)	3.1744(-2)	4.2519(-4)
12	4.4141(-2)	5.5031(-4)	4.4140(-2)	5.4965(-4)
13	1.8729(-2)	3.1938(-4)	1.8729(-2)	3.1901(-4)
14	1.7023(-2)	2.8604(-4)	1.7024(-2)	2.8570(-4)
15	1.2201(-2)	1.9924(-4)	1.2201(-2)	1.9901(-4)
16	3.5378(-3)	5.9937(-5)	3.5379(-3)	5.9867(-5)
17	9.0059(-4)	1.4905(-5)	9.0062(-4)	1.4888(-5)
18	6.2046(-5)	1.0301(-6)	6.2049(-5)	1.0289(-6)
19	9.1048(-6)	1.5020(-7)	9.1123(-6)	1.5013(-7)

5. A SINGLE SLAB WITH L-TH ORDER ANISOTROPIC SCATTERING[2]

5.1 Introduction

In Chapter 4 a solution for the case of isotropic scattering and a triangular transfer matrix was developed, and numerical results were reported. Here we extend the previous analysis to include the important effects of anisotropic scattering. We thus consider, for $i = 1,2,\ldots,M$,

$$\mu \frac{\partial}{\partial z} \psi_i(z,\mu) + \sigma_i \psi_i(z,\mu) = \frac{1}{2} \sum_{j=1}^{i} \sum_{\ell=0}^{L} \sigma_{ij}(\ell) P_\ell(\mu) \phi_{j,\ell}(z) \qquad (5.1)$$

where σ_i is the total cross section for group i and $\sigma_{ij}(\ell) = \sigma_{ij}\beta_{ij}(\ell)$, with $\beta_{ij}(0) = 1$, denote coefficients in Legendre expansions of the transfer cross sections. In addition, $\psi_i(z,\mu)$ represents the angular flux in the i^{th} group and

$$\phi_{j,\ell}(z) = \int_{-1}^{1} \psi_j(z,\mu) P_\ell(\mu) d\mu \quad . \qquad (5.2)$$

We are concerned here with a non-multiplying homogeneous finite slab, $z \in [L,R]$, and thus we seek solutions to equation (5.1) subject to the boundary conditions

[2] This chapter is based on a paper accepted for publication in Journal of Computational Physics (Garcia and Siewert, 1982c).

$$\psi_i(L,\mu) = L_i(\mu) \quad , \mu > 0 \quad , \tag{5.3a}$$

and

$$\psi_i(R,-\mu) = R_i(\mu) \quad , \mu > 0 \quad , \tag{5.3b}$$

where $L_i(\mu)$ and $R_i(\mu)$ are considered specified.

5.2 Analysis

In Chapter 4, full-range orthogonality properties of appropriate elementary solutions and Green's functions were used to deduce a system of singular integral equations and constraints for the boundary fluxes. Here we develop the equivalent expressions, generalized to include the effects of anisotropic scattering, in a more direct manner. We first change μ to $-\mu$ in equation (5.1), multiply the resulting equation by $\exp(-\sigma_i z/s)$ and integrate over all z to obtain

$$s\mu B_i(\mu,s)-\sigma_i(\mu-s) \int_L^R \psi_i(z,-\mu)\exp(-\sigma_i z/s)dz$$
$$= \frac{s}{2} \sum_{j=1}^{i} \sum_{\ell=0}^{L} (-1)^\ell \sigma_{ij}(\ell) P_\ell(\mu) \Phi_{j,\ell}(s/\sigma_i) \tag{5.4}$$

where

$$B_i(\mu,s) = \psi_i(L,-\mu)\exp(-\sigma_i L/s)-\psi_i(R,-\mu)\exp(-\sigma_i R/s) \tag{5.5}$$

and

$$\Phi_{j,\ell}(s/\sigma_i) = \int_L^R \Phi_{j,\ell}(z)\exp(-\sigma_i z/s)dz \quad . \tag{5.6}$$

We can now multiply equation (5.4) by $(\mu-s)^{-1} P_n(\mu)$, $s \notin [-1,1]$, and integrate over all μ to find

$$(-1)^n \sigma_i \Phi_{i,n}(s/\sigma_i) + \frac{s}{2} \sum_{j=1}^{i} \sum_{\ell=0}^{L} (-1)^\ell \sigma_{ij}(\ell)\Phi_{j,\ell}(s/\sigma_i) \int_{-1}^{1} P_n(\mu)P_\ell(\mu) \frac{d\mu}{\mu-s}$$

$$= s \int_{-1}^{1} \mu P_n(\mu)B_i(\mu,s) \frac{d\mu}{\mu-s} \quad . \qquad (5.7)$$

We let $g_{i,n}(\xi)$ denote for the i^{th} group the polynomials introduced for one-group theory by Chandrasekhar (1950), i.e.,

$$h_{i,n}\xi g_{i,n}(\xi) = (n+1)g_{i,n+1}(\xi)+ng_{i,n-1}(\xi) \qquad (5.8)$$

with $g_{i,o}(\xi) = 1$ and

$$h_{i,n} = 2n+1-c_i\beta_{ii}(n) \qquad (5.9)$$

where $c_i = \sigma_{ii}/\sigma_i$. On multiplying equation (5.7) by $\beta_{ii}(n)g_{i,n}(s)$ and summing over n from 0 to L, we find

$$\sigma_i \sum_{\ell=0}^{L} (-1)^\ell \beta_{ii}(\ell)\Phi_{i,\ell}(s/\sigma_i)F_{i,\ell}(s) + \frac{1}{c_i} \sum_{j=1}^{i-1} \sum_{\ell=0}^{L} (-1)^\ell \sigma_{ij}(\ell)\Phi_{j,\ell}(s/\sigma_i)$$

$$\times [F_{i,\ell}(s)-g_{i,\ell}(s)] = s \int_{-1}^{1} \mu G_i(s,\mu)B_i(\mu,s) \frac{d\mu}{\mu-s} \qquad (5.10)$$

where

$$F_{i,\ell}(s) = g_{i,\ell}(s) + \frac{s}{2} c_i \sum_{n=0}^{L} \beta_{ii}(n) g_{i,n}(s) \int_{-1}^{1} P_n(\mu) P_\ell(\mu) \frac{d\mu}{\mu - s} \qquad (5.11)$$

and

$$G_i(s,\mu) = \sum_{\ell=0}^{L} \beta_{ii}(\ell) g_{i,\ell}(s) P_\ell(\mu) \quad . \qquad (5.12)$$

It is not difficult to show that

$$F_{i,\ell}(s) = \Lambda_i(s) P_\ell(s) \qquad (5.13)$$

where

$$\Lambda_i(s) = 1 + s \int_{-1}^{1} \psi_i(\nu) \frac{d\mu}{\mu - s} \quad , \qquad (5.14)$$

with

$$\psi_i(\mu) = \frac{1}{2} c_i G_i(\mu,\mu) \quad , \qquad (5.15)$$

is the one-group dispersion function (Mika, 1961). We find we can write equation (5.10) as

$$\Lambda_i(s) X_{ii}(s) = \frac{s}{\sigma_i} \int_{-1}^{1} \mu G_i(s,\mu) B_i(\mu,s) \frac{d\mu}{\mu - s} + \frac{1}{\sigma_i} \sum_{j=1}^{i-1} \sigma_{ij} [Y_{ij}(s) - \Delta_i(s) X_{ij}(s)] \qquad (5.16)$$

where

$$X_{ij}(s) = \sum_{\ell=0}^{L} (-1)^\ell \beta_{ij}(\ell)\Phi_{j,\ell}(s/\sigma_i)P_\ell(s) \quad , \tag{5.17}$$

$$Y_{ij}(s) = \sum_{\ell=1}^{L} (-1)^\ell \beta_{ij}(\ell)\Phi_{j,\ell}(s/\sigma_i)E_{i,\ell}(s) \quad , \tag{5.18}$$

and

$$\Delta_i(s) = \frac{s}{2}\int_{-1}^{1} G_i(\mu,\mu)\frac{d\mu}{\mu-s} \quad . \tag{5.19}$$

Here the polynomials $E_{i,\ell}(s)$ are defined by

$$E_{i,\ell}(s) = \frac{1}{c_i}\,[g_{i,\ell}(s) - P_\ell(s)] \quad , \tag{5.20}$$

and with $E_{i,o}(s) = 0$, they can be readily computed from

$$(2\ell+1)sE_{i,\ell}(s) = s\beta_{ii}(\ell)g_{i,\ell}(s) + (\ell+1)E_{i,\ell+1}(s) + \ell E_{i,\ell-1}(s) \quad . \tag{5.21}$$

We note that the functions $\Phi_{j,\ell}(s/\sigma_i)$ can have essential singularities at the origin, but otherwise they are analytic in the complex s-plane. The functions $X_{ij}(s)$ and $Y_{ij}(s)$ therefore are, with the exception of the origin, also analytic in the complex s-plane. Thus on investigating equation (5.16) for the first group, i = 1, and assuming that $c_1 \neq 0$, we see that

$$\int_{-1}^{1} \mu G_1(\zeta_{1,m},\mu)B_1(\mu,\zeta_{1,m})\,\frac{d\mu}{\mu-\zeta_{1,m}} = 0 \tag{5.22}$$

where, in general, $\zeta_{i,m}$, $m = 0,1,2,\ldots, 2\kappa_i-1$ are the zeros of $\Lambda_i(s)$. The left- and right-hand sides of equation (5.16) are analytic in the complex s-plane cut from -1 to 1 along the real axis. Thus on letting s approach the branch cut and using the Plemelj formulas (Muskhelishvili, 1953), we find that equation (5.16) yields, for $\nu\epsilon[-1,1]$,

$$\sigma_1[\lambda_1(\nu) \pm \pi i\nu\psi_1(\nu)]X_{11}(\nu) =$$
$$= \nu P \int_{-1}^{1} \mu G_1(\nu,\mu)B_1(\mu,\nu) \frac{d\mu}{\mu-\nu} \pm \pi i\nu^2 G_1(\nu,\nu)B_1(\nu,\nu) \qquad (5.23)$$

where, in general,

$$\lambda_i(\nu) = 1 + \nu P \int_{-1}^{1} \psi_i(\mu) \frac{d\mu}{\mu-\nu} \quad . \qquad (5.24)$$

Thus, for $\nu\epsilon[-1,1]$, it follows that

$$\sigma_1 c_1 X_{11}(\nu) = 2\nu B_1(\nu,\nu) \qquad (5.25)$$

and

$$\lambda_1(\nu)\nu B_1(\nu,\nu) - \frac{1}{2} c_1 \nu P \int_{-1}^{1} \mu G_1(\nu,\mu)B_1(\mu,\nu) \frac{d\mu}{\mu-\nu} = 0 \quad . \qquad (5.26)$$

Equations (5.22) and (5.26) can be seen to be the system of singular integral equations and constraints (Bowden, McCrosson, and Rhodes, 1968; Siewert, 1978) that define the exit fluxes for the first group $\psi_1(L,-\mu)$ and $\psi_1(R,\mu)$, $\mu > 0$, in terms of the incident distributions $L_1(\mu)$ and $R_1(\mu)$. Thus, equations (5.22) and (5.26) can be solved numerically or, for example by the F_N method, to establish $B_1(\mu,s)$. In the event that $c_1 = 0$, equation (5.25) yields $B_1(\mu,\mu) = 0$, $|\mu|\varepsilon(0,1]$.

Considering now that $B_1(\mu,s)$ is known, we note that equation (5.16) yields, for i = 2,

$$\Lambda_2(s)X_{22}(s) = \frac{s}{\sigma_2}\int_{-1}^{1} \mu G_2(s,\mu)B_2(\mu,s)\,\frac{d\mu}{\mu-s} + \frac{1}{\sigma_2}\,\sigma_{21}[Y_{21}(s)-\Lambda_2(s)X_{21}(s)] \qquad (5.27)$$

or, for $\nu\varepsilon[-1,1]$,

$$\lambda_2(\nu)\nu B_2(\nu,\nu) - \frac{1}{2}\,c_2\nu P\int_{-1}^{1} \mu G_2(\nu,\mu)B_2(\mu,\nu)\,\frac{d\mu}{\mu-\nu} = \frac{1}{2}\,\sigma_{21}[c_2Y_{21}(\nu)+X_{21}(\nu)] \qquad (5.28)$$

and

$$\sigma_2c_2X_{22}(\nu) = 2\nu B_2(\nu,\nu) - \sigma_{21}X_{21}(\nu) \quad . \qquad (5.29)$$

For $c_2 \neq 0$ equation (5.27) yields

$$c_2\zeta_{2,m}\int_{-1}^{1} \mu G_2(\zeta_{2,m},\mu)B_2(\mu,\zeta_{2,m})\,\frac{d\mu}{\mu-\zeta_{2,m}} = -\sigma_{21}[c_2Y_{21}(\zeta_{2,m})+X_{21}(\zeta_{2,m})] \quad . \qquad (5.30)$$

For $c_2 = 0$ equation (5.29) yields, for $|\mu|\varepsilon(0,1]$,

$$B_2(\mu,\mu) = \frac{1}{2\mu}\,\sigma_{21}X_{21}(\mu) \quad , c_2 = 0 \quad , \tag{5.31}$$

whereas for $c_2 \neq 0$ we can solve equations (5.28) and (5.30) to find $B_2(\mu,s)$; of course for either case we must first compute

$$W_{21}(s) = c_2Y_{21}(s) + X_{21}(s) \tag{5.32}$$

which can also be written by using equations (5.17) and (5.18) as

$$W_{21}(s) = \sum_{\ell=0}^{L} (-1)^{\ell}\, \beta_{21}(\ell)\Phi_{1,\ell}(s/\sigma_2)g_{2,\ell}(s) \quad . \tag{5.33}$$

If we now multiply equation (5.4) by $P_n(\mu)$ and integrate over all μ there results, for $i = 1$,

$$sh_{1,n}\Phi_{1,n}(s/\sigma_1) + (n+1)\Phi_{1,n+1}(s/\sigma_1) + n\Phi_{1,n-1}(s/\sigma_1)$$
$$= -(-1)^n(2n+1)\,\frac{s}{\sigma_1}\int_{-1}^{1} \mu P_n(\mu)B_1(\mu,s)d\mu \quad , \tag{5.34}$$

which yields

$$\Phi_{1,n}(s/\sigma_1) = (-1)^n\, g_{1,n}(s)\Phi_{1,o}(s/\sigma_1) - (-1)^n\, D_{1,n}(s) \tag{5.35}$$

where $D_{1,o}(s) = 0$ and

$$sh_{1,n}D_{1,n}(s) = (n+1)D_{1,n+1}(s) + nD_{1,n-1}(s)$$
$$+ (2n+1)\,\frac{s}{\sigma_1}\int_{-1}^{1} \mu P_n(\mu)B_1(\mu,s)d\mu \quad . \tag{5.36}$$

Substituting equation (5.35) into equation (5.17), with $i = j = 1$, we find

$$\Phi_{1,o}(s/\sigma_1)$$
$$= G_1^{-1}(s,s)\,[X_{11}(s) + \sum_{\ell=1}^{L} \beta_{11}(\ell)D_{1,\ell}(s)P_\ell(s)] \quad . \qquad (5.37)$$

An expression alternative to equation (5.37) that allows the calculation of $\Phi_{1,o}(s/\sigma_1)$ in the event that $G_1(s,s) = 0$ will be provided later. We see from equations (5.25) and (5.26) that, for $\nu\epsilon[-1,1]$,

$$X_{11}(\nu) = \frac{1}{\sigma_1}\,\{2\nu B_1(\nu,\nu)$$
$$+ \nu \int_{-1}^{1} \mu G_1(\nu,\mu)[B_1(\mu,\nu)-B_1(\nu,\nu)]\,\frac{d\mu}{\mu-\nu}\} \qquad (5.38)$$

and from equation (5.16) that, for $s\notin[-1,1]$,

$$X_{11}(s) = \frac{s}{\sigma_1}\,\Lambda_1^{-1}(s) \int_{-1}^{1} \mu G_1(s,\mu)B_1(\mu,s)\,\frac{d\mu}{\mu-s} \quad . \qquad (5.39)$$

It is apparent from equations (5.22) and (5.39) that a limiting procedure must be used if $X_{11}(\zeta_{1,m})$ is required. For the case $c_1 = 0$ we note, since $B_1(\nu,\nu) = 0$, $|\nu|\epsilon(0,1]$, that equations (5.38) and (5.39) reduce to the following equation for all s:

$$X_{11}(s) = \frac{s}{\sigma_1} \int_{-1}^{1} \mu G_1(s,\mu)B_1(\mu,s)\,\frac{d\mu}{\mu-s} \quad , \ c_1 = 0 \quad . \qquad (5.40)$$

Finally, we use equation (5.35) to conclude that

$$W_{21}(s) = \Phi_{1,o}(s/\sigma_2) \sum_{\ell=0}^{L} \beta_{21}(\ell) g_{1,\ell}(s_{21}s) g_{2,\ell}(s) - \sum_{\ell=1}^{L} \beta_{21}(\ell) D_{1,\ell}(s_{21}s) g_{2,\ell}(s) \quad , \qquad (5.41)$$

where $\Phi_{1,o}(s/\sigma_2)$ is available from equation (5.37) and, in general, $s_{ij} = \sigma_j/\sigma_i$. Equations (5.28) and (5.30) can now be solved to yield the exit distributions for the second group $\psi_2(L,-\mu)$ and $\psi_2(R,\mu)$, $\mu > 0$, in terms of the incident distributions $L_2(\mu)$ and $R_2(\mu)$ and the previously established $B_1(\mu,s)$. Note that in this way we are able to deduce the exit fluxes for the second group directly from the incident distributions for that group and the boundary fluxes of the first group.

We now wish to generalize the foregoing and consider the i^{th} group. We assume that the $B_j(\mu,s)$, $j = 1,2,\ldots,i-1$, have been established, and we deduce from equation (5.16) that, for $\nu \in [-1,1]$,

$$\lambda_i(\nu)\nu B_i(\nu,\nu) - \frac{1}{2} c_i \nu P \int_{-1}^{1} \mu G_i(\nu,\mu) B_i(\mu,\nu) \frac{d\mu}{\mu-\nu} = \frac{1}{2} \sum_{j=1}^{i-1} \sigma_{ij} W_{ij}(\nu) \qquad (5.42)$$

and

$$\sigma_i c_i X_{ii}(\nu) = 2\nu B_i(\nu,\nu) - \sum_{j=1}^{i-1} \sigma_{ij} X_{ij}(\nu) \qquad (5.43)$$

and, for $c_i \neq 0$,

$$c_i \zeta_{i,m} \int_{-1}^{1} \mu G_i(\zeta_{i,m},\mu) B_i(\mu,\zeta_{i,m}) \frac{d\mu}{\mu-\zeta_{i,m}} = - \sum_{j=1}^{i-1} \sigma_{ij} W_{ij}(\zeta_{i,m}) \qquad (5.44)$$

where

$$W_{ij}(s) = \sum_{\ell=0}^{L} (-1)^{\ell} \beta_{ij}(\ell) \Phi_{j,\ell}(s/\sigma_i) g_{i,\ell}(s) \quad . \qquad (5.45)$$

For the special case $c_i = 0$, we see from equation (5.43) that, for $|\mu| \varepsilon (0,1]$,

$$B_i(\mu,\mu) = \frac{1}{2\mu} \sum_{j=1}^{i-1} \sigma_{ij} X_{ij}(\mu) \quad , \ c_i = 0 \quad , \qquad (5.46)$$

whereas for $c_i \neq 0$, the functions $W_{ij}(s)$ rather than just $X_{ij}(\mu)$, $|\mu| \varepsilon (0,1]$, clearly are required before we can solve equations (5.42) and (5.44) to find $B_i(\mu,s)$. In analogy with equations (5.34), (5.35) and (5.36) we now find, for $j = 1,2,\ldots,i-1$,

$$s h_{j,n} \Phi_{j,n}(s/\sigma_j) + (n+1)\Phi_{j,n+1}(s/\sigma_j) + n\Phi_{j,n-1}(s/\sigma_j) = -(-1)^n M_{j,n}(s) \qquad (5.47)$$

where

$$M_{j,n}(s) = (2n+1) \frac{s}{\sigma_j} \int_{-1}^{1} \mu P_n(\mu) B_j(\mu,s) d\mu - (-1)^n \frac{s}{\sigma_j} \sum_{k=1}^{j-1} \sigma_{jk}(n) \Phi_{k,n}(s/\sigma_j) \quad . \qquad (5.48)$$

It is clear that we can write

$$\Phi_{j,n}(s/\sigma_j) = (-1)^n g_{j,n}(s)\Phi_{j,o}(s/\sigma_j) - (-1)^n D_{j,n}(s) \qquad (5.49)$$

where $D_{j,o}(s) = 0$ and

$$sh_{j,n}D_{j,n}(s) = (n+1)D_{j,n+1}(s) + nD_{j,n-1}(s) + M_{j,n}(s) \quad . \qquad (5.50)$$

On substituting equation (5.49) into equation (5.17), with i = j, we find

$$\Phi_{j,o}(s/\sigma_j) = G_j^{-1}(s,s)[X_{jj}(s) + \sum_{\ell=1}^{L} B_{jj}(\ell)D_{j,\ell}(s)P_\ell(s)] \quad . \qquad (5.51)$$

Further, we can deduce from equation (5.16) that for $\nu\epsilon[-1,1]$

$$X_{jj}(\nu) = \frac{1}{\sigma_j}\{2\nu B_j(\nu,\nu) + \nu \int_{-1}^{1} \mu G_j(\nu,\mu)[B_j(\mu,\nu) - B_j(\nu,\nu)]\frac{d\mu}{\mu-\nu} + \sum_{k=1}^{j-1} \sigma_{jk}Y_{jk}(\nu)\} \qquad (5.52)$$

and for $s\notin[-1,1]$

$$X_{jj}(s) = \frac{1}{\sigma_j} \Lambda_j^{-1}(s)\{s \int_{-1}^{1} \mu G_j(s,\mu)B_j(\mu,s)\frac{d\mu}{\mu-s} + \sum_{k=1}^{j-1} \sigma_{jk}[Y_{jk}(s) - \Lambda_j(s)X_{jk}(s)]\} \quad . \qquad (5.53)$$

Again it is apparent that a limiting procedure must be used in the event that $X_{jj}(\zeta_{j,m})$ is required. Finally, if we use equation (5.49) in equation (5.45), we conclude that

$$W_{ij}(s) = \Phi_{j,o}(s/\sigma_i) \sum_{\ell=0}^{L} \beta_{ij}(\ell) g_{j,\ell}(s_{ij}s) g_{i,\ell}(s) - \sum_{\ell=1}^{L} \beta_{ij}(\ell) D_{j,\ell}(s_{ij}s) g_{i,\ell}(s) \quad (5.54)$$

where $\Phi_{j,o}(s/\sigma_i)$ is available from equation (5.51). We recall that $W_{ij}(\mu) = X_{ij}(\mu)$ for $c_i = 0$, and thus for this case equation (5.54) can be used in equation (5.46) to establish the desired result.

We have mentioned before that an alternative expression to equation (5.37) is needed to compute $\Phi_{1,o}(s/\sigma_1)$ in the event that $G_1(s,s) = 0$ for some s. The same is true, in general, for the $\Phi_{j,o}(s/\sigma_j)$ given by equation (5.51) when $G_j(s,s) = 0$. It is clear that equation (5.37) is only a special case of equation (5.51), and thus we now proceed to derive an alternative expression to the latter. If we set n = 0 in equation (5.7) and use equation (5.49) we obtain, for $s \notin [-1,1]$,

$$\sigma_j \Lambda_j(s) \Phi_{j,o}(s/\sigma_j) = s \int_{-1}^{1} \mu B_j(\mu,s) \frac{d\mu}{\mu - s} - s \sum_{\ell=0}^{L} \sigma_{jj}(\ell) D_{j,\ell}(s) Q_\ell(s) + s \sum_{k=1}^{j-1} \sum_{\ell=0}^{L} (-1)^\ell \sigma_{jk}(\ell) \Phi_{k,\ell}(s/\sigma_j) Q_\ell(s) \quad (5.55)$$

where $Q_\ell(s)$ denote the Legendre functions of second kind (Abramowitz and Stegun, 1964), i.e.,

$$(2\ell+1)sQ_\ell(s) = (\ell+1)Q_{\ell+1}(s) + \ell Q_{\ell-1}(s) + \delta_{\ell,0} \tag{5.56}$$

with

$$Q_0(s) = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right) \quad , \; s\notin[-1,1] \quad , \tag{5.57}$$

or

$$Q_0(\nu) = \frac{1}{2}\log\left(\frac{1+\nu}{1-\nu}\right) \quad , \; \nu\in[-1,1] \quad . \tag{5.58}$$

If we demonstrate that $\Lambda_j(s)$ and $G_j(s,s)$ do not have common zeros for $s\notin[-1,1]$, it is clear that we can divide equation (5.55) by $\Lambda_j(s)$ to obtain the desired alternative formula for $\Phi_{j,0}(s/\sigma_j)$, $s\notin[-1,1]$, in the event that $G_j(s,s) = 0$. First we use the summation formulas given by Inönü (1970) to write $\Lambda_j(s)$ and $G_j(s,s)$ in the convenient forms

$$\Lambda_j(s) = (L+1)[Q_L(s)g_{j,L+1}(s) - Q_{L+1}(s)g_{j,L}(s)] \tag{5.59}$$

and

$$G_j(s,s) = \left(\frac{L+1}{c_j s}\right)[P_{L+1}(s)g_{j,L}(s) - P_L(s)g_{j,L+1}(s)] \quad . \tag{5.60}$$

It is easy to show that the following identity holds:

$$(L+1)[P_{L+1}(s)Q_L(s) - P_L(s)Q_{L+1}(s)] = 1 \quad . \tag{5.61}$$

We now multiply equation (5.59) by $P_{L+1}(s)$ and use equations (5.60) and (5.61) to obtain

$$P_{L+1}(s)\Lambda_j(s) = g_{j,L+1}(s) - c_j s Q_{L+1}(s)G_j(s,s) \quad . \tag{5.62}$$

By contradiction, if we suppose that there exists some $s^* \notin [-1,1]$ such that $\Lambda_j(s^*) = G_j(s^*,s^*) = 0$, we conclude immediately from equation (5.62) that $g_{j,L+1}(s^*) = 0$. However from equation (5.60), we see that this would require $g_{j,L}(s^*) = 0$. Clearly this is not possible, otherwise from equation (5.8) we would have $g_{j,\ell}(s^*) = 0$ for all ℓ. We must then conclude that there is no such s^*.

In order to obtain an alternative formula for $\Phi_{j,o}(\nu/\sigma_j)$, $\nu\epsilon[-1,1]$, we let s approach the branch cut to find that equation (5.55) yields, for $\nu\epsilon[-1,1]$,

$$\sigma_j c_j G_j(\nu,\nu)\Phi_{j,o}(\nu/\sigma_j) = 2\nu B_j(\nu,\nu) + \sum_{\ell=0}^{L} \sigma_{jj}(\ell) D_{j,\ell}(\nu) P_\ell(\nu) - \sum_{k=1}^{j-1} \sum_{\ell=0}^{L} (-1)^\ell \sigma_{jk}(\ell)\Phi_{k,\ell}(\nu/\sigma_j) P_\ell(\nu) \qquad (5.63)$$

and

$$\sigma_j \lambda_j(\nu)\Phi_{j,o}(\nu/\sigma_j) = \nu P \int_{-1}^{1} \mu B_j(\mu,\nu) \frac{d\mu}{\mu-\nu} - \nu \sum_{\ell=0}^{L} \sigma_{jj}(\ell) D_{j,\ell}(\nu) Q_\ell(\nu) + \nu \sum_{k=1}^{j-1} \sum_{\ell=0}^{L} (-1)^\ell \sigma_{jk}(\ell)\Phi_{k,\ell}(\nu/\sigma_j) Q_\ell(\nu) \quad . \qquad (5.64)$$

We can use equations (5.63) and (5.64) to show that, for $\nu\epsilon[-1,1]$ and $G_j(\nu,\nu) = 0$,

$$\sigma_j\lambda_j(\nu)\Phi_{j,o}(\nu/\sigma_j) = 2\nu B_j(\nu,\nu) + \nu \int_{-1}^{1} \mu\left[\frac{B_j(\mu,\nu)-B_j(\nu,\nu)}{\mu-\nu}\right] d\mu$$

$$+ \nu \sum_{\ell=1}^{L} \sigma_{jj}(\ell)D_{j,\ell}(\nu)\Gamma_\ell(\nu)$$

$$- \nu \sum_{k=1}^{j-1} \sum_{\ell=1}^{L} (-1)^\ell \sigma_{jk}(\ell)\Phi_{k,\ell}(\nu/\sigma_j)\Gamma_\ell(\nu) \quad , \qquad (5.65)$$

where the polynomials $\Gamma_\ell(\nu)$ can be generated with the recursion formula (Siewert, 1980) for $\ell \geq 0$:

$$(2\ell+1)\nu\Gamma_\ell(\nu) = -\delta_{\ell,o} + (\ell+1)\Gamma_{\ell+1}(\nu) + \ell\Gamma_{\ell-1}(\nu) \qquad (5.66)$$

where

$$\Gamma_o(\nu) = 0 \quad . \qquad (5.67)$$

Again, if we demonstrate that $\lambda_j(\nu)$ and $G_j(\nu,\nu)$ do not have common zeros for $\nu\varepsilon[-1,1]$ we can divide equation (5.65) by $\lambda_j(\nu)$ to obtain the desired alternative formula for $\Phi_{j,o}(\nu/\sigma_j)$, $\nu\varepsilon[-1,1]$, when $G_j(\nu,\nu) = 0$. We let s approach the branch cut to obtain from equation (5.59)

$$\lambda_j(\nu) = (L+1)[Q_L(\nu)g_{j,L+1}(\nu) - Q_{L+1}(\nu)g_{j,L}(\nu)] \qquad (5.68)$$

and

$$G_j(\nu,\nu) = \left(\frac{L+1}{c_j\nu}\right)[P_{L+1}(\nu)g_{j,L}(\nu) - P_L(\nu)g_{j,L+1}(\nu)] \quad . \qquad (5.69)$$

Of course equation (5.61) is still valid for $\nu\varepsilon[-1,1]$:

$$(L+1)[P_{L+1}(\nu)Q_L(\nu) - P_L(\nu)Q_{L+1}(\nu)] = 1 \qquad (5.70)$$

and equation (5.62) yields

$$P_{L+1}(\nu)\lambda_j(\nu) = g_{j,L+1}(\nu) - c_j\nu Q_{L+1}(\nu)G_j(\nu,\nu) \quad . \tag{5.71}$$

By contradiction, if we suppose that there exists some $\nu^*\varepsilon[-1,1]$ such that $\lambda_j(\nu^*) = G_j(\nu^*,\nu^*) = 0$ we see from equation (5.71) that we must have $g_{j,L+1}(\nu^*) = 0$. As before, the possibility that $g_{j,L}(\nu^*) = 0$ has to be ruled out and thus $G_j(\nu^*,\nu^*) = 0$ would require $P_{L+1}(\nu^*) = 0$ in equation (5.69). At the same time $\lambda_j(\nu^*) = 0$ would require $Q_{L+1}(\nu^*) = 0$ in equation (5.68). But $P_{L+1}(\nu^*)$ and $Q_{L+1}(\nu^*)$ cannot be zero simultaneously otherwise equation (5.70) would be violated. We must then conclude that there is no such ν^*. We note that this result also implies that there are no discrete eigenvalues embedded in the continuum.

5.3 The F_N Method

Rather than pursue exact analysis to solve the developed singular integral equations and constraints for the exit distributions $\psi_i(L,-\mu)$ and $\psi_i(R,\mu)$, $\mu > 0$, we prefer to use the F_N method (Siewert and Benoist, 1979) to construct a concise approximate solution. We thus let $\Delta = R - L$, $\Delta_i = \sigma_i\Delta$ and write, for the i^{th} group and $\mu > 0$,

$$\psi_i(L,-\mu) = R_i(\mu)\exp(-\Delta_i/\mu) + \sum_{\alpha=0}^{N} a_{i,\alpha}P_\alpha(2\mu-1) \tag{5.72a}$$

and

$$\psi_i(R,\mu) = L_i(\mu)\exp(-\Delta_i/\mu) + \sum_{\alpha=0}^{N} b_{i,\alpha}P_\alpha(2\mu-1) \quad . \tag{5.72b}$$

If we now use equations (5.3) and (5.72) in (5.5), we can deduce from equations (5.42) and (5.44) that

$$\sum_{\alpha=0}^{N} [a_{i,\alpha}B_{i,\alpha}(\xi) + c_i \exp(-\Delta_i/\xi) b_{i,\alpha}A_{i,\alpha}(\xi)] = c_i I_i(\xi) + \sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(\xi) \quad (5.73a)$$

and

$$\sum_{\alpha=0}^{N} [b_{i,\alpha}B_{i,\alpha}(\xi) + c_i \exp(-\Delta_i/\xi) a_{i,\alpha}A_{i,\alpha}(\xi)] = c_i J_i(\xi) + \sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(\xi) \quad (5.73b)$$

for all $\xi \varepsilon P_i = \{\nu_{i,m}\} \cup [0,1]$. Here $\nu_{i,m}$, $m = 0,1,2,\ldots, \kappa_i-1$, denote the positive discrete eigenvalues relevant to group i,

$$I_i(\xi) = \int_0^1 \mu[L_i(\mu)G_i(-\xi,\mu)S_i(\Delta,\mu,\xi) + R_i(\mu)G_i(\xi,\mu)C_i(\Delta,\mu,\xi)]d\mu \quad (5.74a)$$

and

$$J_i(\xi) = \int_0^1 \mu[L_i(\mu)G_i(\xi,\mu)C_i(\Delta,\mu,\xi) + R_i(\mu)G_i(-\xi,\mu)S_i(\Delta,\mu,\xi)]d\mu \quad (5.74b)$$

where

$$S_i(\Delta,\mu,\xi) = \frac{1-\exp(-\sigma_i\Delta/\mu)\exp(-\sigma_i\Delta/\xi)}{\mu+\xi} \quad (5.75a)$$

and

$$C_i(\Delta,\mu,\xi) = \frac{\exp(-\sigma_i\Delta/\mu)-\exp(-\sigma_i\Delta/\xi)}{\mu-\xi} \quad . \tag{5.75b}$$

We have also introduced

$$\xi I_{ij}(\xi) = e^{\sigma_i L/\xi} W_{ij}(\xi) \tag{5.76a}$$

and

$$\xi J_{ij}(\xi) = e^{-\sigma_i R/\xi} W_{ij}(-\xi) \quad . \tag{5.76b}$$

Finally, the functions $A_{i,\alpha}(\xi)$ and $B_{i,\alpha}(\xi)$ required in equations (5.73) are defined in the following way:

$$A_{i,\alpha}(\xi) = \int_0^1 \mu P_\alpha(2\mu-1)G_i(-\xi,\mu)\,\frac{d\mu}{\mu+\xi} \quad , \ \xi\notin[-1,0) \quad , \tag{5.77}$$

$$B_{i,\alpha}(\xi) = -c_i \int_0^1 \mu P_\alpha(2\mu-1)G_i(\xi,\mu)\,\frac{d\mu}{\mu-\xi} \quad , \ \xi\epsilon\{\nu_{i,m}\} \tag{5.78a}$$

and

$$B_{i,\alpha}(\xi) = 2\lambda_i(\xi)P_\alpha(2\xi-1)-c_i P \int_0^1 \mu P_\alpha(2\mu-1)G_i(\xi,\mu)\,\frac{d\mu}{\mu-\xi} \quad , \quad \xi\epsilon[0,1] \quad . \tag{5.78b}$$

In Section 5.5, we report some recursion relations that establish an accurate and convenient method for evaluating the basic functions $A_{i,\alpha}(\xi)$ and $B_{i,\alpha}(\xi)$.

Considering the right-hand sides of equations (5.73), we note from equations (5.74) that the functions $I_i(\xi)$ and $J_i(\xi)$ are immediately available directly from the boundary data for the i^{th} group. The additional terms $I_{ij}(\xi)$ and $J_{ij}(\xi)$ clearly represent down-scattering contributions to the i^{th} group. We assume now that the constants $\{a_{j,\alpha}\}$ and $\{b_{j,\alpha}\}$, $j = 1,2,...i-1$, have been found so that the approximate results, for $\mu > 0$,

$$B_j(\mu,s) = \exp(-\sigma_j L/s)\ \{R_j(\mu)[\exp(-\Delta_j/\mu)-\exp(-\Delta_j/s)] + \sum_{\alpha=0}^{N} a_{j,\alpha}P_\alpha(2\mu-1)\} \tag{5.79a}$$

and

$$B_j(-\mu,s) = \exp(-\sigma_j L/s)\ \{L_j(\mu)[1-\exp(-\Delta_j/\mu)\exp(-\Delta_j/s)] - \exp(-\Delta_j/s)\sum_{\alpha=0}^{N} b_{j,\alpha}P_\alpha(2\mu-1)\} \tag{5.79b}$$

are available for $j = 1,2,...i-1$. Thus, on considering equations (5.73) at N+1 values of $\xi\varepsilon P_i$, say $\xi_{i,\beta}$, we can solve the system of linear algebraic equations

$$\sum_{\alpha=0}^{N} [a_{i,\alpha}B_{i,\alpha}(\xi_{i,\beta}) + c_i\exp(-\Delta_i/\xi_{i,\beta})b_{i,\alpha}A_{i,\alpha}(\xi_{i,\beta})] = c_i I_i(\xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij}I_{ij}(\xi_{i,\beta}) \tag{5.80a}$$

and

$$\sum_{\alpha=0}^{N} [b_{i,\alpha}B_{i,\alpha}(\xi_{i,\beta}) + c_i\exp(-\Delta_i/\xi_{i,\beta})a_{i,\alpha}A_{i,\alpha}(\xi_{i,\beta})] = c_i J_i(\xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij}J_{ij}(\xi_{i,\beta}) \ , \tag{5.80b}$$

for $\beta = 0,1,2,\ldots,N$, to find the desired constants for the i^{th} group $\{a_{i,\alpha}\}$ and $\{b_{i,\alpha}\}$ provided we first express $I_{ij}(\xi)$ and $J_{ij}(\xi)$ in terms of known quantities. We therefore proceed to use equations (5.76) and the various results developed in the previous section in order to deduce expressions that can be used in a convenient manner to compute the desired $I_{ij}(\xi)$ and $J_{ij}(\xi)$. To be specific, we note that for $i = 1$ the right-hand sides of equations (5.80) are known since $I_1(\xi)$ and $J_1(\xi)$ are given by equations (5.74). We thus can solve the system of linear algebraic equations to find $\{a_{1,\alpha}\}$ and $\{b_{1,\alpha}\}$. Considering equations (5.80) for $i \geq 2$, we see that we must compute $I_{ij}(\xi)$ and $J_{ij}(\xi)$ for $j = 1,2,\ldots i-1$, along with $I_i(\xi)$ and $J_i(\xi)$ as given by equations (5.74), before we can solve the linear system to find $\{a_{i,\alpha}\}$ and $\{b_{i,\alpha}\}$. We find

$$I_{ij}(\xi) = \sum_{\ell=0}^{L} (-1)^{\ell} \beta_{ij}(\ell)\overset{+}{\Phi}_{j,\ell}(\xi/\sigma_i)g_{i,\ell}(\xi) \tag{5.81}$$

where

$$\overset{+}{\Phi}_{j,\ell}(\xi/\sigma_i) = (-1)^{\ell} g_{j,\ell}(s_{ij}\xi)\overset{+}{\Phi}_{j,o}(\xi/\sigma_i) - (-1)^{\ell} \overset{+}{D}_{j,\ell}(s_{ij}\xi) \tag{5.82}$$

with

$$\overset{+}{\Phi}_{j,o}(\xi/\sigma_i) = G_j^{-1}(s_{ij}\xi,s_{ij}\xi) [\overset{+}{X}_{jj}(s_{ij}\xi) + \sum_{\ell=1}^{L} \beta_{jj}(\ell)\overset{+}{D}_{j,\ell}(s_{ij}\xi)P_{\ell}(s_{ij}\xi)] \ . \tag{5.83}$$

Here $\overset{+}{D}_{j,\ell}(s_{ij}\xi)$, with $\overset{+}{D}_{j,o}(s_{ij}\xi) = 0$, are available from

$$s_{ij}\xi h_{j,\ell} \overset{+}{D}_{j,\ell}(s_{ij}\xi) = (\ell+1)\overset{+}{D}_{j,\ell+1}(s_{ij}\xi)$$

$$+ \ell \overset{+}{D}_{j,\ell-1}(s_{ij}\xi) + \overset{+}{M}_{j,\ell}(s_{ij}\xi) \qquad (5.84)$$

where

$$\overset{+}{M}_{j,\ell}(s_{ij}\xi) = \frac{2\ell+1}{\sigma_i} \{K_{j,\ell}(s_{ij}\xi)$$

$$+ \sum_{\alpha=0}^{\ell+1} [a_{j,\alpha} + (-1)^{\ell} \exp(-\Delta_i/\xi) b_{j,\alpha}] T_{\alpha,\ell}\}$$

$$- \frac{\xi}{\sigma_i} U_{j,\ell}(\xi/\sigma_i) \quad . \qquad (5.85)$$

In equation (5.85) we have used the definitions

$$K_{j,\ell}(\xi) = \int_0^1 \mu P_\ell(\mu) \{R_j(\mu)[\exp(-\Delta_j/\mu) - \exp(-\Delta_j/\xi)]$$

$$- (-1)^{\ell} L_j(\mu)[1 - \exp(-\Delta_j/\mu)\exp(-\Delta_j/\xi)]\} d\mu \quad , \qquad (5.86)$$

$$T_{\alpha,\ell} = \int_0^1 \mu P_\alpha(2\mu-1) P_\ell(\mu) d\mu \qquad (5.87)$$

and

$$U_{j,\ell}(\xi/\sigma_i) = (-1)^{\ell} \sum_{k=1}^{j-1} \sigma_{jk}(\ell) \overset{+}{\Phi}_{k,\ell}(\xi/\sigma_i) \quad . \qquad (5.88)$$

We note that Devaux, Siewert, and Yuan (1982) have reported a recursion relation that provides an efficient way to compute the numbers $T_{\alpha,\ell}$ (see Section 5.5). Also we point out that $U_{j,\ell}(\xi/\sigma_i)$ is considered known since all $\overset{+}{\Phi}_{k,\ell}(\xi/\sigma_i)$ required in equation (5.88) have, by necessity, been computed in previous steps. Finally for $s_{ij}\xi\varepsilon[0,1]$, $X^{+}_{jj}(s_{ij}\xi)$ is given by

$$X^{+}_{jj}(s_{ij}\xi) = \frac{1}{\sigma_i} \{I_j(s_{ij}\xi) + [2-A_{j,o}(s_{ij}\xi)] \sum_{\alpha=0}^{N} a_{j,\alpha}P_\alpha(2s_{ij}\xi-1)$$
$$+ \sum_{\alpha=0}^{N} [a_{j,\alpha}G_{j,\alpha}(s_{ij}\xi)-\exp(-\Delta_i/\xi)b_{j,\alpha}A_{j,\alpha}(s_{ij}\xi)]$$
$$+ s_{ji} \sum_{\ell=1}^{L} U_{j,\ell}(\xi/\sigma_i)E_{j,\ell}(s_{ij}\xi)\} \tag{5.89}$$

where

$$G_{j,\alpha}(\xi) = \int_0^1 \mu G_j(\xi,\mu)\left[\frac{P_\alpha(2\mu-1)-P_\alpha(2\xi-1)}{\mu-\xi}\right]d\mu \tag{5.90}$$

can be computed effectively from a recursion relation (see Section 5.5). For $s_{ij}\xi\notin[0,1]$ we find

$$X^{+}_{jj}(s_{ij}\xi) = \frac{1}{\sigma_i} \Lambda_j^{-1}(s_{ij}\xi) \{T_j(s_{ij}\xi)$$
$$- s_{ji}\Delta_j(s_{ij}\xi) \sum_{\ell=0}^{L} U_{j,\ell}(\xi/\sigma_i)P_\ell(s_{ij}\xi)$$
$$+ s_{ji} \sum_{\ell=1}^{L} U_{j,\ell}(\xi/\sigma_i)E_{j,\ell}(s_{ij}\xi)\} \tag{5.91}$$

where

$$T_j(\xi) = I_j(\xi) + \sum_{\alpha=0}^{N} [a_{j,\alpha}A_{j,\alpha}(-\xi)-\exp(-\Delta_j/\xi)b_{j,\alpha}A_{j,\alpha}(\xi)] \ . \tag{5.92}$$

In a similar way we find

$$J_{ij}(\xi) = \sum_{\ell=0}^{L} \beta_{ij}(\ell)\Phi^{-}_{j,\ell}(\xi/\sigma_i)g_{i,\ell}(\xi) \tag{5.93}$$

where

$$\Phi^{-}_{j,\ell}(\xi/\sigma_i) = g_{j,\ell}(s_{ij}\xi)\Phi^{-}_{j,o}(\xi/\sigma_i) - (-1)^{\ell} D^{-}_{j,\ell}(s_{ij}\xi) \tag{5.94}$$

with

$$\Phi^{-}_{j,o}(\xi/\sigma_i) = G_j^{-1}(s_{ij}\xi, s_{ij}\xi)\ [X^{-}_{jj}(s_{ij}\xi) + \sum_{\ell=1}^{L} (-1)^{\ell}\ \beta_{jj}(\ell)D^{-}_{j,\ell}(s_{ij}\xi)P_{\ell}(s_{ij}\xi)] \ . \tag{5.95}$$

Here $D^{-}_{j,\ell}(s_{ij}\xi)$, with $D^{-}_{j,o}(s_{ij}\xi) = 0$, are available from

$$-s_{ij}\xi h_{j,\ell}D^{-}_{j,\ell}(s_{ij}\xi) = (\ell+1)D^{-}_{j,\ell+1}(s_{ij}\xi) + \ell D^{-}_{j,\ell-1}(s_{ij}\xi) + M^{-}_{j,\ell}(s_{ij}\xi) \tag{5.96}$$

where

$$M^{-}_{j,\ell}(s_{ij}\xi) = \frac{2\ell+1}{\sigma_i}\ \{N_{j,\ell}(s_{ij}\xi) - \sum_{\alpha=0}^{\ell+1} [(-1)^{\ell}\ b_{j,\alpha} + \exp(-\Delta_i/\xi)a_{j,\alpha}]T_{\alpha,\ell}\} + (-1)^{\ell}\frac{\xi}{\sigma_i}\ V_{j,\ell}(\xi/\sigma_i) \ . \tag{5.97}$$

In equation (5.97) we have used the definitions

$$N_{j,\ell}(\xi) = \int_0^1 \mu P_\ell(\mu)\{R_j(\mu)[1-\exp(-\Delta_j/\mu)\exp(-\Delta_j/\xi)]$$

$$- (-1)^\ell L_j(\mu)[\exp(-\Delta_j/\mu)-\exp(-\Delta_j/\xi)]\}\, d\mu \qquad (5.98)$$

and

$$V_{j,\ell}(\xi/\sigma_i) = \sum_{k=1}^{j-1} \sigma_{jk}(\ell)\bar{\Phi}_{k,\ell}(\xi/\sigma_i) \qquad (5.99)$$

which, as discussed before, is available from previous steps. Finally, for $s_{ij}\xi\varepsilon[0,1]$, $\bar{X}_{jj}(s_{ij}\xi)$ is given by

$$\bar{X}_{jj}(s_{ij}\xi) = \frac{1}{\sigma_i}\{J_j(s_{ij}\xi) + [2-A_{j,0}(s_{ij}\xi)] \sum_{\alpha=0}^{N} b_{j,\alpha}P_\alpha(2s_{ij}\xi-1)$$

$$+ \sum_{\alpha=0}^{N} [b_{j,\alpha}G_{j,\alpha}(s_{ij}\xi) - \exp(-\Delta_j/\xi)a_{j,\alpha}A_{j,\alpha}(s_{ij}\xi)]$$

$$+ s_{ji} \sum_{\ell=1}^{L} V_{j,\ell}(\xi/\sigma_i)E_{j,\ell}(s_{ij}\xi)\} \quad . \qquad (5.100)$$

For $s_{ij}\xi\notin[0,1]$ we find

$$\bar{X}_{jj}(s_{ij}\xi) = \frac{1}{\sigma_i} A_j^{-1}(s_{ij}\xi)\{\Xi_j(s_{ij}\xi)$$

$$- s_{ji}\Delta_j(s_{ij}\xi) \sum_{\ell=0}^{L} V_{j,\ell}(\xi/\sigma_i)P_\ell(s_{ij}\xi)$$

$$+ s_{ji} \sum_{\ell=1}^{L} V_{j,\ell}(\xi/\sigma_i)E_{j,\ell}(s_{ij}\xi)\} \qquad (5.101)$$

where

$$\Xi_j(\xi) = J_j(\xi) + \sum_{\alpha=0}^{N} [b_{j,\alpha}A_{j,\alpha}(-\xi)-\exp(-\Delta_j/\xi)a_{j,\alpha}A_{j,\alpha}(\xi)] \quad . \qquad (5.102)$$

Having developed the F_N method to find the surface fluxes, we now demonstrate how a slight modification of the analysis of Section 5.2 and the F_N method can be used to compute accurately the angular fluxes for all $z\epsilon(L,R)$.

5.4 The Interior Angular Fluxes

If we change μ to $-\mu$ in equation (5.1), multiply the resulting equation by $\exp(-\sigma_i z/s)$ and integrate over z from z_1 to z_2, with $L<z_1<z_2<R$, we obtain an equation similar to equation (5.4), viz.

$$s\mu B_i^{\dagger}(\mu,s)-\sigma_i(\mu-s)\int_{z_1}^{z_2} \psi_i(z,-\mu)\exp(-\sigma_i z/s)dz$$
$$= \frac{s}{2}\sum_{j=1}^{i}\sum_{\ell=0}^{L} (-1)^{\ell}\, \sigma_{ij}(\ell)P_{\ell}(\mu)\Phi_{j,\ell}^{\dagger}(s/\sigma_i) \qquad (5.103)$$

where

$$B_i^{\dagger}(\mu,s) = \psi_i(z_1,-\mu)\exp(-\sigma_i z_1/s) - \psi_i(z_2,-\mu)\exp(-\sigma_i z_2/s) \qquad (5.104)$$

and

$$\Phi_{j,\ell}^{\dagger}(s/\sigma_i) = \int_{z_1}^{z_2} \phi_{j,\ell}(z)\exp(-\sigma_i z/s)dz \quad . \qquad (5.105)$$

We can follow the development discussed in Section 5.2 to obtain from equation (5.103) generalizations of equations (5.42), (5.43), and (5.44). Thus for the i^{th} group and for $\nu\varepsilon[-1,1]$ we find

$$\lambda_i(\nu)\nu B_i^\dagger(\nu,\nu) - \frac{1}{2}c_i\nu P\int_{-1}^{1}\mu G_i(\nu,\mu)B_i^\dagger(\mu,\nu)\frac{d\mu}{\mu-\nu} = \frac{1}{2}\sum_{j=1}^{i-1}\sigma_{ij}W_{ij}^\dagger(\nu) \quad (5.106)$$

and

$$\sigma_i c_i X_{ii}^\dagger(\nu) = 2\nu B_i^\dagger(\nu,\nu) - \sum_{j=1}^{i-1}\sigma_{ij}X_{ij}^\dagger(\nu) \quad (5.107)$$

and, for $c_i \neq 0$,

$$c_i\zeta_{i,m}\int_{-1}^{1}\mu G_i(\zeta_{i,m},\mu)B_i^\dagger(\mu,\zeta_{i,m})\frac{d\mu}{\mu-\zeta_{i,m}} = -\sum_{j=1}^{i-1}\sigma_{ij}W_{ij}^\dagger(\zeta_{i,m}) \quad (5.108)$$

where

$$W_{ij}^\dagger(s) = \sum_{\ell=0}^{L}(-1)^\ell\beta_{ij}(\ell)\Phi_{j,\ell}^\dagger(s/\sigma_i)g_{i,\ell}(s) \quad (5.109)$$

and

$$X_{ij}^\dagger(s) = \sum_{\ell=0}^{L}(-1)^\ell\beta_{ij}(\ell)\Phi_{j,\ell}^\dagger(s/\sigma_i)P_\ell(s) \quad . \quad (5.110)$$

For the special case $c_i = 0$, we find, for $|\mu|\varepsilon(0,1]$,

$$B_i^\dagger(\mu,\mu) = \frac{1}{2\mu}\sum_{j=1}^{i-1} \sigma_{ij} X_{ij}^\dagger(\mu) \quad . \tag{5.111}$$

We note that the expressions developed in Section 5.2 for the computation of $\Phi_{j,\ell}(s/\sigma_i)$ can be generalized so that $\Phi_{j,\ell}^\dagger(s/\sigma_i)$ can be found in a similar manner.

We now let $z_1 = z$ and $z_2 = R$ and then use equations (5.3b), (5.72b) and the approximations, for $\mu > 0$,

$$\psi_i(z,-\mu) = R_i(\mu)\exp[-\sigma_i(R-z)/\mu] + \sum_{\alpha=0}^{N} c_{i,\alpha}(z)P_\alpha(2\mu-1) \tag{5.112a}$$

and

$$\psi_i(z,\mu) = L_i(\mu)\exp[-\sigma_i(z-L)/\mu] + \sum_{\alpha=0}^{N} d_{i,\alpha}(z)P_\alpha(2\mu-1) \tag{5.112b}$$

in equation (5.104) to deduce from equations (5.106) and (5.108) the first of our F_N equations. Similarly, we can take $z_1 = L$ and $z_2 = z$ and then use equations (5.3a), (5.72a) and (5.112) in equation (5.104) to deduce from equations (5.106) and (5.108) the second of the F_N equations. Thus, for $z\epsilon(L,R)$ and $\xi\epsilon P_i$ we find

$$\sum_{\alpha=0}^{N} [c_{i,\alpha}(z)B_{i,\alpha}(\xi)-c_i d_{i,\alpha}(z)A_{i,\alpha}(\xi)] = c_i I_i(z,\xi)$$
$$+ \sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(z,\xi) - c_i\exp[-\sigma_i(R-z)/\xi] \sum_{\alpha=0}^{N} b_{i,\alpha}A_{i,\alpha}(\xi) \tag{5.113a}$$

and

$$\sum_{\alpha=0}^{N} [d_{i,\alpha}(z)B_{i,\alpha}(\xi)-c_i c_{i,\alpha}(z)A_{i,\alpha}(\xi)] = c_i J_i(z,\xi)$$

$$+ \sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(z,\xi) - c_i \exp[-\sigma_i(z-L)/\xi] \sum_{\alpha=0}^{N} a_{i,\alpha} A_{i,\alpha}(\xi) \quad . \tag{5.113b}$$

Here

$$I_i(z,\xi) = \int_0^1 \mu\{L_i(\mu)G_i(-\xi,\mu)\exp[-\sigma_i(z-L)/\mu]S_i(R-z,\mu,\xi)$$

$$+ R_i(\mu)G_i(\xi,\mu)C_i(R-z,\mu,\xi)\}d\mu \quad , \tag{5.114a}$$

$$J_i(z,\xi) = \int_0^1 \mu\{L_i(\mu)G_i(\xi,\mu)C_i(z-L,\mu,\xi)$$

$$+ R_i(\mu)G_i(-\xi,\mu)\exp[-\sigma_i(R-z)/\mu]S_i(z-L,\mu,\xi)\}d\mu \quad , \tag{5.114b}$$

$$\xi I_{ij}(z,\xi) = e^{\sigma_i z/\xi} W_{ij}^{\dagger}(\xi) \quad , \; z_1 = z \;\text{ and }\; z_2 = R \quad , \tag{5.115a}$$

and

$$\xi J_{ij}(z,\xi) = e^{-\sigma_i z/\xi} W_{ij}^{\dagger}(-\xi) \quad , \; z_1 = L \;\text{ and }\; z_2 = z \quad . \tag{5.115b}$$

Investigating the right-hand sides of equations (5.113), we note from equations (5.114) that $I_i(z,\xi)$ and $J_i(z,\xi)$ are available, for any $z\varepsilon(L,R)$, directly from the boundary data for the i^{th} group. The terms $I_{ij}(z,\xi)$ and $J_{ij}(z,\xi)$ represent, as before, down-scattering contributions to the i^{th} group, and at this point the constants $\{a_{i,\alpha}\}$ and $\{b_{i,\alpha}\}$ have already been established. Thus on considering equations (5.113) at the same N+1 values of $\xi \epsilon P_i$ used in equation (5.80), namely $\xi_{i,\beta}$, we can solve the system of linear algebraic equations

$$\sum_{\alpha=0}^{N} [c_{i,\alpha}(z)B_{i,\alpha}(\xi_{i,\beta})-c_i d_{i,\alpha}(z)A_{i,\alpha}(\xi_{i,\beta})] = c_i I_i(z,\xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij} I_{ij}(z,\xi_{i,\beta}) - c_i \exp[-\sigma_i(R-z)/\xi_{i,\beta}] \sum_{\alpha=0}^{N} b_{i,\alpha}A_{i,\alpha}(\xi_{i,\beta}) \qquad (5.116a)$$

and

$$\sum_{\alpha=0}^{N} [d_{i,\alpha}(z)B_{i,\alpha}(\xi_{i,\beta})-c_i c_{i,\alpha}(z)A_{i,\alpha}(\xi_{i,\beta})] = c_i J_i(z,\xi_{i,\beta}) + \sum_{j=1}^{i-1} \sigma_{ij} J_{ij}(z,\xi_{i,\beta}) - c_i \exp[-\sigma_i(z-L)/\xi_{i,\beta}] \sum_{\alpha=0}^{N} a_{i,\alpha}A_{i,\alpha}(\xi_{i,\beta}) \qquad (5.116b)$$

for $\beta = 0,1,2,\ldots,N$, to find $\{c_{i,\alpha}(z)\}$ and $\{d_{i,\alpha}(z)\}$ for selected values of $z \epsilon (L,R)$ provided we first express $I_{ij}(z,\xi)$ and $J_{ij}(z,\xi)$ in terms of known quantities. We observe that the functions $A_{i,\alpha}(\xi)$ and $B_{i,\alpha}(\xi)$ appearing in equations (5.116) are the same as used in equations (5.80), and thus the only new quantitites to be evaluated are $I_i(z,\xi)$, $J_i(z,\xi)$, $I_{ij}(z,\xi)$, and $J_{ij}(z,\xi)$. Further we note that the matrix of coefficients in equations (5.116) is independent of z so that the solutions to these equations for many values of z can be accomplished with one matrix inversion. We now summarize the equations that can be used to compute the desired $I_{ij}(z,\xi)$ and $J_{ij}(z,\xi)$. We find

$$I_{ij}(z,\xi) = \sum_{\ell=0}^{L} (-1)^{\ell}\, \beta_{ij}(\ell)\overset{+}{\Phi}_{j,\ell}(z,\xi/\sigma_i) g_{i,\ell}(\xi) \tag{5.117}$$

where

$$\overset{+}{\Phi}_{j,\ell}(z,\xi/\sigma_i) = (-1)^{\ell}\, g_{j,\ell}(s_{ij}\xi)\overset{+}{\Phi}_{j,o}(z,\xi/\sigma_i) - (-1)^{\ell}\, \overset{+}{D}_{j,\ell}(z,s_{ij}\xi) \tag{5.118}$$

with

$$\overset{+}{\Phi}_{j,o}(z,\xi/\sigma_i) = G_j^{-1}(s_{ij}\xi,s_{ij}\xi)\, [\overset{+}{X}_{jj}(z,s_{ij}\xi) + \sum_{\ell=1}^{L} \beta_{jj}(\ell)\overset{+}{D}_{j,\ell}(z,s_{ij}\xi)P_{\ell}(s_{ij}\xi)] \ . \tag{5.119}$$

Here $\overset{+}{D}_{j,\ell}(z,s_{ij}\xi)$, with $\overset{+}{D}_{j,o}(z,s_{ij}\xi)$, are available from

$$s_{ij}\xi h_{j,\ell}\overset{+}{D}_{j,\ell}(z,s_{ij}\xi) = (\ell+1)\overset{+}{D}_{j,\ell+1}(z,s_{ij}\xi) + \ell\overset{+}{D}_{j,\ell-1}(z,s_{ij}\xi) + \overset{+}{M}_{j,\ell}(z,s_{ij}\xi) \tag{5.120}$$

where

$$\overset{+}{M}_{j,\ell}(z,s_{ij}\xi) = \frac{2\ell+1}{\sigma_i}\, \{K_{j,\ell}(z,s_{ij}\xi) + \sum_{\alpha=0}^{\ell+1} [c_{j,\alpha}(z) - (-1)^{\ell}\, d_{j,\alpha}(z) + (-1)^{\ell}\, \exp[-\sigma_i(R-z)/\xi]b_{j,\alpha}]T_{\alpha,\ell}\} - \frac{\xi}{\sigma_i}\, U_{j,\ell}(z,\xi/\sigma_i) \ . \tag{5.121}$$

In equation (5.121) we have used the definitions

$$K_{j,\ell}(z,\xi) = \int_0^1 \mu P_\ell(\mu)\Big[R_j(\mu)\{\exp[-\sigma_j(R-z)/\mu]-\exp[-\sigma_j(R-z)/\xi]\}$$
$$- (-1)^\ell L_j(\mu)\exp[-\sigma_j(z-L)/\mu]$$
$$\times \{1-\exp[-\sigma_j(R-z)/\mu]\exp[-\sigma_j(R-z)/\xi]\}\Big]\, d\mu \qquad (5.122)$$

and

$$U_{j,\ell}(z,\xi/\sigma_i) = (-1)^\ell \sum_{k=1}^{j-1} \sigma_{jk}(\ell)\Phi^{+}_{k,\ell}(z,\xi/\sigma_i) \quad . \qquad (5.123)$$

Finally for $s_{ij}\xi\epsilon[0,1]$, $\overset{+}{X}_{jj}(z,s_{ij}\xi)$ is given by

$$\overset{+}{X}_{jj}(z,s_{ij}\xi) = \frac{1}{\sigma_i}\Big\{I_j(z,s_{ij}\xi) + [2-A_{j,0}(s_{ij}\xi)]\sum_{\alpha=0}^{N} c_{j,\alpha}(z)P_\alpha(2s_{ij}\xi-1)$$
$$+ \sum_{\alpha=0}^{N}\Big[c_{j,\alpha}(z)G_{j,\alpha}(s_{ij}\xi) + (d_{j,\alpha}(z)-\exp[-\sigma_i(R-z)/\xi]b_{j,\alpha})A_{j,\alpha}(s_{ij}\xi)\Big]$$
$$+ s_{ji}\sum_{\ell=1}^{L} U_{j,\ell}(z,\xi/\sigma_i)E_{j,\ell}(s_{ij}\xi)\Big\} \quad . \qquad (5.124)$$

For $s_{ij}\xi\notin[0,1]$ we find

$$\overset{+}{X}_{jj}(z,s_{ij}\xi) = \frac{1}{\sigma_j}\Lambda_j^{-1}(s_{ij}\xi)\,\{T_j(z,s_{ij}\xi)$$
$$- s_{ji}\Delta_j(s_{ij}\xi)\sum_{\ell=0}^{L} U_{j,\ell}(z,\xi/\sigma_i)P_\ell(s_{ij}\xi)$$
$$+ s_{ji}\sum_{\ell=1}^{L} U_{j,\ell}(z,\xi/\sigma_i)E_{j,\ell}(s_{ij}\xi)\} \qquad (5.125)$$

where

$$T_j(z,\xi) = I_j(z,\xi) + \sum_{\alpha=0}^{N} [c_{j,\alpha}(z)A_{j,\alpha}(-\xi) + \{d_{j,\alpha}(z) - \exp[-\sigma_j(R-z)/\xi]b_{j,\alpha}\}A_{j,\alpha}(\xi)] \quad . \qquad (5.126)$$

In a similar way we find

$$J_{ij}(z,\xi) = \sum_{\ell=0}^{L} \beta_{ij}(\ell)\Phi^{-}_{j,\ell}(z,\xi/\sigma_i)g_{i,\ell}(\xi) \qquad (5.127)$$

where

$$\Phi^{-}_{j,\ell}(z,\xi/\sigma_i) = g_{j,\ell}(s_{ij}\xi)\Phi^{-}_{j,o}(z,\xi/\sigma_i) - (-1)^{\ell}\, D^{-}_{j,\ell}(z,s_{ij}\xi) \qquad (5.128)$$

with

$$\Phi^{-}_{j,o}(z,\xi/\sigma_i) = G_j^{-1}(s_{ij}\xi,s_{ij}\xi)\,[X^{-}_{jj}(z,s_{ij}\xi) + \sum_{\ell=1}^{L} (-1)^{\ell}\,\beta_{jj}(\ell)D^{-}_{j,\ell}(z,s_{ij}\xi)P_{\ell}(s_{ij}\xi)] \quad . \qquad (5.129)$$

Here $D^{-}_{j,\ell}(z,s_{ij}\xi)$, with $D^{-}_{j,o}(z,s_{ij}\xi) = C$, are available from

$$-s_{ij}\xi h_{j,\ell}D^{-}_{j,\ell}(z,s_{ij}\xi) = (\ell+1)D^{-}_{j,\ell+1}(z,s_{ij}\xi) + \ell D^{-}_{j,\ell-1}(z,s_{ij}\xi) + M^{-}_{j,\ell}(z,s_{ij}\xi) \qquad (5.130)$$

where

$$\bar{M}_{j,\ell}(z,s_{ij}\xi) = \frac{2\ell+1}{\sigma_i}\Big[N_{j,\ell}(z,s_{ij}\xi) + \sum_{\alpha=0}^{\ell+1}(c_{j,\alpha}(z) - (-1)^{\ell}\, d_{j,\alpha}(z)$$

$$- \exp[-\sigma_i(z-L)/\xi]a_{j,\alpha})T_{\alpha,\ell}\Big] + (-1)^{\ell}\frac{\xi}{\sigma_i}\, V_{j,\ell}(z,\xi/\sigma_i) \quad . \qquad (5.131)$$

In equation (5.131) we have used the definitions

$$N_{j,\ell}(z,\xi)$$
$$= \int_0^1 \mu P_\ell(\mu)\Big[R_j(\mu)\exp[-\sigma_j(R-z)/\mu]\{1-\exp[-\sigma_j(z-L)/\mu]\exp[-\sigma_j(z-L)/\xi]\}$$
$$- (-1)^{\ell}\, L_j(\mu)\{\exp[-\sigma_j(z-L)/\mu] - \exp[-\sigma_j(z-L)/\xi]\}\Big]\, d\mu \qquad (5.132)$$

and

$$V_{j,\ell}(z,\xi/\sigma_i) = \sum_{k=1}^{j-1} \sigma_{jk}(\ell)\bar{\phi}_{k,\ell}(z,\xi/\sigma_i) \quad . \qquad (5.133)$$

Finally, for $s_{ij}\xi\epsilon[0,1]$, $\bar{X}_{jj}(z,s_{ij}\xi)$ is given by

$$\bar{X}_{jj}(z,s_{ij}\xi) = \frac{1}{\sigma_i}\Big[J_j(z,s_{ij}\xi) + [2-A_{j,o}(s_{ij}\xi)]\sum_{\alpha=0}^{N} d_{j,\alpha}(z)P_\alpha(2s_{ij}\xi-1)$$

$$+ \sum_{\alpha=0}^{N}[d_{j,\alpha}(z)G_{j,\alpha}(s_{ij}\xi)+(c_{j,\alpha}(z)-\exp[-\sigma_i(z-L)/\xi]a_{j,\alpha})A_{j,\alpha}(s_{ij}\xi)]$$

$$+ s_{ji}\sum_{\ell=1}^{L} V_{j,\ell}(z,\xi/\sigma_i)E_{j,\ell}(s_{ij}\xi)\Big] \quad . \qquad (5.134)$$

For $s_{ij}\xi\notin[0,1]$ we find

$$\bar{X}_{jj}(z,s_{ij}\xi) = \frac{1}{\sigma_i} \Lambda_j^{-1}(s_{ij}\xi) \{\Xi_j(z,s_{ij}\xi)$$

$$- s_{ji}\Delta_j(s_{ij}\xi) \sum_{\ell=0}^{L} V_{j,\ell}(z,\xi/\sigma_i)P_\ell(s_{ij}\xi)$$

$$+ s_{ji} \sum_{\ell=1}^{L} V_{j,\ell}(z,\xi/\sigma_i)E_{j,\ell}(s_{ij}\xi)\} \qquad (5.135)$$

where

$$\Xi_j(z,\xi) = J_j(z,\xi) + \sum_{\alpha=0}^{N}\Big[d_{j,\alpha}(z)A_{j,\alpha}(-\xi) + \{c_{j,\alpha}(z)$$

$$- \exp[-\sigma_j(z-L)/\xi]a_{j,\alpha}\}A_{j,\alpha}(\xi)\Big] \quad . \qquad (5.136)$$

5.5 Computational Aspects and Numerical Results

Initially we would like to show how several of the basic functions introduced in Section 5.3 can be computed efficiently by recursive relations.

The functions $A_{i,\alpha}(\xi)$ defined by equation (5.77) can be shown to satisfy, for $\alpha > 0$, the recursive relation

$$\alpha A_{i,\alpha-1}(\xi) + (2\alpha+1)(2\xi+1)A_{i,\alpha}(\xi) + (\alpha+1)A_{i,\alpha+1}(\xi)$$

$$= 2(2\alpha+1) \sum_{\ell=0}^{L} (-1)^\ell \beta_{ii}(\ell)g_{i,\ell}(\xi)T_{\alpha,\ell} \quad , \qquad (5.137)$$

where for forward recursion the required initial value can be computed from

$$A_{i,o}(\xi) = \sum_{\ell=0}^{L} (-1)^{\ell} \, \beta_{ii}(\ell) g_{i,\ell}(\xi) C_{\ell}(\xi) \quad . \tag{5.138}$$

Here the functions $C_\ell(\xi)$ can be found from

$$\ell C_{\ell-1}(\xi) + (2\ell+1)\xi C_\ell(\xi) + (\ell+1)C_{\ell+1}(\xi) = (2\ell+1)T_{o,\ell} \tag{5.139}$$

with

$$C_o(\xi) = 1 - \xi \log (1+1/\xi) \quad . \tag{5.140}$$

We recall from Sections 5.3 and 5.4 that the functions $A_{i,\alpha}(\xi)$ are required for real $\xi \notin [-1,0)$. We have found that the use of equation (5.137) in the forward direction is stable only for $\xi \in (-1,0]$, and thus an alternative procedure is desired. Using the Christoffel-Darboux formula (Abramowitz and Stegun, 1964) for the Legendre polynomials, we have deduced the alternative recursion relation

$$P_{\alpha+1}(2\xi+1)A_{i,\alpha}(\xi) + P_\alpha(2\xi+1)A_{i,\alpha+1}(\xi) = (-1)^\alpha \left(\frac{2}{\alpha+1}\right) \Gamma_{i,\alpha}(\xi) \tag{5.141}$$

where

$$\Gamma_{i,o}(\xi) = \sum_{\ell=0}^{L} (-1)^\ell \, \beta_{ii}(\ell) g_{i,\ell}(\xi) T_{o,\ell} \quad , \tag{5.142}$$

and, for $1 \leq \alpha \leq L+1$,

$$\Gamma_{i,\alpha}(\xi) = \Gamma_{i,\alpha-1}(\xi) + (-1)^\alpha (2\alpha+1) P_\alpha(2\xi+1) \sum_{\ell=0}^{L} (-1)^\ell \, \beta_{ii}(\ell) g_{i,\ell}(\xi) T_{\alpha,\ell} \quad . \tag{5.143}$$

Since $T_{\alpha,\ell} = 0$ for $\alpha > \ell+1$, we see that equation (5.143) yields

$$\Gamma_{i,\alpha}(\xi) = \Gamma_{i,L+1}(\xi) \quad , \; \alpha > L+1 \quad . \tag{5.144}$$

We have found that backward recursion of equation (5.141) in the manner suggested by Miller (1952) is stable for real $\xi \notin [-1,0)$. However, as discussed before (Garcia and Siewert, 1981b), such a scheme can be time-consuming for ξ close to the transition points -1 and 0, and for this reason we have actually used forward recursion of equation (5.137) for $\xi \varepsilon [-1.001, -1) \cup [0, 0.001]$ without losing too many significant figures. For other ξ we used backward recursion of equation (5.141). The functions $B_{i,\alpha}(\xi)$ defined by equations (5.78) and required for $\xi \varepsilon P_i$ can be deduced from the recursive relation

$$-\alpha B_{i,\alpha-1}(\xi) + (2\alpha+1)(2\xi-1)B_{i,\alpha}(\xi) - (\alpha+1)B_{i,\alpha+1}(\xi)$$
$$= 2(2\alpha+1)c_i \sum_{\ell=0}^{L} \beta_{ii}(\ell) g_{i,\ell}(\xi) T_{\alpha,\ell} \quad , \tag{5.145}$$

with

$$B_{i,0}(\xi) = 2(1-c_i) + c_i A_{i,0}(\xi) \quad . \tag{5.146}$$

Forward recursion of equation (5.145) can be used efficiently to generate $B_{i,\alpha}(\xi)$ for $\xi \varepsilon [0,1]$. For $\xi = \nu_{i,m}$ it is not difficult to see that $B_{i,\alpha}(\nu_{i,m}) = -c_i A_{i,\alpha}(-\nu_{i,m})$, and thus the recursion relations developed for $A_{i,\alpha}(\xi)$ can be readily used to establish $B_{i,\alpha}(\nu_{i,m})$. We note that the strategy adopted here yielded $A_{i,\alpha}(\xi)$ and $B_{i,\alpha}(\xi)$ accurate to at least 13 significant figures for α up to 40 (working in double precision with an IBM 370/165 machine).

We now turn our attention to the constants $T_{\alpha,\ell}$ defined by equation (5.87). We note that the $T_{\alpha,\ell}$ are a special case (m = 0) of the more general $T^m_{\alpha,\ell}$ considered by Devaux, Siewert, and Yuan (1982) and thus we write for $\alpha \geq 0$ and $\ell \geq 0$:

$$T_{\alpha,\ell+1} = \left(\frac{2\ell+1}{2(\ell+1)}\right)\left[\left(\frac{\alpha}{2\alpha+1}\right)T_{\alpha-1,\ell} + T_{\alpha,\ell} + \left(\frac{\alpha+1}{2\alpha+1}\right)T_{\alpha+1,\ell}\right] - \left(\frac{\ell}{\ell+1}\right)T_{\alpha,\ell-1} \quad . \qquad (5.147)$$

In this equation α runs from α=0 to ℓ+2 (note that $T_{\beta,n} = 0$ for $\beta > n+1$) for each ℓ, from ℓ=0 to L-1. To initiate our calculation we use

$$T_{0,0} = \frac{1}{2} \qquad (5.148)$$

and

$$T_{1,0} = \frac{1}{6} \quad . \qquad (5.149)$$

Finally, the polynomials $G_{i,\alpha}(\xi)$ defined by equation (5.90) and required for $\xi \epsilon [0,1]$ can be computed efficiently by forward recursion from

$$\alpha G_{i,\alpha-1}(\xi) - (2\alpha+1)(2\xi-1)G_{i,\alpha}(\xi) + (\alpha+1)G_{i,\alpha+1}(\xi) = 2(2\alpha+1)\sum_{\ell=0}^{L}\beta_{ii}(\ell)g_{i,\ell}(\xi)T_{\alpha,\ell} \quad , \qquad (5.150)$$

with

$$G_{i,o}(\xi) = 0 \quad . \tag{5.151}$$

In order to demonstrate the computational merit of our solution we now consider a 20-group albedo problem with a 10^{th} order Legendre expansion of the scattering law. A 20-cm thick slab has an isotropically incident distribution of radiation only in the first group and only on the surface at $z = L$, i.e., for $\mu > 0$

$$L_i(\mu) = \delta_{i,1} \tag{5.152a}$$

and

$$R_i(\mu) = 0 \quad . \tag{5.152b}$$

To facilitate the data handling we use a fictitious cross-section set (in units of cm^{-1}) defined, for $i = 1,2,\ldots,20$, by

$$\sigma_i = \left(\frac{i}{10}\right) - 0.15\ \delta_{i,5} - 0.15\ \delta_{i,10} \tag{5.153a}$$

and

$$\sigma_{ij}(\ell) = (2\ell+1)\left[\frac{j}{100(i-j+1)}\right](g_{ij})^{\ell} \quad , j = 1, 2, \ldots, i,$$

$$\ell = 0, 1, \ldots, 10, \tag{5.153b}$$

where

$$g_{ij} = 0.7 - \left(\frac{i+j}{200}\right) \quad . \tag{5.153c}$$

The scattering law defined by equations (5.153b) and (5.153c) corresponds to a truncated ($L = 10$) Henyey-Greenstein phase function introduced in the field of radiative transfer (Henyey and Greenstein, 1941). The Henyey-Greenstein phase function is characterized by one parameter g which is a measure of the degree of anisotropy, i.e., $g \to 1$ implies forward scattering while $g \to -1$ implies backward scattering. For a monoenergetic problem, g corresponds to the average cosine of the scattering angle. In our problem the values of g given by equation (5.153c) correspond to moderate forward scattering and were chosen in order to avoid negative values in the scattering law (with L = 10).

In solving the systems of linear algebraic equations given by equations (5.80) and (5.116) we have used, for various orders of the F_N approximation, the collocation scheme

$$\xi_{i,\beta} = \nu_{i,\beta} \quad , \beta = 0, 1, 2, \ldots, \kappa_i - 1 \quad , \tag{5.154a}$$

and

$$\xi_{i,\beta} = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\beta - 2\kappa_i + 1}{2(N+1-\kappa_i)}\pi\right] \quad , \beta = \kappa_i, \kappa_i + 1, \ldots, N \quad . \tag{5.154b}$$

The points given by equation (5.154b) are the zeros of the Chebyshev polynomial of the first kind $T_{N+1-\kappa_i}(2x-1)$. Based on the results of our computations which followed closely the technique discussed by Siewert (1980), we have concluded that there is only one pair of discrete eigenvalues relevant to each group of the considered problem and thus we list in Table 5.1 the positive eigenvalue for each group.

Table 5.1 The positive discrete eigenvalues $\nu_{i,o}$

i	$\nu_{i,o}$	i	$\nu_{i,o}$	i	$\nu_{i,o}$	i	$\nu_{i,o}$
1	1.0146752301B7	6	1.010101983620	11	1.006629121797	16	1.004095374943
2	1.013664030621	7	1.009324801731	12	1.006052161369	17	1.003687842667
3	1.012702645157	8	1.008590208601	13	1.005511523631	18	1.003310515972
4	1.011789497866	9	1.007896906406	14	1.005005991723	19	1.002962148042
5	1.024569285561	10	1.011201112487	15	1.004534348940	20	1.002641480584

We list our converged results for the exit angular fluxes in Tables 5.2 to 5.9. Further converged results for the angular fluxes at various positions inside the slab are reported in Tables 5.10 to 5.21. We note that to compute the angular fluxes accurately for all μ we used a recently proposed technique (Garcia and Siewert, 1982b). First the functions $B_{i,\alpha}(\xi)$ defined by equation (5.78) are expressed as

$$B_{i,\alpha}(\xi) = 2P_\alpha(2\xi-1) - c_i\{[2-A_{i,o}(\xi)]\, P_\alpha(2\xi-1) + G_{i,\alpha}(\xi)\} \quad (5.155)$$

and this relation can be used in equations (5.73) for $\xi = \mu\epsilon[0,1]$ to find the following alternative expressions for the emerging fluxes:

$$\psi_i(L,-\mu) = R_i(\mu)\exp(-\Delta_i/\mu) + \frac{1}{2}c_i\{I_i(\mu) + [2-A_{i,o}(\mu)]\sum_{\alpha=0}^{N} a_{i,\alpha}P_\alpha(2\mu-1)$$

$$+ \sum_{\alpha=0}^{N} [a_{i,\alpha}G_{i,\alpha}(\mu) - \exp(-\Delta_i/\mu)b_{i,\alpha}A_{i,\alpha}(\mu)]\} + \frac{1}{2}\sum_{j=1}^{i-1} \sigma_{ij}I_{ij}(\mu) \quad (5.156a)$$

Table 5.2 The exit angular fluxes $\psi_i(L,-\mu)$ for i = 1 to 5

μ	i = 1	i = 2	i = 3	i = 4	i = 5
0	5.0885(-2)	1.2999(-2)	5.8860(-3)	3.3641(-3)	3.1535(-3)
0.1	2.8195(-2)	8.6681(-3)	4.2785(-3)	2.5789(-3)	2.4685(-3)
0.2	1.8568(-2)	6.2371(-3)	3.2310(-3)	2.0122(-3)	1.9565(-3)
0.3	1.3061(-2)	4.6596(-3)	2.4996(-3)	1.5956(-3)	1.5724(-3)
0.4	9.6343(-3)	3.5983(-3)	1.9835(-3)	1.2912(-3)	1.2862(-3)
0.5	7.3536(-3)	2.8516(-3)	1.6080(-3)	1.0640(-3)	1.0690(-3)
0.6	5.7554(-3)	2.3033(-3)	1.3241(-3)	8.8871(-4)	8.9923(-4)
0.7	4.6063(-3)	1.8921(-3)	1.1055(-3)	7.5096(-4)	7.6454(-4)
0.8	3.7653(-3)	1.5814(-3)	9.3667(-4)	6.4282(-4)	6.5749(-4)
0.9	3.1192(-3)	1.3378(-3)	8.0287(-4)	5.5637(-4)	5.7091(-4)
1	2.6287(-3)	1.1453(-3)	6.9424(-4)	4.8472(-4)	4.9900(-4)

Table 5.3 The exit angular fluxes $\psi_i(L,-\mu)$ for i = 6 to 10

μ	i = 6	i = 7	i = 8	i = 9	i = 10
0	1.5457(-3)	1.1467(-3)	8.8628(-4)	7.0652(-4)	6.8284(-4)
0.1	1.2701(-3)	9.6248(-4)	7.5717(-4)	6.1275(-4)	6.0047(-4)
0.2	1.0397(-3)	7.9986(-4)	6.3758(-4)	5.2196(-4)	5.1654(-4)
0.3	8.5661(-4)	6.6715(-4)	5.3764(-4)	4.4447(-4)	4.4359(-4)
0.4	7.1511(-4)	5.6261(-4)	4.5765(-4)	3.8138(-4)	3.8335(-4)
0.5	6.0504(-4)	4.8015(-4)	3.9355(-4)	3.3036(-4)	3.3407(-4)
0.6	5.1717(-4)	4.1361(-4)	3.4138(-4)	2.8839(-4)	2.9317(-4)
0.7	4.4586(-4)	3.5904(-4)	2.9821(-4)	2.5337(-4)	2.5882(-4)
0.8	3.8823(-4)	3.1451(-4)	2.6267(-4)	2.2431(-4)	2.3009(-4)
0.9	3.4135(-4)	2.7806(-4)	2.3339(-4)	2.0022(-4)	2.0608(-4)
1	3.0128(-4)	2.4661(-4)	2.0793(-4)	1.7914(-4)	1.8504(-4)

Table 5.4 The exit angular fluxes $\psi_i(L,-\mu)$ for i = 11 to 15

μ	i = 11	i = 12	i = 13	i = 14	i = 15
0	4.8231(-4)	4.0797(-4)	3.4993(-4)	3.0368(-4)	2.6619(-4)
0.1	4.2927(-4)	3.6660(-4)	3.1721(-4)	2.7747(-4)	2.4499(-4)
0.2	3.7343(-4)	3.2144(-4)	2.8019(-4)	2.4677(-4)	2.1926(-4)
0.3	3.2384(-4)	2.8070(-4)	2.4627(-4)	2.1823(-4)	1.9501(-4)
0.4	2.8225(-4)	2.4611(-4)	2.1715(-4)	1.9345(-4)	1.7374(-4)
0.5	2.4783(-4)	2.1722(-4)	1.9262(-4)	1.7240(-4)	1.5552(-4)
0.6	2.1900(-4)	1.9286(-4)	1.7179(-4)	1.5442(-4)	1.3987(-4)
0.7	1.9457(-4)	1.7210(-4)	1.5394(-4)	1.3892(-4)	1.2630(-4)
0.8	1.7397(-4)	1.5448(-4)	1.3870(-4)	1.2563(-4)	1.1461(-4)
0.9	1.5668(-4)	1.3962(-4)	1.2578(-4)	1.1429(-4)	1.0459(-4)
1	1.4137(-4)	1.2641(-4)	1.1426(-4)	1.0416(-4)	9.5611(-5)

Table 5.5 The exit angular fluxes $\psi_i(L,-\mu)$ for i = 16 to 20

μ	i = 16	i = 17	i = 18	i = 19	i = 20
0	2.3537(-4)	2.0970(-4)	1.8810(-4)	1.6974(-4)	1.5400(-4)
0.1	2.1806(-4)	1.9547(-4)	1.7632(-4)	1.5994(-4)	1.4582(-4)
0.2	1.9631(-4)	1.7694(-4)	1.6043(-4)	1.4623(-4)	1.3393(-4)
0.3	1.7554(-4)	1.5902(-4)	1.4487(-4)	1.3264(-4)	1.2199(-4)
0.4	1.5714(-4)	1.4299(-4)	1.3082(-4)	1.2026(-4)	1.1103(-4)
0.5	1.4125(-4)	1.2905(-4)	1.1851(-4)	1.0934(-4)	1.0129(-4)
0.6	1.2752(-4)	1.1692(-4)	1.0775(-4)	9.9742(-5)	9.2697(-5)
0.7	1.1557(-4)	1.0633(-4)	9.8307(-5)	9.1283(-5)	8.5089(-5)
0.8	1.0521(-4)	9.7107(-5)	9.0051(-5)	8.3859(-5)	7.8385(-5)
0.9	9.6301(-5)	8.9134(-5)	8.2882(-5)	7.7384(-5)	7.2514(-5)
1	8.8290(-5)	8.1949(-5)	7.6408(-5)	7.1525(-5)	6.7193(-5)

Table 5.6 The exit angular fluxes $\psi_i(R,\mu)$ for i = 1 to 5

μ	i = 1	i = 2	i = 3	i = 4	i = 5
0	1.1146(-3)	3.4317(-4)	1.7392(-4)	1.0772(-4)	1.1525(-4)
0.1	1.5864(-3)	4.5301(-4)	2.2419(-4)	1.3714(-4)	1.4787(-4)
0.2	2.4017(-3)	5.9970(-4)	2.8704(-4)	1.7251(-4)	1.8566(-4)
0.3	4.9489(-3)	8.0657(-4)	3.7054(-4)	2.1806(-4)	2.3364(-4)
0.4	1.2386(-2)	1.0994(-3)	4.8210(-4)	2.7724(-4)	2.9545(-4)
0.5	2.6436(-2)	1.5012(-3)	6.2888(-4)	3.5321(-4)	3.7446(-4)
0.6	4.6527(-2)	2.0197(-3)	8.1612(-4)	4.4840(-4)	4.7327(-4)
0.7	7.1061(-2)	2.6424(-3)	1.0452(-3)	5.6385(-4)	5.9301(-4)
0.8	9.8380(-2)	3.3420(-3)	1.3129(-3)	6.9894(-4)	7.3295(-4)
0.9	1.2713(-1)	4.0868(-3)	1.6124(-3)	8.5165(-4)	8.9074(-4)
1	1.5630(-1)	4.8463(-3)	1.9351(-3)	1.0191(-3)	1.0630(-3)

Table 5.7 The exit angular fluxes $\psi_i(R,\mu)$ for i = 6 to 10

μ	i = 6	i = 7	i = 8	i = 9	i = 10
0	5.8073(-5)	4.4676(-5)	3.5756(-5)	2.9444(-5)	2.9886(-5)
0.1	7.2858(-5)	5.5763(-5)	4.4432(-5)	3.6442(-5)	3.7031(-5)
0.2	8.9704(-5)	6.8195(-5)	5.4026(-5)	4.4088(-5)	4.4666(-5)
0.3	1.1050(-4)	8.3362(-5)	6.5611(-5)	5.3236(-5)	5.3710(-5)
0.4	1.3656(-4)	1.0217(-4)	7.9848(-5)	6.4390(-5)	6.4659(-5)
0.5	1.6903(-4)	1.2541(-4)	9.7307(-5)	7.7979(-5)	7.7929(-5)
0.6	2.0877(-4)	1.5366(-4)	1.1842(-4)	9.4325(-5)	9.3832(-5)
0.7	2.5621(-4)	1.8722(-4)	1.4338(-4)	1.1359(-4)	1.1253(-4)
0.8	3.1124(-4)	2.2603(-4)	1.7218(-4)	1.3576(-4)	1.3403(-4)
0.9	3.7331(-4)	2.6973(-4)	2.0456(-4)	1.6065(-4)	1.5817(-4)
1	4.4168(-4)	3.1785(-4)	2.4019(-4)	1.8803(-4)	1.8473(-4)

Table 5.8 The exit angular fluxes $\psi_i(R,\mu)$ for i = 11 to 15

μ	i = 11	i = 12	i = 13	i = 14	i = 15
0	2.1534(-5)	1.8639(-5)	1.6352(-5)	1.4499(-5)	1.2971(-5)
0.1	2.6462(-5)	2.2831(-5)	1.9968(-5)	1.7653(-5)	1.5748(-5)
0.2	3.1729(-5)	2.7274(-5)	2.3770(-5)	2.0946(-5)	1.8627(-5)
0.3	3.7922(-5)	3.2465(-5)	2.8187(-5)	2.4748(-5)	2.1934(-5)
0.4	4.5360(-5)	3.8664(-5)	3.3432(-5)	2.9242(-5)	2.5823(-5)
0.5	5.4309(-5)	4.6087(-5)	3.9685(-5)	3.4574(-5)	3.0419(-5)
0.6	6.4972(-5)	5.4898(-5)	4.7079(-5)	4.0858(-5)	3.5817(-5)
0.7	7.7453(-5)	6.5183(-5)	5.5688(-5)	4.8158(-5)	4.2073(-5)
0.8	9.1751(-5)	7.6946(-5)	6.5518(-5)	5.6478(-5)	4.9192(-5)
0.9	1.0777(-4)	9.0113(-5)	7.6512(-5)	6.5776(-5)	5.7142(-5)
1	1.2538(-4)	1.0458(-4)	8.8584(-5)	7.5982(-5)	6.5865(-5)

Table 5.9 The exit angular fluxes $\psi_i(R,\mu)$ for i = 16 to 20

μ	i = 16	i = 17	i = 18	i = 19	i = 20
0	1.1692(-5)	1.0610(-5)	9.6829(-6)	8.8825(-6)	8.1854(-6)
0.1	1.4157(-5)	1.2812(-5)	1.1662(-5)	1.0671(-5)	9.8099(-6)
0.2	1.6695(-5)	1.5066(-5)	1.3677(-5)	1.2482(-5)	1.1446(-5)
0.3	1.9596(-5)	1.7630(-5)	1.5958(-5)	1.4523(-5)	1.3282(-5)
0.4	2.2992(-5)	2.0618(-5)	1.8605(-5)	1.6882(-5)	1.5395(-5)
0.5	2.6989(-5)	2.4122(-5)	2.1698(-5)	1.9629(-5)	1.7847(-5)
0.6	3.1670(-5)	2.8212(-5)	2.5298(-5)	2.2817(-5)	2.0686(-5)
0.7	3.7080(-5)	3.2930(-5)	2.9441(-5)	2.6477(-5)	2.3939(-5)
0.8	4.3229(-5)	3.8284(-5)	3.4135(-5)	3.0620(-5)	2.7615(-5)
0.9	5.0089(-5)	4.4252(-5)	3.9365(-5)	3.5232(-5)	3.1704(-5)
1	5.7615(-5)	5.0797(-5)	4.5098(-5)	4.0285(-5)	3.6183(-5)

Table 5.10 The angular fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/4$ and $i = 1$ to 5

μ	i = 1	i = 2	i = 3	i = 4	i = 5
-1	1.0471(-3)	4.7532(-4)	2.9442(-4)	2.0836(-4)	2.2548(-4)
-0.8	1.4632(-3)	6.3596(-4)	3.8406(-4)	2.6721(-4)	2.8916(-4)
-0.6	2.1471(-3)	8.8218(-4)	5.1690(-4)	3.5252(-4)	3.8060(-4)
-0.4	3.3761(-3)	1.2884(-3)	7.2703(-4)	4.8356(-4)	5.1900(-4)
-0.2	5.7980(-3)	1.9994(-3)	1.0739(-3)	6.9193(-4)	7.3619(-4)
0	1.1621(-2)	3.3843(-3)	1.6919(-3)	1.0444(-3)	1.0971(-3)
0.2	1.0604(-1)	6.1309(-3)	2.7893(-3)	1.6302(-3)	1.6888(-3)
0.4	3.1493(-1)	8.9005(-3)	4.1886(-3)	2.4158(-3)	2.4442(-3)
0.6	4.6327(-1)	1.0270(-2)	5.2044(-3)	3.1081(-3)	3.0428(-3)
0.8	5.6269(-1)	1.0647(-2)	5.7110(-3)	3.5443(-3)	3.3742(-3)
1	6.3224(-1)	1.0504(-2)	5.8661(-3)	3.7561(-3)	3.4995(-3)

Table 5.11 The angular fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/4$ and $i = 6$ to 10

μ	i = 6	i = 7	i = 8	i = 9	i = 10
-1	1.3465(-4)	1.1058(-4)	9.3621(-5)	8.1008(-5)	8.4882(-5)
-0.8	1.6854(-4)	1.3719(-4)	1.1526(-4)	9.9037(-5)	1.0339(-4)
-0.6	2.1598(-4)	1.7403(-4)	1.4490(-4)	1.2351(-4)	1.2833(-4)
-0.4	2.8558(-4)	2.2731(-4)	1.8724(-4)	1.5808(-4)	1.6323(-4)
-0.2	3.9049(-4)	3.0636(-4)	2.4921(-4)	2.0807(-4)	2.1325(-4)
0	5.5598(-4)	4.2861(-4)	3.4339(-4)	2.8291(-4)	2.8730(-4)
0.2	8.1014(-4)	6.1246(-4)	4.8257(-4)	3.9185(-4)	3.9406(-4)
0.4	1.1457(-3)	8.5297(-4)	6.6300(-4)	5.3192(-4)	5.3084(-4)
0.6	1.4886(-3)	1.1061(-3)	8.5588(-4)	6.8292(-4)	6.7815(-4)
0.8	1.7593(-3)	1.3194(-3)	1.0257(-3)	8.1998(-4)	8.1111(-4)
1	1.9369(-3)	1.4720(-3)	1.1550(-3)	9.2922(-4)	9.1575(-4)

Table 5.12 The angular fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/4$ and $i = 11$ to 15

μ	i = 11	i = 12	i = 13	i = 14	i = 15
-1	6.4895(-5)	5.8213(-5)	5.2815(-5)	4.8333(-5)	4.4547(-5)
-0.8	7.8313(-5)	6.9873(-5)	6.3073(-5)	5.7443(-5)	5.2700(-5)
-0.6	9.6221(-5)	8.5334(-5)	7.6589(-5)	6.9376(-5)	6.3321(-5)
-0.4	1.2098(-4)	1.0654(-4)	9.4989(-5)	8.5506(-5)	7.7585(-5)
-0.2	1.5600(-4)	1.3629(-4)	1.2060(-4)	1.0779(-4)	9.7152(-5)
0	2.0702(-4)	1.7916(-4)	1.5715(-4)	1.3931(-4)	1.2459(-4)
0.2	2.7932(-4)	2.3930(-4)	2.0792(-4)	1.8270(-4)	1.6205(-4)
0.4	3.7090(-4)	3.1505(-4)	2.7154(-4)	2.3679(-4)	2.0854(-4)
0.6	4.7031(-4)	3.9743(-4)	3.4080(-4)	2.9574(-4)	2.5922(-4)
0.8	5.6385(-4)	4.7583(-4)	4.0729(-4)	3.5271(-4)	3.0848(-4)
1	6.4306(-4)	5.4362(-4)	4.6572(-4)	4.0342(-4)	3.5278(-4)

Table 5.13 The angular fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/4$ and $i = 16$ to 20

μ	i = 16	i = 17	i = 18	i = 19	i = 20
-1	4.1304(-5)	3.8496(-5)	3.6043(-5)	3.3881(-5)	3.1962(-5)
-0.8	4.8650(-5)	4.5153(-5)	4.2105(-5)	3.9427(-5)	3.7055(-5)
-0.6	5.8170(-5)	5.3738(-5)	4.9888(-5)	4.6514(-5)	4.3537(-5)
-0.4	7.0875(-5)	6.5128(-5)	6.0156(-5)	5.5817(-5)	5.2003(-5)
-0.2	8.8191(-5)	8.0555(-5)	7.3982(-5)	6.8274(-5)	6.3278(-5)
0	1.1228(-4)	1.0185(-4)	9.2930(-5)	8.5228(-5)	7.8524(-5)
0.2	1.4491(-4)	1.3049(-4)	1.1823(-4)	1.0772(-4)	9.8623(-5)
0.4	1.8522(-4)	1.6572(-4)	1.4924(-4)	1.3518(-4)	1.2307(-4)
0.6	2.2918(-4)	2.0416(-4)	1.8308(-4)	1.6515(-4)	1.4977(-4)
0.8	2.7212(-4)	2.4185(-4)	2.1638(-4)	1.9474(-4)	1.7619(-4)
1	3.1106(-4)	2.7628(-4)	2.4699(-4)	2.2208(-4)	2.0073(-4)

Table 5.14 The angular fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/2$ and i = 1 to 5

μ	i = 1	i = 2	i = 3	i = 4	i = 5
-1	4.6637(-4)	2.2300(-4)	1.4098(-4)	1.0055(-4)	1.0980(-4)
-0.8	6.5719(-4)	3.0006(-4)	1.8405(-4)	1.2860(-4)	1.4081(-4)
-0.6	9.7087(-4)	4.1608(-4)	2.4616(-4)	1.6813(-4)	1.8419(-4)
-0.4	1.5206(-3)	5.9792(-4)	3.3951(-4)	2.2623(-4)	2.4721(-4)
-0.2	2.5630(-3)	9.0213(-4)	4.8879(-4)	3.1638(-4)	3.4365(-4)
0	4.7967(-3)	1.4586(-3)	7.4368(-4)	4.6406(-4)	4.9928(-4)
0.2	1.7611(-2)	2.5687(-3)	1.2010(-3)	7.1509(-4)	7.5999(-4)
0.4	1.0153(-1)	4.5595(-3)	1.9722(-3)	1.1178(-3)	1.1749(-3)
0.6	2.1428(-1)	6.8622(-3)	2.9949(-3)	1.6625(-3)	1.7253(-3)
0.8	3.1513(-1)	8.7448(-3)	4.0077(-3)	2.2513(-3)	2.2971(-3)
1	3.9787(-1)	1.0026(-2)	4.8357(-3)	2.7862(-3)	2.7919(-3)

Table 5.15 The angular fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/2$ and i = 6 to 10

μ	i = 6	i = 7	i = 8	i = 9	i = 10
-1	6.5688(-5)	5.3896(-5)	4.5600(-5)	3.9433(-5)	4.1400(-5)
-0.8	8.1781(-5)	6.6465(-5)	5.5781(-5)	4.7897(-5)	5.0105(-5)
-0.6	1.0381(-4)	8.3510(-5)	6.9474(-5)	5.9192(-5)	6.1650(-5)
-0.4	1.3507(-4)	1.0743(-4)	8.8500(-5)	7.4746(-5)	7.7431(-5)
-0.2	1.8139(-4)	1.4239(-4)	1.1596(-4)	9.6954(-5)	9.9770(-5)
0	2.5304(-4)	1.9558(-4)	1.5716(-4)	1.2986(-4)	1.3254(-4)
0.2	3.6685(-4)	2.7848(-4)	2.2033(-4)	1.7961(-4)	1.8155(-4)
0.4	5.3863(-4)	4.0154(-4)	3.1281(-4)	2.5157(-4)	2.5190(-4)
0.6	7.6632(-4)	5.6338(-4)	4.3358(-4)	3.4496(-4)	3.4299(-4)
0.8	1.0242(-3)	7.4781(-4)	5.7151(-4)	4.5170(-4)	4.4733(-4)
1	1.2807(-3)	9.3504(-4)	7.1320(-4)	5.6215(-4)	5.5567(-4)

Table 5.16 The angular fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/2$ and i = 11 to 15

μ	i = 11	i = 12	i = 13	i = 14	i = 15
-1	3.1587(-5)	2.8323(-5)	2.5691(-5)	2.3508(-5)	2.1664(-5)
-0.8	3.7875(-5)	3.3784(-5)	3.0493(-5)	2.7771(-5)	2.5480(-5)
-0.6	4.6145(-5)	4.0923(-5)	3.6736(-5)	3.3286(-5)	3.0392(-5)
-0.4	5.7338(-5)	5.0525(-5)	4.5082(-5)	4.0615(-5)	3.6885(-5)
-0.2	7.3005(-5)	6.3863(-5)	5.6592(-5)	5.0657(-5)	4.5725(-5)
0	9.5688(-5)	8.3007(-5)	7.2978(-5)	6.4843(-5)	5.8122(-5)
0.2	1.2911(-4)	1.1094(-4)	9.6668(-5)	8.5174(-5)	7.5748(-5)
0.4	1.7643(-4)	1.5016(-4)	1.2967(-4)	1.1329(-4)	9.9953(-5)
0.6	2.3717(-4)	2.0027(-4)	1.7167(-4)	1.4894(-4)	1.3053(-4)
0.8	3.0662(-4)	2.5757(-4)	2.1968(-4)	1.8968(-4)	1.6548(-4)
1	3.7912(-4)	3.1754(-4)	2.7006(-4)	2.3251(-4)	2.0228(-4)

Table 5.17 The angular fluxes $\psi_i(z,\mu)$ for $z = L + \Delta/2$ and i = 16 to 20

μ	i = 16	i = 17	i = 18	i = 19	i = 20
-1	2.0087(-5)	1.8722(-5)	1.7530(-5)	1.6480(-5)	1.5549(-5)
-0.8	2.3525(-5)	2.1839(-5)	2.0369(-5)	1.9079(-5)	1.7937(-5)
-0.6	2.7931(-5)	2.5815(-5)	2.3977(-5)	2.2368(-5)	2.0947(-5)
-0.4	3.3725(-5)	3.1018(-5)	2.8677(-5)	2.6633(-5)	2.4835(-5)
-0.2	4.1568(-5)	3.8023(-5)	3.4969(-5)	3.2315(-5)	2.9990(-5)
0	5.2492(-5)	4.7716(-5)	4.3624(-5)	4.0085(-5)	3.7000(-5)
0.2	6.7903(-5)	6.1293(-5)	5.5663(-5)	5.0823(-5)	4.6628(-5)
0.4	8.8931(-5)	7.9705(-5)	7.1895(-5)	6.5220(-5)	5.9467(-5)
0.6	1.1541(-4)	1.0281(-4)	9.2203(-5)	8.3179(-5)	7.5437(-5)
0.8	1.4566(-4)	1.2921(-4)	1.1540(-4)	1.0369(-4)	9.3668(-5)
1	1.7757(-4)	1.5710(-4)	1.3994(-4)	1.2542(-4)	1.1302(-4)

Table 5.18 The angular fluxes $\psi_i(z,\mu)$ for $z = L + 3\Delta/4$ and $i = 1$ to 5

μ	i = 1	i = 2	i = 3	i = 4	i = 5
-1	1.7860(-4)	9.3645(-5)	6.2715(-5)	4.6495(-5)	4.9553(-5)
-0.8	2.5809(-4)	1.3050(-4)	8.4778(-5)	6.1291(-5)	6.5936(-5)
-0.6	3.9570(-4)	1.8906(-4)	1.1763(-4)	8.2318(-5)	8.9400(-5)
-0.4	6.5510(-4)	2.8481(-4)	1.6683(-4)	1.1233(-4)	1.2293(-4)
-0.2	1.1985(-3)	4.4257(-4)	2.4142(-4)	1.5663(-4)	1.7143(-4)
0	2.2618(-3)	7.0354(-4)	3.6161(-4)	2.2667(-4)	2.4631(-4)
0.2	5.4530(-3)	1.2061(-3)	5.7458(-4)	3.4520(-4)	3.7093(-4)
0.4	3.4292(-2)	2.2024(-3)	9.5455(-4)	5.4592(-4)	5.7968(-4)
0.6	9.9505(-2)	3.7912(-3)	1.5582(-3)	8.5481(-4)	8.9957(-4)
0.8	1.7614(-1)	5.6253(-3)	2.3367(-3)	1.2609(-3)	1.3159(-3)
1	2.4965(-1)	7.3271(-3)	3.1650(-3)	1.7184(-3)	1.7754(-3)

Table 5.19 The angular fluxes $\psi_i(z,\mu)$ for $z = L + 3\Delta/4$ and $i = 6$ to 10

μ	i = 6	i = 7	i = 8	i = 9	i = 10
-1	3.1722(-5)	2.6391(-5)	2.2549(-5)	1.9639(-5)	2.0604(-5)
-0.8	4.0308(-5)	3.3052(-5)	2.7903(-5)	2.4057(-5)	2.5181(-5)
-0.6	5.1863(-5)	4.1893(-5)	3.4944(-5)	2.9829(-5)	3.1112(-5)
-0.4	6.7677(-5)	5.3904(-5)	4.4458(-5)	3.7591(-5)	3.9017(-5)
-0.2	9.0429(-5)	7.1062(-5)	5.7945(-5)	4.8510(-5)	5.0033(-5)
0	1.2494(-4)	9.6756(-5)	7.7905(-5)	6.4499(-5)	6.6013(-5)
0.2	1.7989(-4)	1.3697(-4)	1.0869(-4)	8.8846(-5)	9.0085(-5)
0.4	2.6734(-4)	1.9988(-4)	1.5615(-4)	1.2590(-4)	1.2636(-4)
0.6	3.9606(-4)	2.9133(-4)	2.2441(-4)	1.7871(-4)	1.7778(-4)
0.8	5.6355(-4)	4.0973(-4)	3.1239(-4)	2.4652(-4)	2.4372(-4)
1	7.5725(-4)	5.4699(-4)	4.1448(-4)	3.2522(-4)	3.2042(-4)

Table 5.20 The angular fluxes $\psi_i(z,\mu)$ for $z = L + 3\Delta/4$ and i = 11 to 15

μ	i = 11	i = 12	i = 13	i = 14	i = 15
-1	1.5874(-5)	1.4278(-5)	1.2982(-5)	1.1902(-5)	1.0986(-5)
-0.8	1.9123(-5)	1.7087(-5)	1.5444(-5)	1.4082(-5)	1.2934(-5)
-0.6	2.3322(-5)	2.0704(-5)	1.8603(-5)	1.6870(-5)	1.5416(-5)
-0.4	2.8907(-5)	2.5495(-5)	2.2768(-5)	2.0530(-5)	1.8660(-5)
-0.2	3.6635(-5)	3.2082(-5)	2.8460(-5)	2.5502(-5)	2.3043(-5)
0	4.7726(-5)	4.1465(-5)	3.6510(-5)	3.2488(-5)	2.9161(-5)
0.2	6.4217(-5)	5.5291(-5)	4.8272(-5)	4.2612(-5)	3.7962(-5)
0.4	8.8735(-5)	7.5661(-5)	6.5454(-5)	5.7283(-5)	5.0621(-5)
0.6	1.2312(-4)	1.0405(-4)	8.9251(-5)	7.7488(-5)	6.7959(-5)
0.8	1.6695(-4)	1.4013(-4)	1.1943(-4)	1.0304(-4)	8.9839(-5)
1	2.1784(-4)	1.8202(-4)	1.5445(-4)	1.3271(-4)	1.1524(-4)

Table 5.21 The angular fluxes $\psi_i(z,\mu)$ for $z = L + 3\Delta/4$ and i = 16 to 20

μ	i = 16	i = 17	i = 18	i = 19	i = 20
-1	1.0199(-5)	9.5172(-6)	8.9201(-6)	8.3934(-6)	7.9254(-6)
-0.8	1.1952(-5)	1.1105(-5)	1.0365(-5)	9.7154(-6)	9.1398(-6)
-0.6	1.4178(-5)	1.3113(-5)	1.2188(-5)	1.1377(-5)	1.0662(-5)
-0.4	1.7075(-5)	1.5717(-5)	1.4542(-5)	1.3515(-5)	1.2612(-5)
-0.2	2.0969(-5)	1.9199(-5)	1.7674(-5)	1.6347(-5)	1.5184(-5)
0	2.6371(-5)	2.4003(-5)	2.1971(-5)	2.0213(-5)	1.8678(-5)
0.2	3.4087(-5)	3.0817(-5)	2.8029(-5)	2.5628(-5)	2.3544(-5)
0.4	4.5106(-5)	4.0483(-5)	3.6565(-5)	3.3212(-5)	3.0318(-5)
0.6	6.0122(-5)	5.3591(-5)	4.8087(-5)	4.3403(-5)	3.9382(-5)
0.8	7.9028(-5)	7.0060(-5)	6.2534(-5)	5.6156(-5)	5.0702(-5)
1	1.0098(-4)	8.9177(-5)	7.9305(-5)	7.0960(-5)	6.3842(-5)

and

$$\psi_i(R,\mu) = L_i(\mu)\exp(-\Delta_i/\mu) + \frac{1}{2}c_i\{J_i(\mu) + [2-A_{i,o}(\mu)]\sum_{\alpha=0}^{N} b_{i,\alpha}P_\alpha(2\mu-1)$$

$$+ \sum_{\alpha=0}^{N}[b_{i,\alpha}G_{i,\alpha}(\mu)-\exp(-\Delta_i/\mu)a_{i,\alpha}A_{i,\alpha}(\mu)]\} + \frac{1}{2}\sum_{j=1}^{i-1}\sigma_{ij}J_{ij}(\mu) \ . \quad (5.156b)$$

In a similar way we find for the interior fluxes:

$$\psi_i(z,-\mu) = R_i(\mu)\exp[-\sigma_i(R-z)/\mu] + \frac{1}{2}c_i\Big\{I_i(z,\mu)$$

$$+ [2-A_{i,o}(\mu)]\sum_{\alpha=0}^{N} c_{i,\alpha}(z)P_\alpha(2\mu-1) + \sum_{\alpha=0}^{N}\Big[c_{i,\alpha}(z)G_{i,\alpha}(\mu)$$

$$+ \{d_{i,\alpha}(z) - \exp[-\sigma_i(R-z)/\mu]b_{i,\alpha}\}A_{i,\alpha}(\mu)\Big]\Big\}$$

$$+ \frac{1}{2}\sum_{j=1}^{i-1}\sigma_{ij}I_{ij}(z,\mu) \quad (5.157a)$$

and

$$\psi_i(z,\mu) = L_i(\mu)\exp[-\sigma_i(z-L)/\mu] + \frac{1}{2}c_i\Big\{J_i(z,\mu)$$

$$+ [2-A_{i,o}(\mu)]\sum_{\alpha=0}^{N} d_{i,\alpha}(z)P_\alpha(2\mu-1) + \sum_{\alpha=0}^{N}\Big[d_{i,\alpha}(z)G_{i,\alpha}(\mu)$$

$$+ \{c_{i,\alpha}(z) - \exp[-\sigma_i(z-L)/\mu]a_{i,\alpha}\}A_{i,\alpha}(\mu)\Big]\Big\}$$

$$+ \frac{1}{2}\sum_{j=1}^{i-1}\sigma_{ij}J_{ij}(z,\mu) \ . \quad (5.157b)$$

We have found that these expressions are significant improvements, especially as $\mu \to 0$, over the usual (and simpler) expressions given by equations (5.72) and (5.112).

In addition, we report in Table 5.22 converged results for the group fluxes

$$\phi_i(z) = \int_{-1}^{1} \psi_i(z,\mu)d\mu \tag{5.158}$$

which, for z = L and z = R, can be expressed by using equations (5.72) and (5.152) as

$$\phi_i(L) = \delta_{i,1} + a_{i,o} \tag{5.159a}$$

and

$$\phi_i(R) = \delta_{i,1}E_2(\Delta_i) + b_{i,o} \quad , \tag{5.159b}$$

where, in general, $E_n(x)$ denotes exponential integral functions. For $z\varepsilon(L,R)$ we use equations (5.112) and (5.152) in (5.158) to obtain

$$\phi_i(z) = \delta_{i,1}E_2[\sigma_i(z-L)] + c_{i,o}(z) + d_{i,o}(z) \quad . \tag{5.159c}$$

In Table 5.23 we show our converged results for the group albedos

$$A_i^* = 2\int_0^1 \mu\psi_i(L,-\mu)d\mu \tag{5.160a}$$

and the group transmission factors

$$B_i^* = 2\int_0^1 \mu\psi_i(R,\mu)d\mu \quad . \tag{5.160b}$$

Table 5.22 The group fluxes $\phi_i(z)$

i	z = L	z = L+Δ/4	z = L+Δ/2	z = L+3Δ/4	z = R
1	1.0117	3.5594(-1)	1.7050(-1)	8.7855(-2)	4.6716(-2)
2	3.9657(-3)	9.9306(-3)	6.2916(-3)	3.6317(-3)	1.9091(-3)
3	2.0883(-3)	5.0739(-3)	2.9221(-3)	1.5888(-3)	7.8103(-4)
4	1.3228(-3)	3.1027(-3)	1.6910(-3)	8.9818(-4)	4.2724(-4)
5	1.3007(-3)	3.0869(-3)	1.7578(-3)	9.4748(-4)	4.5042(-4)
6	7.0807(-4)	1.5694(-3)	8.2100(-4)	4.3208(-4)	1.9734(-4)
7	5.5208(-4)	1.1895(-3)	6.1336(-4)	3.2196(-4)	1.4496(-4)
8	4.4579(-4)	9.3681(-4)	4.7874(-4)	2.5096(-4)	1.1154(-4)
9	3.6946(-4)	7.5937(-4)	3.8576(-4)	2.0207(-4)	8.8744(-5)
10	3.6961(-4)	7.6078(-4)	3.8710(-4)	2.0269(-4)	8.8221(-5)
11	2.7104(-4)	5.3840(-4)	2.7184(-4)	1.4228(-4)	6.1011(-5)
12	2.3568(-4)	4.6029(-4)	2.3188(-4)	1.2133(-4)	5.1520(-5)
13	2.0748(-4)	3.9910(-4)	2.0072(-4)	1.0500(-4)	4.4159(-5)
14	1.8450(-4)	3.5000(-4)	1.7580(-4)	9.1954(-5)	3.8308(-5)
15	1.6545(-4)	3.0989(-4)	1.5550(-4)	8.1325(-5)	3.3569(-5)
16	1.4946(-4)	2.7666(-4)	1.3871(-4)	7.2537(-5)	2.9673(-5)
17	1.3588(-4)	2.4877(-4)	1.2465(-4)	6.5175(-5)	2.6427(-5)
18	1.2423(-4)	2.2511(-4)	1.1273(-4)	5.8941(-5)	2.3692(-5)
19	1.1415(-4)	2.0486(-4)	1.0254(-4)	5.3609(-5)	2.1365(-5)
20	1.0536(-4)	1.8737(-4)	9.3748(-5)	4.9008(-5)	1.9367(-5)

Table 5.23 A_i^* and B_i^* for the 20-group problem

	Present Work		DTF69	
i	A_i^*	B_i^*	A_i^*	B_i^*
1	6.4394(-3)	7.3100(-2)	6.4399(-3)	7.2983(-2)
2	2.4468(-3)	2.6667(-3)	2.4467(-3)	2.6646(-3)
3	1.3718(-3)	1.0693(-3)	1.3719(-3)	1.0678(-3)
4	9.0655(-4)	5.7560(-4)	9.0668(-4)	5.7472(-4)
5	9.1002(-4)	6.0465(-4)	9.1011(-4)	6.0380(-4)
6	5.1696(-4)	2.5976(-4)	5.1708(-4)	2.5935(-4)
7	4.1123(-4)	1.8942(-4)	4.1134(-4)	1.8912(-4)
8	3.3795(-4)	1.4482(-4)	3.3805(-4)	1.4458(-4)
9	2.8449(-4)	1.1456(-4)	2.8459(-4)	1.1437(-4)
10	2.8836(-4)	1.1340(-4)	2.8844(-4)	1.1321(-4)
11	2.1483(-4)	7.7912(-5)	2.1492(-4)	7.7785(-4)
12	1.8886(-4)	6.5506(-5)	1.8894(-4)	6.5399(-5)
13	1.6797(-4)	5.5914(-5)	1.6805(-4)	5.5822(-5)
14	1.5080(-4)	4.8312(-5)	1.5088(-4)	4.8233(-5)
15	1.3645(-4)	4.2175(-5)	1.3652(-4)	4.2106(-5)
16	1.2430(-4)	3.7144(-5)	1.2437(-4)	3.7082(-5)
17	1.1390(-4)	3.2964(-5)	1.1397(-4)	3.2909(-5)
18	1.0491(-4)	2.9452(-5)	1.0497(-4)	2.9402(-5)
19	9.7071(-5)	2.6472(-5)	9.7135(-5)	2.6427(-5)
20	9.0188(-5)	2.3920(-5)	9.0250(-5)	2.3879(-5)

If we use equations (5.72) and (5.152) in (5.160), we find

$$A_i^* = a_{i,o} + \frac{1}{3} a_{i,1} \tag{5.161a}$$

and

$$B_i^* = 2\delta_{i,1}E_3(\Delta_i) + b_{i,o} + \frac{1}{3} b_{i,1} \quad . \tag{5.161b}$$

All our numerical results are accurate to within ±1 in the last digit shown. In Table 5.23 we show also the results of a calculation by Renken (1981) who used the code DTF69 (Renken and Adams, 1969), with 40 space points and eight directions for each half-range of μ. We observe here, as in the isotropic scattering case, what we believe to be a slight deterioration in the DTF69 results for increasing absorption (as the group number increases).

Regarding the convergence of our method, we have found that to establish $\psi_i(L,-\mu)$ and $\psi_i(R,\mu)$ accurate to five significant figures for all μ required in this case $N = 20$; for the interior angular fluxes and the integrated quantities $\phi_i(z)$, A_i^*, and B_i^* we have found that $N = 15$ was sufficient to obtain five figures of accuracy.

Finally, we would like to mention that we have also generated numerical results for the 19-group problem considered in Chapter 4, but generalized to include anisotropic scattering effects of the Klein-Nishina differential scattering cross section. A set of P_5 multigroup transfer cross sections was provided by Renken (1981). However due to the truncation of the Legendre expansion of the cross section the resulting multigroup transfer cross sections turned out to be negative for some values of the scattering angle. Thus, the familiar and challenging question of how to deal with the solution of a strictly non-physical problem was encountered (Brockmann, 1981). From a mathematical point of view, a transport equation based on cross sections that can be negative is a perfectly valid candidate for study and clearly can yield a solution that can be negative. One can (as we did) solve such a problem and accept, at least on a mathematical basis, the solution--be it positive or otherwise. We note that in comparing our results in Table 5.24 for the considered problem with P_5 scattering with those obtained by Renken (1981) with and without the use of the negative flux fix-up option in the DTF69 code we found excellent agreement with Renken's results only when he did not use the negative flux fix-up option. A separate question is what is the relationship of the solution obtained in this way to the physically correct solution that satisfies the transport equation for which the Klein-Nishina cross section has not been truncated.

Table 5.24 A_i^* and B_i^* for the 19-group problem with P_5 scattering

	Present Work		DTF69[a]		DTF69[b]	
i	A_i^*	B_i^*	A_i^*	B_i^*	A_i^*	B_i^*
1	1.5570(-4)	3.0755(-3)	1.5608(-4)	3.0718(-3)	6.1672(-4)	3.0717(-3)
2	1.3607(-3)	1.7104(-3)	1.3612(-3)	1.7092(-3)	1.5699(-3)	1.7088(-3)
3	2.2129(-3)	1.3852(-3)	2.2131(-3)	1.3844(-3)	2.4448(-3)	1.3838(-3)
4	3.5136(-3)	1.4811(-3)	3.5137(-3)	1.4802(-3)	3.7808(-3)	1.4795(-3)
5	5.2552(-3)	1.5822(-3)	5.2552(-3)	1.5814(-3)	5.3582(-3)	1.5804(-3)
6	7.8082(-3)	1.6861(-3)	7.8081(-3)	1.6852(-3)	7.8424(-3)	1.6841(-3)
7	7.3321(-3)	1.1797(-3)	7.3320(-3)	1.1791(-3)	7.3781(-3)	1.1782(-3)
8	1.0003(-2)	1.2220(-3)	1.0003(-2)	1.2214(-3)	9.9918(-3)	1.2205(-3)
9	1.4220(-2)	1.2593(-3)	1.4220(-2)	1.2587(-3)	1.4190(-2)	1.2576(-3)
10	2.1450(-2)	1.2876(-3)	2.1450(-2)	1.2870(-3)	2.1408(-2)	1.2859(-3)
11	3.5287(-2)	1.2995(-3)	3.5286(-2)	1.2989(-3)	3.5224(-2)	1.2978(-3)
12	6.5003(-2)	1.2754(-3)	6.5003(-2)	1.2748(-3)	6.4895(-2)	1.2736(-3)
13	2.2359(-2)	7.1754(-4)	2.2359(-2)	7.1716(-4)	2.2306(-2)	7.1656(-4)
14	1.7355(-2)	6.2526(-4)	1.7355(-2)	6.2491(-4)	1.7320(-2)	6.2442(-4)
15	1.0196(-2)	4.3287(-4)	1.0196(-2)	4.3262(-4)	1.0174(-2)	4.3228(-4)
16	3.0154(-3)	1.2458(-4)	3.0153(-3)	1.2451(-4)	3.0094(-3)	1.2441(-4)
17	7.3692(-4)	2.7222(-5)	7.3692(-4)	2.7207(-5)	7.3547(-4)	2.7185(-5)
18	4.8899(-5)	1.8750(-6)	4.8898(-5)	1.8740(-6)	4.8800(-5)	1.8725(-6)
19	6.0834(-6)	2.3489(-7)	6.0847(-6)	2.3478(-7)	6.0725(-6)	2.3459(-7)

[a] No negative flux fix-up

[b] With negative flux fix-up

6. MULTISLABS WITH L-TH ORDER ANISOTROPIC SCATTERING

6.1 Introduction

In Chapter 5, a method for solving the multigroup transport equation with a triangular transfer matrix including L^{th} order anisotropic scattering was proposed and numerical results were reported for the case of a single slab. The purpose of this chapter is to extend the method to multislab problems. Because the previously developed analysis can be readily applied in this case, we do not repeat here the derivation reported in Chapter 5. We consider the following multigroup transport equation to be applicable in each of R regions, $z\epsilon[z_{r-1}, z_r]$, $r = 1,2,\ldots,R$, and for each group $i = 1,2,\ldots,M$:

$$\mu \frac{\partial}{\partial z} \psi_i(z,\mu) + \sigma_{i,r} \psi_i(z,\mu)$$

$$= \frac{1}{2} \sum_{j=1}^{i} \sum_{\ell=0}^{L} \sigma_{ij}^{r}(\ell) P_\ell(\mu) \phi_{j,\ell}(z) \quad . \qquad (6.1)$$

Here $\sigma_{i,r}$ is the total cross section for group i and region r and $\sigma_{ij}^{r}(\ell) = \sigma_{ij}^{r} \beta_{ij}^{r}(\ell)$, with $\beta_{ij}^{r}(0) = 1$, denote coefficients in Legendre expansions of the transfer cross sections. As usual, $\psi_i(z,\mu)$ represents the angular flux in the i^{th} group and

$$\phi_{j,\ell}(z) = \int_{-1}^{1} \psi_j(z,\mu)P_\ell(\mu)d\mu \quad . \tag{6.2}$$

We are concerned in this chapter with non-multiplying multislabs and boundary conditions of the type

$$\psi_i(z_0,\mu) = f_{i,0}(\mu) \quad , \mu > 0 \quad , \tag{6.3a}$$

and

$$\psi_i(z_R,-\mu) = f_{i,R}(\mu) \quad , \mu > 0 \quad , \tag{6.3b}$$

where $f_{i,0}(\mu)$ and $f_{i,R}(\mu)$ are considered specified.

6.2 The F_N Method

We note that a generalization of the technique applied in Section 5.2 could be used here to derive singular integral equations and constraints for each group, involving only the boundary data and boundary and interface angular fluxes established for previous groups. By using the F_N method in a manner similar to that reported previously for one-speed problems (Devaux, Grandjean, Ishiguro, and Siewert, 1979), the problem of finding the unknown boundary and interface angular fluxes could be reduced to the solution of a system of linear algebraic equations for each group. Since the size of each of these M systems would be, in the present case, 2(N+1)R, it is clear that for large R the solution of very large systems would be required. We thus prefer to attack the problem in an alternative way: we consider one slab at a time, solve, for each group, R systems of 2(N+1) linear algebraic equations and iterate these solutions on adjacent slabs until convergence is achieved.

We consider, for each $r = 1,2,\ldots,R$, the problem defined by equation (6.1) and boundary conditions written formally as

$$\psi_i(z_{r-1},\mu) = L_{i,r}(\mu) \quad , \mu > 0 \quad , \tag{6.4a}$$

and

$$\psi_i(z_r,-\mu) = R_{i,r}(\mu) \quad , \mu > 0 \quad . \tag{6.4b}$$

Of course, only

$$L_{i,1}(\mu) = f_{i,o}(\mu) \quad , \mu > 0 \quad , \tag{6.5a}$$

and

$$R_{i,R}(\mu) = f_{i,R}(\mu) \quad , \mu > 0 \quad , \tag{6.5b}$$

are presently known. We now follow the analysis of Section 5.2 to derive, for the i^{th} group, the system of singular integral equations and constraints given by equations (5.42) and (5.44) where an index r should be included when defining quantities that depend on the particular region being considered. We let $\Delta_r = z_r - z_{r-1}$, $\Delta_{i,r} = \sigma_{i,r}\,\Delta_r$ and use the approximations

$$\psi_i(z_{r-1},-\mu) = R_{i,r}(\mu)\exp(-\Delta_{i,r}/\mu) + \sum_{\alpha=0}^{N} a_{i,\alpha}^{r}\, P_\alpha(2\mu-1) \tag{6.6a}$$

and

$$\psi_i(z_r,\mu) = L_{i,r}(\mu)\exp(-\Delta_{i,r}/\mu) + \sum_{\alpha=0}^{N} b_{i,\alpha}^{r}\, P_\alpha(2\mu-1) \quad , \tag{6.6b}$$

for $\mu > 0$, to deduce from equations (5.42) and (5.44) that

$$\sum_{\alpha=0}^{N} [a^r_{i,\alpha} B^r_{i,\alpha}(\xi) + c_{i,r} \exp(-\Delta_{i,r}/\xi) b^r_{i,\alpha} A^r_{i,\alpha}(\xi)] = c_{i,r} I_{i,r}(\xi) + \sum_{j=1}^{i-1} \sigma^r_{ij} I^r_{ij}(\xi) \qquad (6.7a)$$

and

$$\sum_{\alpha=0}^{N} [b^r_{i,\alpha} B^r_{i,\alpha}(\xi) + c_{i,r} \exp(-\Delta_{i,r}/\xi) a^r_{i,\alpha} A^r_{i,\alpha}(\xi)] = c_{i,r} J_{i,r}(\xi) + \sum_{j=1}^{i-1} \sigma^r_{ij} J^r_{ij}(\xi) \quad , \qquad (6.7b)$$

for all $\xi \varepsilon P_{i,r} = \{\nu^r_{i,m}\} \cup [0,1]$, where $\nu^r_{i,m}$, $m = 0,1,2,\ldots,\kappa_{i,r} - 1$, denote the positive discrete eigenvalues relevant to group i and region r. All quantities in equations (6.7) are defined precisely in the same way as in Chapter 5 (except for the inclusion of an index r as previously noted). If we now consider equations (6.7) at N+1 values of $\xi \varepsilon P_{i,r}$, say $\xi^r_{i,\beta}$, we obtain a system of linear algebraic equations which can be written in matrix notation as

$$\underset{\sim}{C}_{i,r}\, \underset{\sim}{X}_{i,r} = \underset{\sim}{K}_{i,r} \quad , \qquad (6.8)$$

where

$$\underset{\sim}{C}_{i,r} = \begin{vmatrix} \underset{\sim}{B}_{i,r} & \underset{\sim}{A}_{i,r} \\ \underset{\sim}{A}_{i,r} & \underset{\sim}{B}_{i,r} \end{vmatrix} \quad , \qquad (6.9)$$

$$\underset{\sim}{B}_{i,r} = \begin{vmatrix} B^r_{i,o}(\xi^r_{i,o}) & B^r_{i,1}(\xi^r_{i,o}) & \cdots & B^r_{i,N}(\xi^r_{i,o}) \\ B^r_{i,o}(\xi^r_{i,1}) & B^r_{i,1}(\xi^r_{i,1}) & \cdots & B^r_{i,N}(\xi^r_{i,1}) \\ \vdots & \vdots & & \vdots \\ B^r_{i,o}(\xi^r_{i,N}) & B^r_{i,1}(\xi^r_{i,N}) & \cdots & B^r_{i,N}(\xi^r_{i,N}) \end{vmatrix} . \qquad (6.10a)$$

$$\underset{\sim}{A}_{i,r}=\underset{\sim}{E}_{i,r} \begin{vmatrix} A^r_{i,o}(\xi^r_{i,o}) & A^r_{i,1}(\xi^r_{i,o}) & \cdots & A^r_{i,N}(\xi^r_{i,o}) \\ A^r_{i,o}(\xi^r_{i,1}) & A^r_{i,1}(\xi^r_{i,1}) & \cdots & A^r_{i,N}(\xi^r_{i,1}) \\ \vdots & \vdots & & \vdots \\ A^r_{i,o}(\xi^r_{i,N}) & A^r_{i,1}(\xi^r_{i,N}) & \cdots & A^r_{i,N}(\xi^r_{i,N}) \end{vmatrix} , \qquad (6.10b)$$

$$\underset{\sim}{E}_{i,r} = c_{i,r}\{\exp(-\Delta_{i,r}/\xi^r_{i,o})\ \underset{\sim}{k}_o + \exp(-\Delta_{i,r}/\xi^r_{i,1})\ \underset{\sim}{k}_1$$
$$+ \ldots + \exp(-\Delta_{i,r}/\xi^r_{i,N})\ \underset{\sim}{k}_N\} \quad , \qquad (6.11)$$

$\underset{\sim}{k}_j$ is, in general, a square matrix of order N+1 whose elements are all zero, except for a one in the (j+1)th diagonal position,

$$\underset{\sim}{X}_{i,r} = \begin{vmatrix} a^r_{i,o} \\ a^r_{i,1} \\ \vdots \\ a^r_{i,N} \\ b^r_{i,o} \\ b^r_{i,1} \\ \vdots \\ b^r_{i,N} \end{vmatrix} , \tag{6.12}$$

and

$$\underset{\sim}{K}_{i,r} = \begin{vmatrix} c_{i,r} I_{i,r}(\xi^r_{i,o}) + \sum_{j=1}^{i-1} \sigma^r_{ij} I^r_{ij}(\xi^r_{i,o}) \\ c_{i,r} I_{i,r}(\xi^r_{i,1}) + \sum_{j=1}^{i-1} \sigma^r_{ij} I^r_{ij}(\xi^r_{i,1}) \\ \vdots \\ c_{i,r} I_{i,r}(\xi^r_{i,N}) + \sum_{j=1}^{i-1} \sigma^r_{ij} I^r_{ij}(\xi^r_{i,N}) \\ c_{i,r} J_{i,r}(\xi^r_{i,o}) + \sum_{j=1}^{i-1} \sigma^r_{ij} J^r_{ij}(\xi^r_{i,o}) \\ c_{i,r} J_{i,r}(\xi^r_{i,1}) + \sum_{j=1}^{i-1} \sigma^r_{ij} J^r_{ij}(\xi^r_{i,1}) \\ \vdots \\ c_{i,r} J_{i,r}(\xi^r_{i,N}) + \sum_{j=1}^{i-1} \sigma^r_{ij} J^r_{ij}(\xi^r_{i,N}) \end{vmatrix} . \tag{6.13}$$

If $\underset{\sim}{C}_{i,r}^{-1}$ exists, equation (6.8) yields

$$\underset{\sim}{X}_{i,r} = \underset{\sim}{C}_{i,r}^{-1} \underset{\sim}{K}_{i,r} \quad , \ r = 1, 2, \ldots, R \quad , \tag{6.14}$$

where $\underset{\sim}{K}_{i,r}$, given by equation (6.13), is not completely known because, as can be seen from equations (5.74), $L_{i,r}(\mu)$ and $R_{i,r}(\mu)$ are required to compute $I_{i,r}(\xi)$ and $J_{i,r}(\xi)$ and, as discussed before, $L_{i,r}(\mu)$ and $R_{i,r}(\mu)$ are only formal representations as yet undetermined, with the exceptions of $L_{i,1}(\mu)$ and $R_{i,R}(\mu)$. Therefore, only the summation terms corresponding to down-scattering contributions from previous groups are supposed known in the right-hand side of equation (6.13). From the discussion above it is clear that an iterative solution is needed to establish $\underset{\sim}{X}_{i,r}$ and consequently the emerging angular fluxes given by equations (6.6). In the next section we illustrate such an iterative solution for a specific problem.

6.3 A Test Problem

We now consider a 20-group, 5-region albedo problem with a 10^{th} order Legendre expansion of the scattering law. A 20-cm thick slab has an isotropically incident distribution of radiation only in the first group and only on the surface at $z = z_0$, i.e., for $\mu > 0$

$$L_{i,1}(\mu) = \delta_{i,1} \tag{6.15a}$$

and

$$R_{i,R}(\mu) = 0 \quad . \tag{6.15b}$$

In addition, the thickness of each layer is specified by $\Delta_r = (r+1)$ cm, $r = 1,2,\ldots,5$. By using the fictitious cross-section set that follows, we intend to facilitate the data handling. We define, for $i = 1,2,\ldots,20$ and $r = 1,2,\ldots,5$,

$$\sigma_i^r = \left(\frac{r+20}{21}\right)^5 \left[\left(\frac{i}{10}\right) - 0.15\ \delta_{i,5} - 0.15\ \delta_{i,10}\right] \tag{6.16a}$$

and

$$\sigma_{ij}^r(\ell) = (2\ell+1)\left(\frac{r+20}{21}\right)\left(\frac{j}{100(i-j+1)}\right)(g_{ij})^\ell \ ,$$

$$j = 1,2,\ldots,i \text{ and } \ell = 0,1,\ldots,10, \tag{6.16b}$$

where

$$g_{ij} = 0.7 - \left(\frac{i+j}{200}\right) \ . \tag{6.16c}$$

We note that the cross-section set proposed here is a simple modification of that used in Chapter 5. The same collocation scheme is also used here, i.e.,

$$\xi_{i,\beta}^r = \nu_{i,\beta}^r \ , \ \beta = 0,\ 1,\ 2,\ \ldots,\ \kappa_{i,r} - 1 \ , \tag{6.17a}$$

and

$$\xi_{i,\beta}^r = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\beta-2\kappa_{i,r}+1}{2(N+1-\kappa_{i,r})}\ \pi\right] \ ,$$

$$\beta = \kappa_{i,r},\ \kappa_{i,r}+1,\ \ldots,\ N \ . \tag{6.17b}$$

Based on computations which followed closely the technique discussed by Siewert (1980), we have concluded that there is only one pair of discrete eigenvalues relevant to each group in each region of the considered problem; we list in Table 6.1 the positive eigenvalue $\nu_{i,o}^{r}$ for $i = 1,2,\ldots,20$ and $r = 1,2,\ldots,5$.

We now discuss our iterative procedure to determine $\underset{\sim}{X}_{i,r}$. Initially, for $r = 1$, we use equation (6.15a) and, as a first guess, we set $R_{i,1}(\mu) = 0$ to compute, from equation (6.13), a crude approximation to $\underset{\sim}{K}_{i,1}$ which we denote by $\underset{\sim}{K}_{i,1}^{(o)}$. Accordingly, from equation (6.14), we compute $\underset{\sim}{X}_{i,1}^{(o)}$, i.e., approximate values for $\{a_{i,\alpha}^{1}\}$ and $\{b_{i,\alpha}^{1}\}$. We now note that equations (6.4a) and (6.6b) yield, for $r = 2,3,\ldots,R$,

$$L_{i,r}(\mu) = L_{i,r-1}(\mu)\exp(-\Delta_{i,r-1}/\mu) + \sum_{\alpha=0}^{N} b_{i,\alpha}^{r-1} P_\alpha(2\mu-1) \quad , \qquad \mu > 0 \quad , \tag{6.18a}$$

which together with $R_{i,r}(\mu) = 0$ can be used successively to compute approximations $\underset{\sim}{K}_{i,r}^{(o)}$ and $\underset{\sim}{X}_{i,r}^{(o)}$ for $r = 2,3,\ldots,R$. Likewise, equations (6.4b) and (6.6a) yield, for $r = 1,2,\ldots,R-1$,

$$R_{i,r}(\mu) = R_{i,r+1}(\mu)\exp(-\Delta_{i,r+1}/\mu) + \sum_{\alpha=0}^{N} a_{i,\alpha}^{r+1} P_\alpha(2\mu-1) \quad , \qquad \mu > 0 \quad . \tag{6.18b}$$

The approximations for $\{a_{i,\alpha}^{r}\}$ and $\{b_{i,\alpha}^{r}\}$, $r = 1,2,\ldots R$, found in the previous step can now be used in equations (6.18), and equation (6.14)

Table 6.1 The positive discrete eigenvalues $\nu^{r}_{i,o}$

i	r = 1	r = 2	r = 3	r = 4	r = 5
1	1.014675230187	1.009627226459	1.006235579588	1.003967210867	1.002466094633
2	1.013664030621	1.008862139834	1.005664560862	1.003548929674	1.002166992837
3	1.012702645157	1.008140817000	1.005131686945	1.003163422111	1.001895462297
4	1.011789497866	1.007461768696	1.004635489487	1.002809198427	1.001649975857
5	1.024569285561	1.016300706575	1.010733319846	1.006980415171	1.004462235734
6	1.010101983620	1.006224775612	1.003747380593	1.002188702349	1.001231062637
7	1.009324801731	1.005664061769	1.003352641311	1.001919496742	1.001054616960
8	1.008590208601	1.005140068079	1.002988911056	1.001675703293	1.000898180681
9	1.007896906406	1.004651465922	1.002654795130	1.001455862714	1.000760271267
10	1.011201112487	1.006780184327	1.004002486528	1.002286943006	1.001254754508
11	1.006629121797	1.003775170108	1.002069829189	1.001082218593	1.000534201606
12	1.006052161369	1.003384852919	1.001816186142	1.000925519124	1.000443194621
13	1.005511523631	1.003024672419	1.001586570418	1.000786988284	1.000365036116
14	1.005005991723	1.002693310408	1.001379580893	1.000665212080	1.000298406573
15	1.004534348940	1.002389442258	1.001193817737	1.000558800707	1.000242042897
16	1.004095374943	1.002111736239	1.001027885874	1.000466395680	1.000194746410
17	1.003687842667	1.001858853819	1.000880399460	1.000386677447	1.000155390132
18	1.003310515972	1.001629450936	1.000749987263	1.000318373189	1.000122925006
19	1.002962148042	1.001422180241	1.000635298816	1.000260264497	1.000096384752
20	1.002641480584	1.001235694282	1.000535011145	1.000211194616	1.000074889135

can be solved successively for $r = R-1, R-2, \ldots, 2, 1$ to provide improved approximations for $\{a^r_{i,\alpha}\}$ and $\{b^r_{i,\alpha}\}$. This completes what we call the first sweep of the multislab system. Second, third, and higher order sweeps can be repeated in a similar way by using the updated approximations for $\{a^r_{i,\alpha}\}$ and $\{b^r_{i,\alpha}\}$ until the relative error in these constants for two successive sweeps is as small as desired. In our problem we have found that, in general, five or six sweeps were sufficient to achieve a relative error smaller than 10^{-10}. Once the converged values of $\{a^r_{i,\alpha}\}$ and $\{b^r_{i,\alpha}\}$ are available, appropriate versions of equations (5.156) can be used to compute the angular fluxes emerging from each region, and interior fluxes can also be computed by the method of Section 5.4. We list our converged results for $\psi_i(z_0,-\mu)$ and $\psi_i(z_5,\mu)$, $\mu > 0$, $i = 1,2,\ldots,20$, in Tables 6.2 to 6.9. Converged results for the angular flux at the center of region 3, $z = z_2 + \Delta_3/2$, are reported in Tables 6.10 to 6.13 and for the group fluxes

$$\phi_i(z) = \int_{-1}^{1} \psi_i(z,\mu)d\mu \tag{6.19}$$

in Table 6.14. Finally, converged results for the group albedos

$$A_i^* = 2 \int_0^1 \mu\, \psi_i(z_0,-\mu)d\mu \tag{6.20a}$$

and the group transmission factors

$$B_i^* = 2 \int_0^1 \mu\, \psi_i(z_5,\mu)d\mu \tag{6.20b}$$

are reported in Table 6.15, along with those found by Renken (1981) with the code DTF69 (Renken and Adams, 1969), with 75 space points and eight discrete directions for each half-range of μ. All our numerical results are accurate to within ± 1 in the last digit and the degree of agreement with the DTF69 results is essentially the same as in Chapter 5.

Finally, in regard to the convergence of our method, we have found that $N = 20$ was required to establish the boundary and interface fluxes accurate to five significant figures; for the integrated quantities $\phi_i(z)$, A_i^* and B_i^* and interior angular fluxes $N = 15$ was sufficient to obtain five figures of accuracy.

Table 6.2 The exit angular fluxes $\psi_i(z_0,-\mu)$ for i = 1 to 5

μ	i = 1	i = 2	i = 3	i = 4	i = 5
0	5.0856(-2)	1.2986(-2)	5.8781(-3)	3.3590(-3)	3.1468(-3)
0.1	2.7785(-2)	8.6312(-3)	4.2658(-3)	2.5715(-3)	2.4585(-3)
0.2	1.7783(-2)	6.1190(-3)	3.1971(-3)	1.9979(-3)	1.9352(-3)
0.3	1.2213(-2)	4.4866(-3)	2.4404(-3)	1.5695(-3)	1.5346(-3)
0.4	8.8379(-3)	3.4029(-3)	1.9074(-3)	1.2545(-3)	1.2351(-3)
0.5	6.6435(-3)	2.6534(-3)	1.5233(-3)	1.0202(-3)	1.0094(-3)
0.6	5.1352(-3)	2.1121(-3)	1.2366(-3)	8.4089(-4)	8.3517(-4)
0.7	4.0672(-3)	1.7123(-3)	1.0186(-3)	7.0136(-4)	6.9883(-4)
0.8	3.2968(-3)	1.4145(-3)	8.5238(-4)	5.9298(-4)	5.9203(-4)
0.9	2.7135(-3)	1.1849(-3)	7.2266(-4)	5.0750(-4)	5.0707(-4)
1	2.2726(-3)	1.0048(-3)	6.1816(-4)	4.3720(-4)	4.3734(-4)

Table 6.3 The exit angular fluxes $\psi_i(z_0,-\mu)$ for i = 6 to 10

μ	i = 6	i = 7	i = 8	i = 9	i = 10
0	1.5425(-3)	1.1443(-3)	8.8437(-4)	7.0497(-4)	6.8124(-4)
0.1	1.2658(-3)	9.5926(-4)	7.5467(-4)	6.1074(-4)	5.9839(-4)
0.2	1.0337(-3)	7.9559(-4)	6.3435(-4)	5.1942(-4)	5.1387(-4)
0.3	8.4721(-4)	6.6099(-4)	5.3325(-4)	4.4115(-4)	4.4002(-4)
0.4	7.0147(-4)	5.5382(-4)	4.5146(-4)	3.7693(-4)	3.7844(-4)
0.5	5.8743(-4)	4.6861(-4)	3.8553(-4)	3.2454(-4)	3.2753(-4)
0.6	4.9644(-4)	3.9966(-4)	3.3154(-4)	2.8118(-4)	2.8501(-4)
0.7	4.2292(-4)	3.4321(-4)	2.8682(-4)	2.4490(-4)	2.4920(-4)
0.8	3.6388(-4)	2.9734(-4)	2.5006(-4)	2.1479(-4)	2.1927(-4)
0.9	3.1630(-4)	2.6004(-4)	2.1993(-4)	1.8990(-4)	1.9432(-4)
1	2.7596(-4)	2.2808(-4)	1.9388(-4)	1.6821(-4)	1.7260(-4)

Table 6.4 The exit angular fluxes $\psi_i(z_0,-\mu)$ for i = 11 to 15

μ	i = 11	i = 12	i = 13	i = 14	i = 15
0	4.8119(-4)	4.0701(-4)	3.4911(-4)	3.0296(-4)	2.6556(-4)
0.1	4.2784(-4)	3.6539(-4)	3.1617(-4)	2.7657(-4)	2.4420(-4)
0.2	3.7165(-4)	3.1996(-4)	2.7892(-4)	2.4568(-4)	2.1831(-4)
0.3	3.2164(-4)	2.7888(-4)	2.4475(-4)	2.1692(-4)	1.9388(-4)
0.4	2.7946(-4)	2.4386(-4)	2.1530(-4)	1.9189(-4)	1.7241(-4)
0.5	2.4428(-4)	2.1441(-4)	1.9034(-4)	1.7051(-4)	1.5393(-4)
0.6	2.1461(-4)	1.8941(-4)	1.6901(-4)	1.5213(-4)	1.3795(-4)
0.7	1.8934(-4)	1.6797(-4)	1.5061(-4)	1.3620(-4)	1.2404(-4)
0.8	1.6796(-4)	1.4970(-4)	1.3484(-4)	1.2245(-4)	1.1197(-4)
0.9	1.4999(-4)	1.3426(-4)	1.2141(-4)	1.1069(-4)	1.0159(-4)
1	1.3412(-4)	1.2054(-4)	1.0944(-4)	1.0015(-4)	9.2254(-5)

Table 6.5 The exit angular fluxes $\psi_i(z_0,-\mu)$ for i = 16 to 20

μ	i = 16	i = 17	i = 18	i = 19	i = 20
0	2.3480(-4)	2.0920(-4)	1.8765(-4)	1.6933(-4)	1.5362(-4)
0.1	2.1736(-4)	1.9485(-4)	1.7576(-4)	1.5944(-4)	1.4536(-4)
0.2	1.9547(-4)	1.7620(-4)	1.5977(-4)	1.4564(-4)	1.3339(-4)
0.3	1.7455(-4)	1.5815(-4)	1.4410(-4)	1.3195(-4)	1.2137(-4)
0.4	1.5598(-4)	1.4197(-4)	1.2992(-4)	1.1946(-4)	1.1032(-4)
0.5	1.3988(-4)	1.2786(-4)	1.1747(-4)	1.0842(-4)	1.0048(-4)
0.6	1.2590(-4)	1.1554(-4)	1.0655(-4)	9.8688(-5)	9.1767(-5)
0.7	1.1366(-4)	1.0470(-4)	9.6908(-5)	9.0067(-5)	8.4023(-5)
0.8	1.0299(-4)	9.5223(-5)	8.8435(-5)	8.2461(-5)	7.7165(-5)
0.9	9.3768(-5)	8.6980(-5)	8.1036(-5)	7.5790(-5)	7.1127(-5)
1	8.5449(-5)	7.9528(-5)	7.4329(-5)	6.9729(-5)	6.5630(-5)

Table 6.6 The exit angular fluxes $\psi_i(z_5,\mu)$ for i = 1 to 5

μ	i = 1	i = 2	i = 3	i = 4	i = 5
0	6.3366(-5)	1.7999(-5)	8.6499(-6)	5.1625(-6)	5.1734(-6)
0.1	8.7378(-5)	2.3228(-5)	1.0926(-5)	6.4481(-6)	6.5023(-6)
0.2	1.2533(-4)	3.0390(-5)	1.3902(-5)	8.0858(-6)	8.1596(-6)
0.3	1.9871(-4)	4.0727(-5)	1.8012(-5)	1.0296(-5)	1.0379(-5)
0.4	4.4550(-4)	5.6122(-5)	2.3833(-5)	1.3357(-5)	1.3434(-5)
0.5	1.3050(-3)	7.9503(-5)	3.2169(-5)	1.7639(-5)	1.7691(-5)
0.6	3.4157(-3)	1.1508(-4)	4.4073(-5)	2.3613(-5)	2.3616(-5)
0.7	7.3116(-3)	1.6810(-4)	6.0800(-5)	3.1819(-5)	3.1758(-5)
0.8	1.3259(-2)	2.4387(-4)	8.3692(-5)	4.2820(-5)	4.2697(-5)
0.9	2.1265(-2)	3.4652(-4)	1.1401(-4)	5.7130(-5)	5.6985(-5)
1	3.1162(-2)	4.7836(-4)	1.5281(-4)	7.5179(-5)	7.5100(-5)

Table 6.7 The exit angular fluxes $\psi_i(z_5,\mu)$ for i = 6 to 10

μ	i = 6	i = 7	i = 8	i = 9	i = 10
0	2.5806(-6)	1.9573(-6)	1.5457(-6)	1.2575(-6)	1.2485(-6)
0.1	3.1828(-6)	2.4032(-6)	1.8906(-6)	1.5329(-6)	1.5226(-6)
0.2	3.9235(-6)	2.9460(-6)	2.3067(-6)	1.8626(-6)	1.8463(-6)
0.3	4.8942(-6)	3.6513(-6)	2.8433(-6)	2.2848(-6)	2.2582(-6)
0.4	6.2000(-6)	4.5920(-6)	3.5536(-6)	2.8401(-6)	2.7967(-6)
0.5	7.9782(-6)	5.8625(-6)	4.5060(-6)	3.5797(-6)	3.5104(-6)
0.6	1.0398(-5)	7.5780(-6)	5.7835(-6)	4.5659(-6)	4.4577(-6)
0.7	1.3649(-5)	9.8683(-6)	7.4795(-6)	5.8685(-6)	5.7041(-6)
0.8	1.7926(-5)	1.2865(-5)	9.6887(-6)	7.5584(-6)	7.3170(-6)
0.9	2.3405(-5)	1.6689(-5)	1.2498(-5)	9.7006(-6)	9.3578(-6)
1	3.0229(-5)	2.1436(-5)	1.5976(-5)	1.2348(-5)	1.1878(-5)

Table 6.8 The exit angular fluxes $\psi_i(z_5,\mu)$ for i = 11 to 15

μ	i = 11	i = 12	i = 13	i = 14	i = 15
0	8.9495(-7)	7.6920(-7)	6.7021(-7)	5.9049(-7)	5.2518(-7)
0.1	1.0844(-6)	9.2949(-7)	8.0777(-7)	7.0992(-7)	6.2989(-7)
0.2	1.3080(-6)	1.1176(-6)	9.6841(-7)	8.4872(-7)	7.5102(-7)
0.3	1.5907(-6)	1.3544(-6)	1.1696(-6)	1.0218(-6)	9.0142(-7)
0.4	1.9579(-6)	1.6603(-6)	1.4283(-6)	1.2433(-6)	1.0930(-6)
0.5	2.4410(-6)	2.0608(-6)	1.7654(-6)	1.5305(-6)	1.3404(-6)
0.6	3.0781(-6)	2.5865(-6)	2.2058(-6)	1.9042(-6)	1.6610(-6)
0.7	3.9116(-6)	3.2715(-6)	2.7775(-6)	2.3875(-6)	2.0740(-6)
0.8	4.9849(-6)	4.1508(-6)	3.5091(-6)	3.0042(-6)	2.5995(-6)
0.9	6.3381(-6)	5.2567(-6)	4.4272(-6)	3.7763(-6)	3.2560(-6)
1	8.0043(-6)	6.6163(-6)	5.5541(-6)	4.7226(-6)	4.0596(-6)

Table 6.9 The exit angular fluxes $\psi_i(z_5,\mu)$ for i = 16 to 20

μ	i = 16	i = 17	i = 18	i = 19	i = 20
0	4.7090(-7)	4.2522(-7)	3.8636(-7)	3.5299(-7)	3.2410(-7)
0.1	5.6347(-7)	5.0765(-7)	4.6025(-7)	4.1959(-7)	3.8444(-7)
0.2	6.7009(-7)	6.0222(-7)	5.4467(-7)	4.9540(-7)	4.5285(-7)
0.3	8.0194(-7)	7.1870(-7)	6.4826(-7)	5.8808(-7)	5.3621(-7)
0.4	9.6920(-7)	8.6585(-7)	7.7861(-7)	7.0426(-7)	6.4032(-7)
0.5	1.1843(-6)	1.0543(-6)	9.4486(-7)	8.5185(-7)	7.7208(-7)
0.6	1.4618(-6)	1.2965(-6)	1.1577(-6)	1.0402(-6)	9.3958(-7)
0.7	1.8181(-6)	1.6064(-6)	1.4293(-6)	1.2795(-6)	1.1519(-6)
0.8	2.2701(-6)	1.9985(-6)	1.7718(-6)	1.5808(-6)	1.4183(-6)
0.9	2.8337(-6)	2.4863(-6)	2.1972(-6)	1.9541(-6)	1.7478(-6)
1	3.5226(-6)	3.0818(-6)	2.7157(-6)	2.4085(-6)	2.1483(-6)

Table 6.10 The angular fluxes $\psi_i(z,\mu)$ for $z = z_2 + \Delta_3/2$ and $i = 1$ to 5

μ	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
-1	3.6184(-4)	1.5638(-4)	9.3646(-5)	6.4533(-5)	6.6068(-5)
-0.8	5.0171(-4)	2.0930(-4)	1.2277(-4)	8.3348(-5)	8.5646(-5)
-0.6	7.2982(-4)	2.9089(-4)	1.6621(-4)	1.1075(-4)	1.1420(-4)
-0.4	1.1346(-3)	4.2492(-4)	2.3438(-4)	1.5236(-4)	1.5755(-4)
-0.2	1.9474(-3)	6.6092(-4)	3.4641(-4)	2.1809(-4)	2.2566(-4)
0	3.7995(-3)	1.0988(-3)	5.4058(-4)	3.2815(-4)	3.3814(-4)
0.2	2.1021(-2)	2.0301(-3)	9.0271(-4)	5.2139(-4)	5.3497(-4)
0.4	1.2432(-1)	3.8384(-3)	1.5599(-3)	8.4639(-4)	8.7182(-4)
0.6	2.4736(-1)	5.8053(-3)	2.4324(-3)	1.2967(-3)	1.3293(-3)
0.8	3.5123(-1)	7.2855(-3)	3.2564(-3)	1.7745(-3)	1.7904(-3)
1	4.3378(-1)	8.2115(-3)	3.8945(-3)	2.1939(-3)	2.1725(-3)

Table 6.11 The angular fluxes $\psi_i(z,\mu)$ for $z = z_2 + \Delta_3/2$ and $i = 6$ to 10

μ	$i = 6$	$i = 7$	$i = 8$	$i = 9$	$i = 10$
-1	3.9539(-5)	3.1976(-5)	2.6682(-5)	2.2774(-5)	2.3252(-5)
-0.8	4.9959(-5)	4.0050(-5)	3.3160(-5)	2.8108(-5)	2.8655(-5)
-0.6	6.4587(-5)	5.1241(-5)	4.2047(-5)	3.5361(-5)	3.5967(-5)
-0.4	8.5842(-5)	6.7294(-5)	5.4664(-5)	4.5574(-5)	4.6189(-5)
-0.2	1.1797(-4)	9.1278(-5)	7.3334(-5)	6.0556(-5)	6.1062(-5)
0	1.6929(-4)	1.2903(-4)	1.0233(-4)	8.3552(-5)	8.3687(-5)
0.2	2.5349(-4)	1.8976(-4)	1.4822(-4)	1.1942(-4)	1.1866(-4)
0.4	3.8345(-4)	2.8147(-4)	2.1636(-4)	1.7198(-4)	1.6966(-4)
0.6	5.5908(-4)	4.0352(-4)	3.0587(-4)	2.4029(-4)	2.3615(-4)
0.8	7.5900(-4)	5.4334(-4)	4.0840(-4)	3.1833(-4)	3.1261(-4)
1	9.5642(-4)	6.8511(-4)	5.1381(-4)	3.9913(-4)	3.9222(-4)

Table 6.12 The angular fluxes $\psi_i(z,\mu)$ for $z = z_2 + \Delta_3/2$ and i = 11 to 15

μ	i = 11	i = 12	i = 13	i = 14	i = 15
-1	1.7696(-5)	1.5718(-5)	1.4123(-5)	1.2805(-5)	1.1699(-5)
-0.8	2.1572(-5)	1.9061(-5)	1.7044(-5)	1.5385(-5)	1.3996(-5)
-0.6	2.6766(-5)	2.3519(-5)	2.0921(-5)	1.8793(-5)	1.7020(-5)
-0.4	3.3976(-5)	2.9672(-5)	2.6246(-5)	2.3451(-5)	2.1133(-5)
-0.2	4.4376(-5)	3.8491(-5)	3.3828(-5)	3.0044(-5)	2.6920(-5)
0	5.9990(-5)	5.1616(-5)	4.5020(-5)	3.9700(-5)	3.5333(-5)
0.2	8.3727(-5)	7.1365(-5)	6.1701(-5)	5.3963(-5)	4.7657(-5)
0.4	1.1776(-4)	9.9441(-5)	8.5227(-5)	7.3932(-5)	6.4792(-5)
0.6	1.6132(-4)	1.3517(-4)	1.1502(-4)	9.9120(-5)	8.6325(-5)
0.8	2.1080(-4)	1.7565(-4)	1.4871(-4)	1.2754(-4)	1.1060(-4)
1	2.6230(-4)	2.1779(-4)	1.8377(-4)	1.5713(-4)	1.3587(-4)

Table 6.13 The angular fluxes $\psi_i(z,\mu)$ for $z = z_2 + \Delta_3/2$ and i = 16 to 20

μ	i = 16	i = 17	i = 18	i = 19	i = 20
-1	1.0758(-5)	9.9483(-6)	9.2458(-6)	8.6311(-6)	8.0892(-6)
-0.8	1.2819(-5)	1.1810(-5)	1.0937(-5)	1.0175(-5)	9.5048(-6)
-0.6	1.5522(-5)	1.4243(-5)	1.3139(-5)	1.2178(-5)	1.1336(-5)
-0.4	1.9182(-5)	1.7522(-5)	1.6095(-5)	1.4857(-5)	1.3775(-5)
-0.2	2.4303(-5)	2.2086(-5)	2.0188(-5)	1.8547(-5)	1.7119(-5)
0	3.1697(-5)	2.8631(-5)	2.6020(-5)	2.3774(-5)	2.1827(-5)
0.2	4.2440(-5)	3.8070(-5)	3.4370(-5)	3.1206(-5)	2.8477(-5)
0.4	5.7282(-5)	5.1031(-5)	4.5769(-5)	4.1295(-5)	3.7457(-5)
0.6	7.5871(-5)	6.7214(-5)	5.9961(-5)	5.3822(-5)	4.8578(-5)
0.8	9.6807(-5)	8.5428(-5)	7.5928(-5)	6.7913(-5)	6.1087(-5)
1	1.1860(-4)	1.0440(-4)	9.2568(-5)	8.2607(-5)	7.4141(-5)

Table 6.14 The group fluxes $\phi_i(z)$

i	$z = z_0$	$z = z_2+\Delta_3/2$	$z = z_5$
1	1.0112	1.9270(-1)	6.2131(-3)
2	3.8166(-3)	5.1651(-3)	1.3399(-4)
3	2.0235(-3)	2.3027(-3)	4.7871(-5)
4	1.2877(-3)	1.2863(-3)	2.4981(-5)
5	1.2533(-3)	1.3082(-3)	2.4979(-5)
6	6.9222(-4)	5.8515(-4)	1.0738(-5)
7	5.4114(-4)	4.2893(-4)	7.7753(-6)
8	4.3784(-4)	3.2930(-4)	5.9019(-6)
9	3.6347(-4)	2.6157(-4)	4.6376(-6)
10	3.6290(-4)	2.5884(-4)	4.5126(-6)
11	2.6722(-4)	1.7926(-4)	3.1007(-6)
12	2.3262(-4)	1.5150(-4)	2.5970(-6)
13	2.0497(-4)	1.2999(-4)	2.2080(-6)
14	1.8241(-4)	1.1293(-4)	1.9006(-6)
15	1.6369(-4)	9.9125(-5)	1.6534(-6)
16	1.4796(-4)	8.7789(-5)	1.4514(-6)
17	1.3459(-4)	7.8354(-5)	1.2842(-6)
18	1.2311(-4)	7.0410(-5)	1.1443(-6)
19	1.1317(-4)	6.3655(-5)	1.0259(-6)
20	1.0449(-4)	5.7858(-5)	9.2482(-7)

Table 6.15 The group albedos A_i^* and the transmission factors B_i^*

	Present Work		DTF69	
i	A_i^*	B_i^*	A_i^*	B_i^*
1	5.8809(-3)	1.0453(-2)	5.8821(-3)	1.0439(-2)
2	2.2791(-3)	1.9993(-4)	2.2798(-3)	1.9965(-4)
3	1.2939(-3)	6.9012(-5)	1.2944(-3)	6.8915(-5)
4	8.6280(-4)	3.5393(-5)	8.6319(-4)	3.5346(-5)
5	8.5170(-4)	3.5350(-5)	8.5202(-4)	3.5300(-5)
6	4.9662(-4)	1.4899(-5)	4.9692(-4)	1.4879(-5)
7	3.9706(-4)	1.0716(-5)	3.9733(-4)	1.0702(-5)
8	3.2763(-4)	8.0863(-6)	3.2787(-4)	8.0757(-6)
9	2.7671(-4)	6.3202(-6)	2.7693(-4)	6.3119(-6)
10	2.7956(-4)	6.1271(-6)	2.7977(-4)	6.1191(-6)
11	2.0989(-4)	4.1837(-6)	2.1007(-4)	4.1782(-6)
12	1.8491(-4)	3.4892(-6)	1.8508(-4)	3.4847(-6)
13	1.6476(-4)	2.9545(-6)	1.6491(-4)	2.9506(-6)
14	1.4814(-4)	2.5332(-6)	1.4828(-4)	2.5299(-6)
15	1.3423(-4)	2.1953(-6)	1.3435(-4)	2.1924(-6)
16	1.2242(-4)	1.9200(-6)	1.2253(-4)	1.9174(-6)
17	1.1229(-4)	1.6927(-6)	1.1240(-4)	1.6904(-6)
18	1.0352(-4)	1.5029(-6)	1.0362(-4)	1.5009(-6)
19	9.5859(-5)	1.3428(-6)	9.5952(-5)	1.3409(-6)
20	8.9125(-5)	1.2064(-6)	8.9210(-5)	1.2047(-6)

7. CONCLUSIONS

In this work we have successfully used the F_N method to solve basic multigroup transport problems in plane geometry. We have concluded from our studies that the F_N method is capable of producing accurate results for the considered multigroup model and that the most interesting aspect of the method seems to be the capability of finding the angular fluxes emerging from a slab for a given group by using only the boundary data and established emerging fluxes for preceding groups. This feature of the F_N method is a particularly attractive one for shielding calculations where frequently the interior angular fluxes are not of primary interest. In a few situations where the knowledge of the interior fluxes is important, e.g., gamma-ray heating, these can be readily computed in our method from the previously determined boundary fluxes.

We note that for most cases the results deduced from the method of discrete ordinates are clearly adequate--especially when we consider the magnitude of the uncertainties normally associated with the input data. However, for strong absorption and/or optically thick slabs, increased computer time will be required by strictly numerical methods to achieve a desired degree of accuracy--a characteristic not shared by the F_N method. In fact, we have observed that slabs with strong absorption and/or large optical thickness is precisely the most favorable situation for the F_N method, in the sense that accurate results can be produced with a small N.

The primary objective of our work on the numerical aspects of the F_N method was to extract the maximum obtainable accuracy from the method with a reasonable amount of computation time. Thus, several improvements were incorporated into the numerical framework of the F_N method during the course of this research:

a. the use of the orthogonal basis $P_\alpha(2\mu-1)$ instead of powers μ^α

b. the use of a collocation scheme based on the zeros of Chebyshev polynomials in contrast to the previously used equally spaced schemes

c. introduction of a new technique for computing the angular fluxes accurately for all μ.

It was not our aim to compare the computational efficiency of the F_N method with that of existing methods in this work; clearly this is an important aspect that needs to be studied, and topics such as a study of alternative techniques for computing the required functions, the viability of using single precision throughout the developed computer programs and the task of optimizing the developed computer programs are interesting aspects that deserve consideration. It is our feeling, however, that our method is competitive, especially if only boundary quantities are desired in a calculation.

Additional recommendations for future research include the extension of the method for multiplying media and inclusion of upscattering in the transfer matrix. These should be rather simple extensions but very valuable for applications in reactor analysis. The extension of the method to spherical and cylindrical geometries and time-dependent problems seems possible, at least for simple problems.

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