INTEGRAL EQUATION FORMULATION FOR BOUNDARY DATA RECONSTRUCTION IN ELASTOSTATICS

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Introduction

This paper deals with the solution of the Inverse Elastostatics Problem (IESP) of reconstruction of boundary tractions using response measurements from sensors located at discrete, interior or exterior, locations of a solid. Computational techniques for the solution of such problems may provide non-destructive evaluation tools, such as identifying contact regions in neighboring objects, as well as hybrid experimental and numerical methods for the analyses of solids [1, 2].

In solid mechanics, given sufficient traction and displacement boundary conditions, one can solve the boundary value problem for displacement, strain, and stress fields [3]. The same is not true when a boundary data is missing. This type of problem is defined as an inverse problem [4]. Much of the literature on the solution of inverse problems has been devoted to Inverse Heat Conduction Problems IHCP [5]. A limited literature exists on the solution of IESP's. The IESP's fall under two main categories, namely the reconstruction problems and the identification problems [6]. Solutions for inverse problems do not necessarily satisfy conditions of existence, uniqueness, and stability [7]. The numerical solution of inverse problems may be obtained using the finite element method (FEM) or the boundary element method (BEM). The significant advantages offered by the BEM for such problems were outlined in Bezerra and Saigal [8] where a BEM based formulation for inverse identification problems was presented. The inverse problem of boundary traction reconstruction in elastostatics, despite its numerous physical applications, has received scant attention in the literature. Such formulations, may be employed in characterizing tractions at inaccessible regions of critical components in sensitive mechanical equipment. Maniatty, Zabaras, Stelson, Molerras, and Schnur [9, 10, 11] have used the FEM and the BEM along with the spatial "key-node" regularization procedure for the solution of such problems. These contributions have been presented only recently and deal with the determination of the magnitude of simple distributions of tractions at a given location on the surface of the body. For realistic applications, it is necessary to develop further formulations to treat more general traction distributions as well as to include the treatment of a priori unknown regions of the application of these tractions. In this paper, the IESP of boundary traction reconstruction is first explained and briefly defined in terms of mathematical equations. The problem is then formulated as a constrained non-linear least-squares optimization problem in a BEM framework. Using function specifications for the unknown boundary tractions and opitimization methods, the solution procedure adopted seeks to minimize, in the least-squares sense, the difference between the vector $\hat{\psi}$ consisting of simulated experimental data and the vector ψ of corresponding computed quantities. The geometric constraints that the boundary tractions lie within a certain given portion of the boundary of the solid

Structural Optimization 93 - World Congress, Rio de Janeiro, RJ, August 2-6, 1993 are imposed when necessary by a simple direct search approach. The design sensitivities required in the numerical optimization procedure are obtained by the implicit differentiation [12] of the boundary integral equations. An example involving a rectangular panel is presented to demonstrate the effectiveness of the present solution procedures in reconstructing boundary tractions. Good results were solution was obtained for this case including with the introduction of small Gaussian errors in the simulated experimental data. It is noted that the present study deals with the determination of: (a) the magnitude and extent of the missing boundary traction data, and (b) the location of the distribution. The latter has not been attempted previously in the literature to the best knowledge of the authors.

Definition of the Problem

Consider a homogeneous, isotropic, linear elastic, two-dimensional solid, Ω , bounded by its boundary Γ. The direct field problems in elastostatics involve the determination of displacement, strain, and stress fields in Ω , provided the following are known [4]: (a) the domain, Ω , and the boundaries, Γ , of the solid, (b) the governing equations in the domain, (c) the appropriate boundary conditions on Γ, (d) the material properties involved in the governing equations, and (e) the forces or other inputs acting on the solid. Under these conditions the solution may be calculated by a direct analysis using analytical or numerical schemes. If any of the above information is lacking, incomplete, or overdefined, a direct analysis cannot be carried out, and the problem is regarded as an inverse problem. In this study, the reconstruction of a missing boundary traction, along the boundary, Γ , of the solid, Ω , constitutes an inverse problem since: (a) the magnitude and the location of the boundary traction are not known, (b) internal measurement data within the solid, Ω , is available as additional information to overcome the lack of sufficient boundary conditions. In mathematical notation, this can be expressed as

$$\sigma_{ij,i}(\mathbf{x}) = -b_j(\mathbf{x});\ \mathbf{x} \in \Omega \tag{1}$$

$$\sigma_{ij}(\mathbf{x}) = \lambda\delta_{ij}\varepsilon_{ll}(\mathbf{x}) + 2\mu\varepsilon_{ij}(\mathbf{x}) \tag{2}$$

$$\sigma_{ij}(\mathbf{y})n_j(\mathbf{y}) = \bar{t}_i;\ \mathbf{y} \in \Gamma \tag{3}$$

$$u_i(\mathbf{y}) = \bar{u}_i \tag{4}$$

$$\psi_i(\mathbf{v}_k) = \hat{\psi}_{ik};\ \mathbf{v}_k \in \Omega \tag{5}$$

where σ_{ij} is the stress tensor; $i,j,l = 1,2$; b_j are the body forces; ε_{ij} is the strain tensor; λ and μ are the Lamé's constants; δ_{ij} is the Kroneker's delta; n_j denotes the outer normal to the boundary Γ; t_i and u_i are the tractions and displacements, respectively; an overbar ($^-$) denotes prescribed quantities; a tilda ($\hat{}$) denotes experimental measurements; ψ_i are the measured displacements, strains, or stresses along direction i at location k ($k = 1,2,3....m$); and m is the total number of experimental measurements available. In this study, the m observations $\hat{\psi}_{ik}$ lie inside Ω . Eqs. (1) and (2) denote, respectively, the equilibrium equations and the constitutive relations; Eqs (3) and (4) denote the traction and displacement boundary conditions, respectively; and Eq. (5) denotes the measured data. The solution of the inverse problem of boundary traction reconstruction involves the determination of the design variables, $\mathbf{z}$, such

that the quantities ψ match the experimentally measured quantities $\hat{\psi}$ in the least-squares sense. This is accomplished by minimizing the difference between the mapping $\psi = \mathbf{Az}$ and the data vector $\hat{\psi}$ and is expressed as

$$f(\mathbf{z}) = w\sum_{k=1}^{m}\sum_{i=1}^{2}\left(\psi_{ik} - \hat{\psi}_{ik}\right)^2 \tag{6}$$

where w is a weighting parameter included to enhance numerical sensitivity in the minimization process. Due to the ill-posed nature of the problem, small changes in the given data may produce distinctly answers unless restrictions on the smoothness of the solution are imposed [5, 7, 11]. To overcome such instabilities, *a-priori* information on the desired solution is often introduced in the form of smoothness conditions that are implemented through the use of approximating functions or by regularization of the objective function to be minimized [5, 7]. In the present study, the unknown tractions, $\Phi_j = \sigma_{ij}n_j$, that are required to be reconstructed are assumed in the form $\Phi_j = \Phi_j(\mathbf{z})$; where ω, η, and v are a magnitude, a span, and a position that completely define the missing boundary traction using the smooth function Φ_j. The form of the smooth function, Φ_j, is assumed to be known.

Minimization Procedure

The numerical procedure adopted in this study for the solution of the IESP of boundary traction reconstruction involves the determination of the model vector, $\mathbf{z}$, such that $f(\mathbf{z})$ in Eq. (6) is a minimum. The vector $\mathbf{z}$ contains the parameters that completely define the position and the amplitude of the function describing the missing boundary tractions. The possible location of the missing tractions is limited to a bounded set of locations on the body, say $\bar{\Gamma}_1$. This condition of limiting the location of the missing boundary tractions to a feasible geometrical region is expressed in the form of constraint equations as

$$C_j(z_i) = \pm z_i \mp (a \pm \epsilon) \geq 0;\ i = 1,p;\ j = 1,L \tag{7}$$

where z_i are the components of the vector $\mathbf{z}$; L is the number of geometry constraints; p is the number of design variables used to define the missing boundary tractions; a is a constant and represents the bounds of the domain; and ϵ is a small number to ensure that the reconstructed tractions lie within an $\epsilon -$ *neighborhood* inside the prescribed portion, $\bar{\Gamma}_1$, of the object boundary. The constraints in Eq. (7), together with the minimization of Eq. (6), lead to a constrained minimization problem in the theory of optimization. In this paper, when a constraint is violated, the technique of retracting the pattern of the step-length is used. Such a procedure must start from a feasible point and be exercised with care or the minimization process may terminate prematurely [13, 14]. An alternative strategy using inverse penalty functions was presented by Bezerra and Saigal in Refs. [15]. Numerous other direct search techniques are available in the literature. Further discussions on these methods may be found in Ref. [13, 14]. In this study, an unconstrained method to search for the minimum is adopted after the step-length reduction is performed to avoid the violation of a constraint. The unconstrianed variable metric method is adopted for this purpose. This method involves constructing in each iteration n, a good approximation, $\mathbf{A}^{(n)}$, to the inverse Hessian matrix $\mathbf{H}^{-1}$ by computing only the gradient $\nabla f(\mathbf{z})$. The BFGS algorithm was adopted to update $\mathbf{A}^{(n)}$ [16]. This algorithm starts with an initial feasible guess for the missing tractions defined by the vector $\mathbf{z}^{(0)}$, and generates subsequent updates to this vector according to the following relations

$$\mathbf{z}^{(n+1)} = \mathbf{z}^{(n)} + \alpha^{(n)} S\left(\mathbf{z}^{(n)}\right) \tag{8}$$

$$S\left(\mathbf{z}^{(n)}\right) = S^{(n)} = -\mathbf{A}^{(n)}\mathbf{g}^{(n)} \tag{9}$$

$$\mathbf{g}^{(n)} = \nabla f(\mathbf{z}^{(n)}) = \frac{\partial f(\mathbf{z}^{(n)})}{\partial \mathbf{z}} = 2w\sum_{k=1}^{m}\sum_{i=1}^{2}\left(\psi_{ik} - \hat{\psi}_{ik}\right)\frac{\partial \psi_{ik}}{\partial \mathbf{z}} \tag{10}$$

$$\mathbf{A}^{(n+1)} = \mathbf{A}^{(n)} - \frac{\mathbf{A}^{(n)}\Delta\mathbf{z}^{(n)}\Delta\mathbf{g}^{(n)T}\mathbf{A}^{(n)}}{\Delta\mathbf{z}^{(n)T}\mathbf{A}^{(n)}\Delta\mathbf{g}^{(n)}} + \frac{\Delta\mathbf{g}^{(n)}\Delta\mathbf{g}^{(n)T}}{\Delta\mathbf{g}^{(n)T}\Delta\mathbf{z}^{(n)}} \tag{11}$$

$$\Delta\mathbf{z}^{(n)} = \mathbf{z}^{(n+1)} - \mathbf{z}^{(n)} \tag{12}$$

$$\Delta\mathbf{g}^{(n)} = \mathbf{g}\left(\mathbf{z}^{(n+1)}\right) - \mathbf{g}\left(\mathbf{z}^{(n)}\right) \tag{13}$$

where $S^{(n)}$ is the search direction; $\alpha^{(n)}$ is the step-length along $S^{(n)}$; ∇ is the gradient operator; and the sequence $\mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)} \ldots \mathbf{A}^{(n+1)}$ approaches $\mathbf{H}^{-1}$, starting with $\mathbf{A}^{(0)} = I$. The above procedure reduces the problem to a unidimensional optimization problem of determining the scalar $\alpha^{(n)}$, in Eq. (8), that minimizes the objective function along the direction $S^{(n)}$. To ensure that the components of the vector $\mathbf{z}^{(n)}$ lie inside the feasible boundary domain, $\bar{\Gamma}_1$, the step-length $\alpha^{(n)}$ is retracted by 10%, whenever $\mathbf{z}^{(n)}$ lies outside the feasible domain. Given three feasible points, $\alpha^{(a)} < \alpha^{(b)} < \alpha^{(c)}$ that isolate the minimum along direction $S^{(n)}$, Brent's method [16] is applied to find the minimun of $f(\mathbf{z})$. With $f(\alpha^{(a)}) = f(\mathbf{z}^{(a)})$, $f(\alpha^{(b)}) = f(\mathbf{z}^{(b)})$, and $f(\alpha^{(c)}) = f(\mathbf{z}^{(c)})$, solving the inverse interpolation problem, the variable $\alpha_m^{(n)}$ denoting the minimum of the interpolating parabola, is found as

$$\alpha_m^{(n)} = \alpha^{(b)} + \frac{1}{2}\frac{\left(\alpha^{(b)} - \alpha^{(a)}\right)^2\left[f(\alpha^{(b)}) - f(\alpha^{(c)})\right] - \left(\alpha^{(b)} - \alpha^{(c)}\right)^2\left[f(\alpha^{(b)}) - f(\alpha^{(a)})\right]}{\left(\alpha^{(b)} - \alpha^{(a)}\right)\left[f(\alpha^{(b)}) - f(\alpha^{(c)})\right] - \left(\alpha^{(b)} - \alpha^{(c)}\right)\left[f(\alpha^{(b)}) - f(\alpha^{(a)})\right]} \tag{14}$$

The above relation fails only if the three points are collinear. Brent's method takes care of this situation by shifting the search for the minimum to the Golden Section method [17] whenever necessary. Upon determining the appropriate $\alpha_m^{(n)}$ that minimizes $f(\mathbf{z})$ in the search direction corresponding to iteration n, Eqs. (11), (9), and (8) are used to update $\mathbf{A}$, S, and $\mathbf{z}$, respectively. If convergence has not been achieved, the next iteration then starts with these updated values.

BEM and Sensitivity Analysis

The sensitivities, $\partial\psi/\partial\mathbf{z}$, in Eq. (10), are determined in this study using the boundary element method (BEM). The compelling advantages of the BEM for sensitivity analysis have been demonstrated in, Ref. [12], among others. The analytical formulation and numerical implementation considerations for two-dimensional elastostatics sensitivity analysis considered here are available in Ref. [12] and are only briefly discussed below.

The BEM equations, starting from the Somigliana's identity [18] for elastostatics and after discretization using interpolation functions, are written in the matrix form as

$$[\mathrm{F}]\{\mathrm{u}\} = [\mathrm{G}]\{\mathrm{t}\} + \{\mathrm{q}\} \tag{15}$$

where [F] and [G] are the system matrices; {u} and {t} are the vectors of displacements and tractions, respectively; and {q} is the vector of other influences such as body forces, etc. The implicit differentiation of Eq. (15) with respect to the design variable, $\mathbf{z}$, leads to

$$[\mathrm{F}]\{\mathrm{u}\}_{,\mathbf{z}} = [\mathrm{G}]\{\mathrm{t}\}_{,\mathbf{z}} + [\mathrm{G}]_{,\mathbf{z}}\{\mathrm{t}\} - [\mathrm{F}]_{,\mathbf{z}}\{\mathrm{u}\} + \{\mathrm{q}\}_{,\mathbf{z}} \tag{16}$$

where the subscript $(,\mathbf{z})$ denotes differentiation with respect to $\mathbf{z}$. The derivations of the matrices [F] and [G] and their respective sensitivities as well as techniques for their numerical evaluations may be found in, for example, Ref [12]. It is noted that the model vector, $\mathbf{z}$, contains parameters that define the location, the distribution, and the magnitude of the missing tractions. The sensitivities $[\mathrm{F}]_{,\mathbf{z}}$ and $[\mathrm{G}]_{,\mathbf{z}}$ exist only for the components of $\mathbf{z}$ which are related to the location of the missing tractions. These matrix sensitivities vanish for those component of $\mathbf{z}$ that are related to the magnitude of the applied loading. In the latter case the sensitivities of $\{\mathrm{t}\}_{,\mathbf{z}}$ in Eq. (16) are, however, non zero.

The sensitivities are obtained by first solving Eq. (15) for the unknown displacements and tractions, substituting these quantities in Eq. (16), and finally solving Eq. (16) for the unknown displacement and traction sensitivitie.. For the cases where strains and stresses are the measured quantities, the sensitivities of strains and stresses may be obtained from the displacement and traction sensitivities obtained from Eq. (16), following the procedure given by Kane and Saigal [19].

Example

A simply supported panel was considered to evaluate the effectiveness and the limitations of the formulation presented here. The experimental measurements, $\hat{\psi}_{ik}$, required in the formulation were obtained from a prior direct BEM analysis with the actual boundary tractions imposed on the structure. These boundary tractions then also served as the "exact" solutions for the purposes of comparison of the accuracy of the present procedures. In this example, the measured quantities $\hat{\psi}_{ik}$ are strains. The algorithm for the minimization of the functional, $f(\mathbf{z})$, in Eq. (6) proceeds in an iterative fashion and is considered to have converged when two successive evaluations, f_1 and f_2, of the functional are such that $2 \mid f_1 - f_2 \mid \leq \hat{\epsilon} \times (\mid f_1 \mid + \mid f_2 \mid + \bar{\epsilon})$, where $\hat{\epsilon}$ is a prescribed tolerance and $\bar{\epsilon}$ is a small number to account for the special case of converging to exactly zero function value. A value of $\hat{\epsilon} = 10^{-3}$ and $\bar{\epsilon} = 10^{-10}$ was used in the present study. The weighting parameter, w, was selected such that $wM >> \epsilon$, where M is a typical magnitude of the experimental data. In the present study, M ranges in magnitude from 10^{-2} to 10^{3}, and the parameter, w, was accordingly chosen such that $w \times M \approx 10^{5}$. The geometric dimensions for the panel are shown in Fig. 1. The panel is subjected to a normal stress with a parabolic distribution at its top edge. The location, Z, of the parabolic distribution, the span, W, of the parabola, and the peak magnitude, P, of the parabola are unknown and are desired to be reconstructed. The normal parabolic distribution may be expressed as

$$\Phi(s) = -\frac{4P}{W^2}s^2 + \frac{8PZ}{W^2}s + P - \frac{4PZ^2}{W^2} \qquad (17)$$

where s is the distance along the span of the parabola, and $(Z - 0.5W) \leq s \leq (Z + 0.5W)$. The measured data consists of strains along the orthogonal x and y directions at 39 locations within the body. These locations are shown by cross symbols in Figure 3. The parameters P, W, and Z constitute the model vector, $\mathbf{z}$, for the present case. The initial guess for these parameters was selected as $P = 500psi$, $W = 75in$ and $Z = 50in$. The evolution of the missing traction distribution, starting from this initial distribution, as the iterations in the present analysis proceed, is shown in Fig. 1. The exact traction distribution for this case is also shown in bold line in Fig. 1 for comparison and a good agreement is observed. As the missing tractions varied in position and span length after each iteration, the BEM mesh for the upper boundary edge was modified to accommodate such evolutions. Normally distributed random numbers were added to the experimental data to simulate experimental errors. The errors were considered to be uncorrelated, and were assumed to have a zero mean, and a constant variance. The errors for stress in direction i are picked randomly, with a 99% probability, from the interval $(-\hat{\eta}\hat{\delta}_i, +\hat{\eta}\hat{\delta}_i)$, where $\hat{\delta}_i$ is the average of all strains along i in the strain data set. A higher value of $\hat{\eta}$ corresponds to larger measurement errors. The results were obtained for increasing values of $\hat{\eta}$ and are shown in Table 1 (inserted in Fig. 1). It is noted from these results that for the example studied, the proposed procedures are stable with respect to small errors in the experimental data.

Conclusion

An optimization based integral formulation has been presented for the solution of inverse problems in elastostatics that involve the reconstruction of a missing or an inaccessible boundary data. The objective function used for the minimization in the optimization procedure was taken to be the square of the difference between a set of experimental measurements and their corresponding computed quantities. The minimization is performed using a variable metric method. The response sensitivities required in this algorithm are computed by an analytical approach by performing the implicit differentiation of the boundary integral equations. Using the present developments, a parabolic boundary tractions configurations, its location, extent of spread, and amplitude were closely predicted. The example demonstrated the validity of the present aproach to reconstruct boundary data from experimental measurements that may be contaminated with usual order of experimental errors. The prime limitations of the present developments arise from: (a) the optimization procedure may converge to a local minimum especially for cases for which the initial guess is far away from the actual boundary data, (b) the retraction of the step length to accommodate the inequality constraints offers no rigorous guarantee of convergence.

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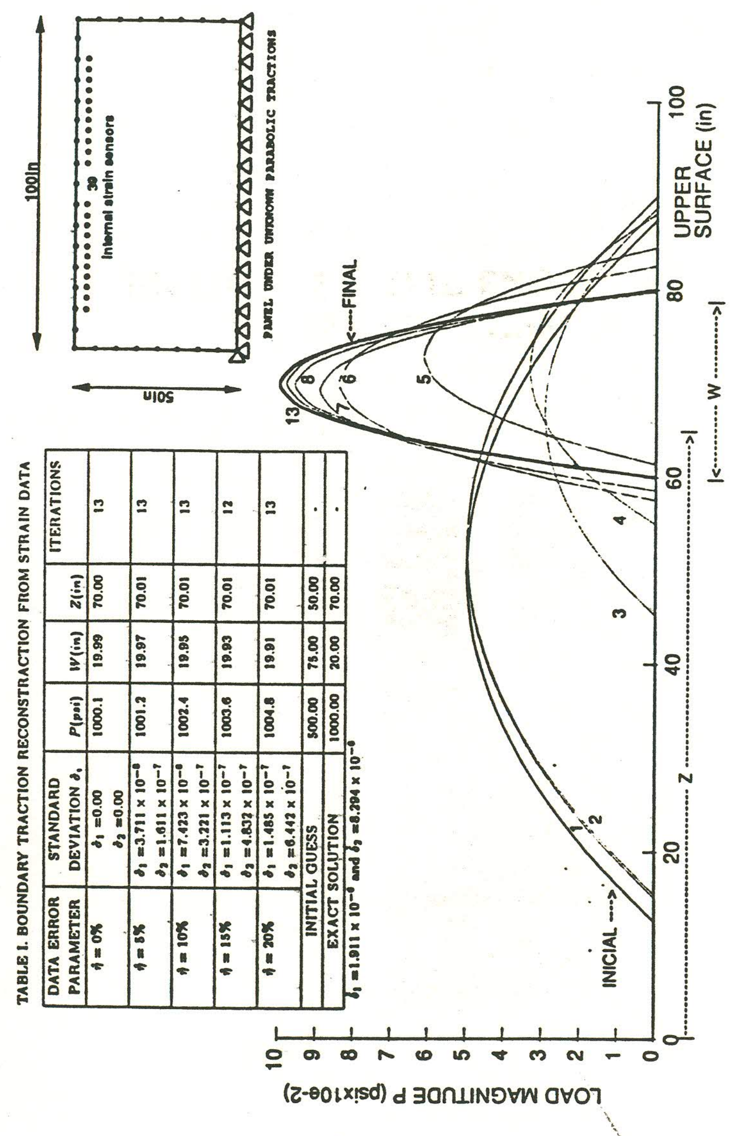

TABLE I. BOUNDARY TRACTION RECONSTRACTION FROM STRAIN DATA

DATA ERROR PARAMETER	STANDARD DEVIATION δ_i	$P(psi)$	$W(in)$	$Z(in)$	ITERATIONS
$\hat{\eta} = 0\%$	$\delta_1 = 0.00$, $\delta_2 = 0.00$	1000.1	19.99	70.00	13
$\hat{\eta} = 5\%$	$\delta_1 = 3.711 \times 10^{-8}$, $\delta_2 = 1.611 \times 10^{-7}$	1001.2	19.97	70.01	13
$\hat{\eta} = 10\%$	$\delta_1 = 7.423 \times 10^{-8}$, $\delta_2 = 3.221 \times 10^{-7}$	1002.4	19.95	70.01	13
$\hat{\eta} = 15\%$	$\delta_1 = 1.113 \times 10^{-7}$, $\delta_2 = 4.832 \times 10^{-7}$	1003.6	19.93	70.01	12
$\hat{\eta} = 20\%$	$\delta_1 = 1.485 \times 10^{-7}$, $\delta_2 = 6.442 \times 10^{-7}$	1004.8	19.91	70.01	13
INITIAL GUESS		500.00	75.00	50.00	-
EXACT SOLUTION		1000.00	20.00	70.00	-

$\hat{\delta}_1 = 1.911 \times 10^{-6}$ and $\hat{\delta}_2 = 8.294 \times 10^{-6}$

FIGURE 1. PARABOLIC TRACTION DISTRIBUTION RECONSTRUCTION - CASE WITH NO ERROR IN STRAIN DATA - MAGNITUDE (P), SPAN (W), AND LOCATION (Z).

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