# DYNAMICS OF THE MASONRY ARCH

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## ABSTRACT

The unreinforced masonry arch responds to horizontal ground acceleration as a four-link mechanism. The resulting governing equation is comparable to that of a single rocking block, with additional terms and with highly non-linear behavior resulting from the mechanism kinematics. A typical arch displays a high excitation threshold, below which it is not excited, but it possesses relatively little resistance to excitations exceeding that threshold. A number of additional mechanics questions are outlined in this article.

## INTRODUCTION

Figure 1 depicts a kinematically admissible motion for an arch as a planar four-link mechanism. We assume the arch to be composed of rigid blocks, which possess no tensile strength, which are incompressible, and which rotate without slipping at points A, B, C, and D; we believe this to be a reasonable characterization of voussoir masonry arch construction. We next assume that this mechanism motion constitutes the response to horizontal ground acceleration. Examining Figure 1, the four-link mechanism is a single DOF system and we choose as generalized co-ordinate the angle $\theta$ measured to link AB. The original (unexcited) geometry corresponds to a repose angle $\theta_o$; the sense of motion shown, for which $\theta$ is less than $\theta_o$, would be initiated by a ground acceleration in the negative x-direction. In earlier work [1] a rectangular portal mechanism of precast concrete construction was studied, and the behavior was described as a more-general case of the dynamics displayed by the simple rocking block [2]. The response of the arch as a four-link mechanism has recently been studied [3], and this paper now compares the mechanics to that of a single rocking block, and discusses some additional mechanics questions which emerge.

## ANALYSIS

The analysis is offered for an example part-circular arch with radius $a$ to the middle surface. The equation of motion for the system in Figure 1 is expressed as:

$$M(\theta)ma^3\ddot{\theta} + L(\theta)ma^2\dot{\theta}^2 + F(\theta)ma^2g = P(\theta)ma^2\ddot{x}_g \qquad (1)$$

Here $m$ is the mass per unit length, and all coefficients $M, L, F, P$ are non-linear in $\theta$. The coefficient formed by $M$ can be interpreted as a generalized inertia multiplying the angular acceleration; it is highly non-linear because of the changing mechanism geometry. The coefficient formed by $L$ includes the effects of centrifugal and coriolis accelerations, and further captures large contributions to the generalized force resulting from the changing mechanism geometry. The coefficient formed by $F$ represents the generalized gravitational force which the structure experiences at any displaced state, and in equation (1) that generalized force is a function of $\theta$ but not a term in $\theta$. Having chosen $\theta$ to be the generalized co-ordinate, the left-hand side of equation (1) equals the external torque applied to link $AB$; mechanical engineers experienced with the force-acceleration analysis of mechanisms, or with the dynamics equations for a robot manipulator, will recognize the l.h.s. of equation (1) to constitute that same analysis. In this case the r.h.s. of equation (1) constitutes the external generalized force (torque) resulting from horizontal ground acceleration; the forcing function is formed with ground acceleration $\ddot{x}_g$, which varies with time, and with coefficient $P$, which varies with $\theta$. Finally, we observe that equation (1) is continuous only for values of $\theta$ less than $\theta_o$. For values of $\theta$ greater than $\theta_o$ a different mechanism geometry (not shown) will apply and a different governing equation will be established, in the same manner with which a single rocking block changes its toes of rotation.

**Numerical Example** It is instructive to examine equation (1) with instantaneous (constant) values of its coefficients, forming a tangent approximation of system response at displacement state $\theta$. When evaluated at the repose angle, $\theta_o$, the resulting equation describes small rotations about the original geometry. We examine a representative arch with $a$ equal to $10m$, thickness equal to $1.5m$, angle of embrace equal to $7\pi/8$, and hinge locations at central angle spacings of $3\pi/8, \pi/4$ and $\pi/4$ measured clockwise from $A$ to $D$. (Note: This precise geometry is not the one portrayed in Figure 1, which depicts only generally the four-link mechanism and the motion sense.) For such an arch the original geometry corresponds to $\theta_o$ equal to 0.8972 radians. Cancelling the unit mass $m$ from each term, equation (1) becomes:

$$4370\,\ddot{\theta} + 59000\,\dot{\theta}^2 - 867 = 239\,\ddot{x}_g \tag{2}$$

We can also write equation (2) in terms of rotation $\phi$ measured with respect to the original geometry, defined as $(\theta_o - \theta)$, and the coefficient formed by $F(\theta)$ can then be expressed as a Taylor series expansion about $\theta_o$. Doing so, and also dividing by gravity, equation (2) becomes:

$$445\,\ddot{\phi} + 6010\,\dot{\phi}^2 - 239\,\phi = -88.4 - 239\,\ddot{x}_g/g \tag{3}$$

The instantaneous form of the governing equation is meaningful only for a very small range of rotation; the non-linearities in the coefficients, introduced by the mechanism kinematics, are far more pronounced than in typical structural applications or in preceding studies of rigid block systems.

**Engineering Results** Two results of engineering interest can be extracted. Examining the r.h.s. of equation (3), we note that excitation (positive acceleration in $\phi$) commences only when the magnitude of $\ddot{x}_g$, in the $-x$ direction, exceeds $0.370g$. This constitutes a threshold excitation, below which the structure is not excited into mechanism motion. The threshold is observed to increase with greater arch thickness and to decrease with greater arch embrace, reflecting basic funicular arch behavior, and most practical arch geometries will display a high acceleration threshold. In our opinion this threshold condition is one major reason that masonry arches have withstood ground motion. Time-history analysis of the example arch [3] then shows it to possess relatively little resistance to excitation once that threshold is exceeded, because the mechanism reaches a point of gravitational destabilization after relatively little rotation, about 0.07 radians in this example.

## COMPARISON TO THE ROCKING BLOCK

Equation (3), derived from equation (1), can be compared to the equation for simple rocking motion of a single block. The following observations can be made:

- For initial rotation from rest, for which $\dot{\phi}$ is zero, equation (3) is identical in form to that for a single block. Equation (3) would correspond to a block with aspect ratio 0.37, rectangular dimensions $0.520m$ by $1.045m$, but with mass radius of gyration about the block centroid of $2.31m$.

- Equation (3) is an instantaneous form of the governing equation, but it cannot be interpreted as a linearized form applicable over any useful range of motion, because the governing equation for the four-link mechanism is highly non-linear. In contrast, for a single slender rocking block, a linearized equation can commonly be used as a reasonable approximation throughout the range of motion.

- The inertia coefficient for a single rocking block is a constant, independent of rotation, but the generalized inertia coefficient for the four-link mechanism captures kinematic conditions, and varies rapidly with rotation $\theta$. The single rocking block has no term in $\dot{\phi}^2$, but such a term is present for the four-link mechanism, capturing coriolis and centrifugal effects, and its coefficient varies rapidly with rotation. Finally, the terms resulting from gravitation and base excitation for a single rocking block involve simple trigonometric quantities, allowing small-angle approximations, but the corresponding terms for the four-link mechanism are again capturing full kinematic conditions and vary rapidly with rotation.

Other differences between the four-link mechanism and the rocking block are discussed in the next section.

# ADDITIONAL MECHANICS QUESTIONS

**Establishment of Mechanism Geometry** We commenced this analysis assuming, but not proving, that the four-link mechanism characterizes arch response. Moreover, the masonry arch is constructed with some greater number of voussoirs; each of the three rigid links in Figure 1 is therefore formed by multiple voussoirs, requiring the assumption that within each such link the forces between voissours are resolved as compressive resultants contained within the thickness of the masonry. Observations on these matters are offered as follows:

- For any given arch there exist many different kinematically admissible four-link mechanisms; our practice was to analyze the full set, and then select the mechanism forming at the lowest value of $\ddot{x}_g$ as the governing one.
- The arguments were then tested with model experiments on a part-circular jointed voussoir arch in which the base plane was inclined to create an equivalent constant base acceleration. Inclination was increased until motion developed, and the predicted four-link mechanism was observed. We note that model experiments are inherently valid for rigid block structures. In particular, the threshold condition for onset of motion is identical in the scaled model as in the prototype.
- The force conditions at joints within each link could be monitored to trace the existence and persistence of the four-link mechanism. While the assumption of four-link mechanism motion is partly confirmed above for initial motions, it seems certain that the conditions could be violated during response to excitation, implying that subsequent motion (and failure, most likely) would develop with additional hinge formation. We have not yet performed such analyses.
- The force resultant within the masonry could be formulated analytically, and the initial governing mechanism could be identified by the extremal fit of that resultant within the masonry geometry. This leads us to conclude that the formation of the four-link mechanism may be a reasonable expected outcome for the common arch geometries, but is certainly not a general mechanical principle for all rigid body structures. It appears that geometries could be found to produce any kinematically admissible mechanism. Moreover, our study has considered only horizontal ground acceleration, and if vertical excitation is incorporated then the mechanism conditions can clearly be altered.

**Pre-motion Conditions** We assumed that the arch takes up the mechanism geometry at the onset of motion, and no mention was made of motions, or state transitions, prior to that instant. However, prior to excitation any such arch is in repose as a three-hinged arch, in a typical arch geometry there will be two contact points at the intrados near the springing points, and one contact point at the extrados near the crown. It is clear that some transitions must occur to bring the arch from that static geometry into the four-link mechanism kinematic geometry. It is possible that those early transitions have negligible influence on subsequent response, because the change in energy (potential and kinetic) as such transitions occur may be negligible in a well-fitted arch, but that condition awaits study.

**Sliding** In this study sliding has been ignored. Where blocks meet at planar non-keyed joints, the internal force can always be expressed as a tangential-normal force pair. In general the joint integrity is maintained by friction, provided that the tangential-normal ratio is less than the friction coefficient. Analysis of sliding requires the time history solution for that force pair at each joint. During dynamic response the tangential-normal ratio will vary, and when it exceeds the friction coefficient a sliding motion is possible. In principle the equations of motion can be extended to include that additional degree-of-freedom, and numerical integration will produce the time history of the resulting net motion. When the tangential-normal ratio decreases and drops below the friction coefficient the sliding motion halts. This appears to be reflected in many surviving structures where a block is seen displaced (by sliding) from its original position. Failure would occur if a block were to slide beyond the geometric limits of contact. Sliding can develop during rigid body motions, under ground acceleration less than the excitation threshold, and does not require the formation of the four-link mechansim motion. Finally, the local condition for sliding is a necessary but not a sufficient condition; there must also be a kinematically admissible form for the sliding motion.

**Conditions at Impact** The mechanism pictured in Figure 1 is kinematically admissible only for motion in the sense shown. If the arch responds as a four-link mechanism, at some time it will return to its original geometry but with non-zero kinetic energy. We assume its motion may continue, but with the formation of a different mechanism geometry, kinematically admissible for the new sense of motion. This condition is comparable to the establishment of a new toe of rotation for a single rocking block, and it contributes the discontinuity which characterizes these problems. However, the conditions at impact for the arch mechanism differ fundamentally from those observed for the single rocking block, in both theoretical and practical terms. The major issues are outlined as follows:

- For the single rocking block, there exists a well-known theoretical argument for loss of kinetic energy at impact. That argument results from the instantaneous establishment of a new toe of rotation with the observation that the only external impulsive force is applied at that toe. The moment of momentum is conserved about that toe, and therefore the velocity after impact can be calculated. For a four-link mechanism, no such theoretical argument applies, because every body is a three-force body.

- In practical terms, we believe that an arch will display significant corner rounding (crushing, fracturing, chamfering) at impact. In contrast, most single rocking blocks, such as tombstones, are relatively low in mass, and the analyst may be justified in neglecting the effects on stone of corner rounding for impact loadings of low magnitude. However, the arch possesses a total mass greater by orders of magnitude than that in any single block, while the impact is still applied to a single stone. Therefore we expect that the actual response of arches will be marked by significant corner rounding at impact, and that the structure will therefore display a degrading geometry. This is an important and untouched problem, but its study should await physical or experimental evidence of stone material behavior under corner impacts.

## REFERENCES


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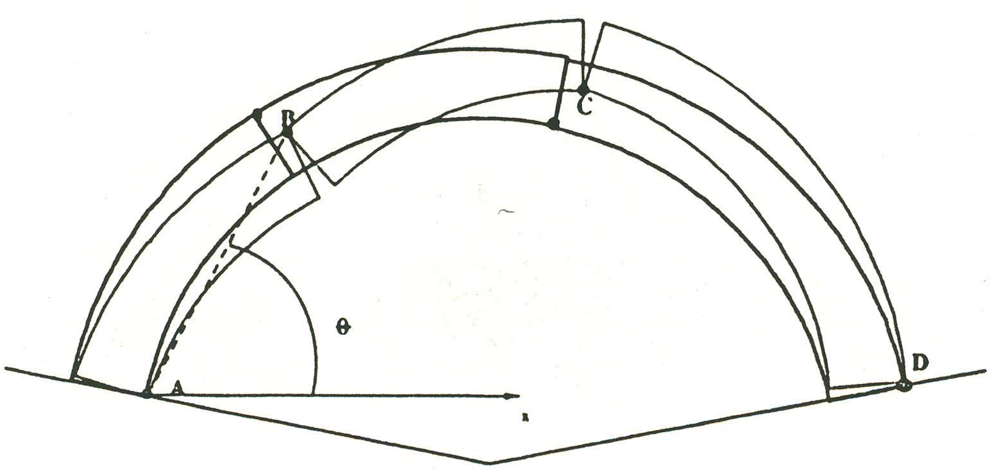


**Figure 1. The Masonry Arch as a Four-Link Mechanism**