MATRIX REPRESENTATION FOR CALCULATION OF THE SECOND ORDER ABERRATIONS IN PARTICLE OPTICS FOR HOMOGENEOUS AND INHOMOGENEOUS (n=1) MAGNETIC FIELDS

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PUBLICAÇÃO IEA N.º 144
Abril — 1967

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SÃO PAULO — BRASIL

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Matrix Representation for Calculation of the Second Order Aberrations in Particle Optics for Homogeneous and Inhomogeneous (n = 1) Magnetic Fields

A. A. Suarez and F. A. B. Coutinho

(Received January 15, 1968; presented by Marcello Damy de Souza Santos)

INTRODUCTION

In the calculation of the properties of magnetic systems to analyse the energy of charged particles beams, the use of analytical procedures is tedious and heavily time consuming.

In analogy with the geometrical optics, S. Penner [1] introduced a matrix method to calculate the properties of magnetic deflection systems involving a first order aberration. I. Takeshita [3] has extended this formalism to the second order in the case of a two dimensional motion, utilising a nine-dimensional matrix for an homogeneous magnetic field. Later on, K. L. Brown [1, 4] developed the general theory of first and second order aberrations of deflecting magnets.

We have used that formalism to calculate the elements of a twelve-dimensional matrix, extended to a three dimensional particle motion close to the central orbit (paraxial orbits), in two special cases: homogeneous and inhomogeneous magnetic fields (n = 1). This calculation was extended for the case of a rotation fo the field boundary, neglecting however the effects due to fringing fields, which will be published in a future paper.

METHOD OF CALCULATION

Since the particle trajectories near to the central axis can be expressed by linear functions, one can use the matrix representation to express the transformation of object to image coordinates as the usual procedures in geometrical optics.

Using such procedure, several deflecting elements can be coupled easily to each other by multiplying matrices and then analysing optical properties such as the object magnification, particle dispersion and resolution of the system. This formalism is also convenient for digital computation of the particle trajectories in complicated systems where the analytical evaluation is time consuming and tedious.

Let's suppose that we have a symmetric magnetic field relative to a plane situated between the pole faces of a magnet (figure 1) (median plane). One particle with momentum p_o will describe a trajectory AB (central trajectory) and a particle with momentum $p_o + \Delta p$ will be displaced in relation to the central orbit, describing a trajectory CD. Let the entrance and exit coordinates be: x_o and x, the distances of the CD trajectory from the central trajectory measured on a perpendicular to AB in the entrance and exit point respectively; Θ_o and Θ, the angle between the central trajectory and CD in the entrance and exit of the magnet respectively and $\frac{\Delta p}{p_o}$ the momentum variation of the particle.

Analogously we define the coordinates y and ϕ on the perpendicular plane to the meridian plane.

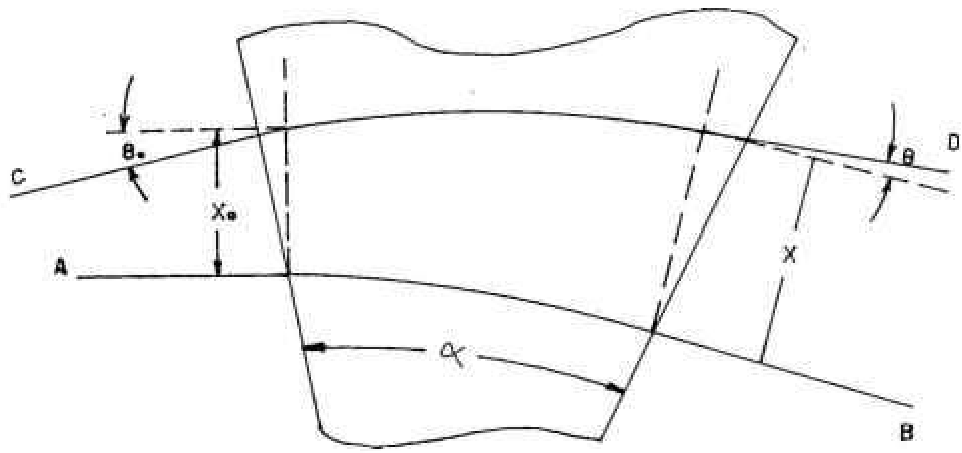

Figure 1

For paraxial rays one can expand x in terms of $\left(x_o, y_o, \Theta_o, \phi_o, \frac{\Delta p}{p_o}\right)$ until the second order approximation, dropping out all the higher order terms.

$$x = \langle x|x_o\rangle x_o + \langle x|\Theta_o\rangle \Theta_o + \langle x|\delta\rangle \delta + \langle x|x_o^2\rangle x_o^2 + \langle x|x_o\,\Theta_o\rangle x_o\Theta_o + \langle x|x_o\,\delta\rangle x_o\,\delta +$$
$$+ \langle x|\Theta_o^2\rangle \Theta_o^2 + \langle x|\Theta_o.\delta\rangle \Theta_o\delta + \langle x|\delta^2\rangle \delta^2 + \langle x|y_o^2\rangle y_o^2 + \langle x|y_o\,\phi_o\rangle y_o\,\phi_o + \langle x|\phi_o^2\rangle \phi_o^2$$

The coefficients of y_o, $y_o\,x_o$, $y_o\,\Theta_o$, $y_o\,\delta$, ϕ_o and $\phi_o\,\delta$ are identically equal to zero as will be demonstrated later on.

These initial coordinates span a twelve dimensional vector space giving for the transfer matrix the following aspect:

A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}
C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}
E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}
F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}	G_{12}
H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}
I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}	I_{11}	I_{12}
J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}	J_{11}	J_{12}
K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}
L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}

Defining a coordinate system on the central trajectory as given by figure 2, the second row can be obtained of the first row observing that:

$$\Theta = \frac{x'}{1 + hx}$$

where $h^{-1} = \rho_0$ (curvature radius of the central orbit) and $x' = \frac{dx}{dt}$, where t is the distance travelled by the particle from its origin in the central orbit.

Since the velocity of the particles is not changed by the magnetic field, the matrix elements of the third row are zero except $C_{33} = 1$.

All the other rows may be obtained by means of the combination of the three first rows and the tenth and eleventh rows.

In the space free of magnetic field, the matrices become

	x	Θ	δ	x^2	$x\Theta$	$x\delta$	Θ^2	$\Theta\delta$	δ^2	y^2	$y\phi$	ϕ^2
x	1	L	0	0	0	0	0	0	0	0	0	0
Θ	0	1	0	0	0	0	0	0	0	0	0	0
$\frac{\Delta p}{p} = \delta$	0	0	1	0	0	0	0	0	0	0	0	0
x^2	0	0	0	1	2L	0	L^2	0	0	0	0	0
$x\Theta$	0	0	0	0	1	0	L	0	0	0	0	0
$x\delta$	0	0	0	0	0	1	0	L	0	0	0	0
Θ^2	0	0	0	0	0	0	1	0	0	0	0	0
$\Theta\delta$	0	0	0	0	0	0	0	1	0	0	0	0
δ^2	0	0	0	0	0	0	0	0	1	0	0	0
y^2	0	0	0	0	0	0	0	0	0	1	2L	L^2
$y\phi$	0	0	0	0	0	0	0	0	0	0	1	L
ϕ^2	0	0	0	0	0	0	0	0	0	0	0	1

Translation Matrix (x)

	y	ϕ	xy	$x\phi$	Θy	$\Theta\phi$	$y\delta$	$\phi\delta$
y	1	L	0	0	0	0	0	0
ϕ	0	1	0	0	0	0	0	0
xy	0	0	1	L	L	L^2	0	0
$x\phi$	0	0	0	1	0	L	0	0
Θy	0	0	0	0	1	L	0	0
$\Theta\phi$	0	0	0	0	0	1	0	0
$y\delta$	0	0	0	0	0	0	1	L
$\phi\delta$	0	0	0	0	0	0	0	1

Translation Matrix (y)

since the trajectory is simply a straight line.

Let's now describe succintly the procedure developped by K. L. Brown [1] to find the matrix elements.

Taking the relativistic equation of motion of a particle of momentum p placed in an arbitrarily shaped magnetic field.

$$\frac{d\mathbf{p}}{dt} = e\ (\mathbf{v} \times \mathbf{B})$$

and eliminating the time in that equation using the coordinate system described in figure 2 one obtains

$$\frac{d^2\mathbf{s}}{dt^2} - 1/2\ \frac{\frac{d\dot{\mathbf{s}}}{dt}}{\left(\frac{d\mathbf{s}}{dt}\right)^2}\ \frac{d}{dt}\left(\frac{d\mathbf{s}}{dt}\right)^2 = \frac{e}{p}\ \frac{ds}{dt}\left(\frac{d\mathbf{s}}{dt} \times \mathbf{B}\right),$$

where t is the distance along the central trajectory and s is the path along an arbitrary trajectory.

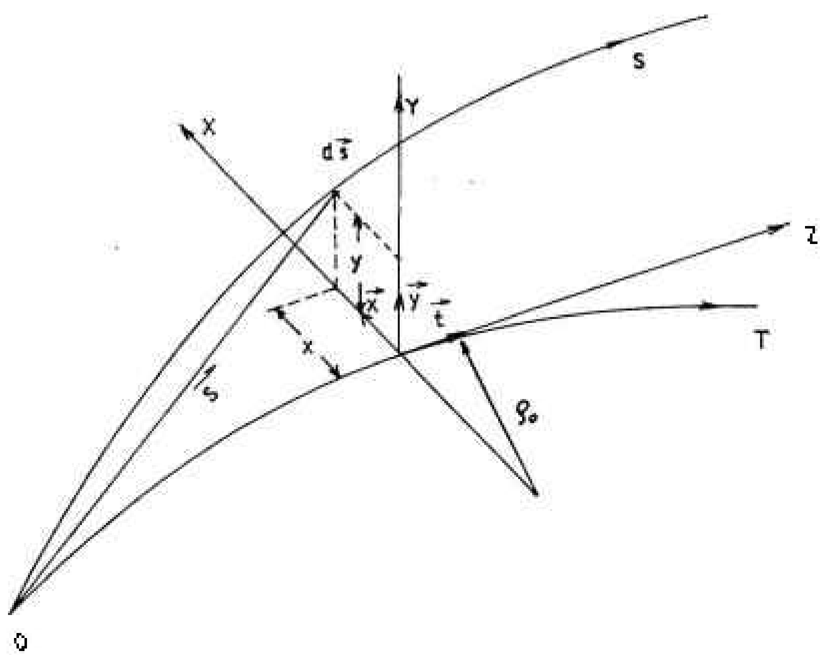

Figure 2

Expressing this equation in terms of the coordinates x and y we get:

(1) $$x'' - h(1 + hx) - x'(hx' + h'x) = \frac{e}{p}\ s'\left[y' B_t - (1 + hx) B_y\right]$$

(2) $$y'' - y'(hx' + h'x) = \frac{e}{p}\ s'\left[(1 + hx) B_x - x' B_t\right]$$

where (′) means $\frac{d}{dt}$

The central trajectory is defined by the initial conditions

$$x = x' = y = y' = 0; \text{ i.e., } h = \frac{e}{p_o} = B_y\,(0, 0, t).$$

One can derive the magnetic field components B_x, B_y and B_t from a scalar potential ϕ which is a simetric odd function of y.

Expanding these magnetic field components B_x, B_y and B_t we get

$$B_x\,(x, y, t) = \frac{\partial \phi}{\partial x}\, A_{11}\, y + A_{12}\, x y + \dots\dots\dots$$

$$B_y\,(x, y, t) = \frac{\partial \phi}{\partial y} = A_{10} + A_{11}\, x + \frac{A_{12}}{2!}\, x^2 + \frac{A_{30}}{2!}\, y^2 + \dots\dots$$

$$B_t\,(x, y, t) = \frac{1}{(1 + ht)}\, \frac{\partial \phi}{\partial t} = \frac{1}{(1 + ht)}\left[A'_{10}\, y + A'_{11}\, x y + \dots\right]$$

where

$$A_{1n} = \frac{\partial^n}{\partial x^n}\, B_y\,(x, 0, t) \Bigg|_{\substack{x = 0 \\ y = 0}}$$

and

$$A_{30} = -\left[A'_{10} + h\, A_{11} + A_{12}\right]$$

Defining two dimensionless quantities $n\,(t)$ and $\beta\,(t)$ by

$$n\,(t) = -\left[\frac{1}{h B_y}\left(\frac{\partial B_y}{\partial x}\right)\right] \Bigg|_{\substack{x = 0 \\ y = 0}}$$

and

$$\beta\,(t) = \left[\frac{1}{2!\, h^2 B_y}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\right] \Bigg|_{\substack{x = 0 \\ y = 0}}$$

which characterize the magnetic field and using the equation of motion for the central trajectory one gets for the two equations (1) and (2) the following expressions:

(1a) $$x'' + (1 - n)\, h^2 x = h\,\delta + (2n - 1 - \beta)\, h^3 x^2 + h' x x' + 1/2\, h x'^2 +$$
$$+ (2 - n)\, h^2 x \delta + 1/2\,(h'' - n h^3 + 2 \beta h^3)\, y^2 + h' y y' - 1/2\, h y'^2 - h \delta^2 +$$
$$+ \text{ higher order terms.}$$

(2a) $$y'' + n h^2 y = 2(\beta - n)\, h^3 x y + h' x y' - h' x' y + h x' y' + n h^2 y +$$
$$+ \text{ higher order terms.}$$

Now expanding x and y until the second order in terms of x_o, x'_o, $\delta = \frac{\Delta p}{p_o}$, y_o, y'_o with all the cross terms and substituting in (1a) and (2a) the coefficient of y_o, y'_o x_o, $y_o \Theta_o$, $y_o \delta$, ϕ_o, $\phi_o \delta$ will be identically zero and the others will be defined by equations

$$c'' + k^2 c = 0 \tag{3}$$

$$s'' + k^2 s = 0 \tag{4}$$

with the boundary conditions

$$c(0) = 1 \qquad c'(0) = 0$$

$$s(0) = 0 \qquad s'(0) = 1$$

and

$$q'' + k^2 q = f \tag{5}$$

where the coefficient k^2 is equal to $(1-n)h^2$ for the x motion and equal to nh^2 to the y motion. c is the coefficient of x_o, s is the coefficient of Θ_o and the third equation is the differential equation of the dispersion and of all the second order aberrations. f is a function of h, n, β and of the coefficients of the first order aberrations which can be obtained from substitutions of the Taylor's expansions into the general differential equations (1a) and (2a).

This equation may be solved using the integral Green's function.

$$q = \int_0^t f(\tau)\, G(t - \tau)\, d\tau$$

where the convenient Green's function is:

$$G(t - \tau) = s(t)\, c(\tau) - s(\tau)\, c(t),$$

giving to q the following expression:

$$q = s(t) \int_0^t f(\tau)\, c(\tau)\, d\tau - c(t) \int_0^t f(\tau)\, s(\tau)\, d\tau$$

In order to construct the transfer matrix in terms of convenient parameters one should remember the relations:

$$\Theta = \frac{x'}{1 + hx} \qquad \text{and} \qquad \phi = \frac{y'}{1 + hx}$$

TRANSFER MATRIX

Since our interest lies in the transfer matrix for homogeneous magnetic field and for fields varying as 1/r we should use the proper values of $n(t)$ and $\beta(t)$ in equations (3), (4), (5) and solve them.

For an homogeneous magnetic field these values are $n(t) = 0$ and $\beta(t) = 0$ and for a field varying as 1/r; $n(t) = 1$ and $\beta(t) = 1$.

Thus, the obtained transfer matrices are:

	x_0	θ_0	δ	x_0^2	$x_0\theta_0$	$x_0\delta$	θ_0^2	$\theta_0\delta$	δ^2	y_0^2	$y_0\phi_0$	ϕ_0^2
x	$\cos\alpha$	$\rho\,\mathrm{sen}\,\alpha$	$\rho(1-\cos\alpha)$	$-\frac{1}{2\rho}\,\mathrm{sen}^2\alpha$	$\mathrm{sen}\,\alpha\cos\alpha$	$\mathrm{sen}^2\alpha$	$\frac{\rho}{2}\cos\alpha(1-\cos\alpha)$	$\rho\,\mathrm{sen}\,\alpha(1-\cos\alpha)$	$\frac{\rho}{2}\,\mathrm{sen}^2\alpha$	0	0	$-\frac{\rho}{2}\,[illegible](1-\cos\alpha)$
θ	$-\frac{1}{\rho}\,\mathrm{sen}\,\alpha$	$\cos\alpha$	$\mathrm{sen}\,\alpha$	0	0	$\frac{1}{2}\,\mathrm{sen}\,\alpha$	$\frac{1}{2}\,\mathrm{sen}\,\alpha$	0	$\mathrm{sen}\,\alpha$	0	0	$\frac{1}{2}\,\mathrm{sen}\,\alpha$
δ	0	0	1	0	0	0	0	0	0	0	0	[illegible]
x^2	0	0	0	$\cos^2\alpha$	$\rho\,\mathrm{sen}\,2\alpha$	$2\rho\cos\alpha(1-\cos\alpha)$	$\rho^2\,\mathrm{sen}^2\alpha$	$2\rho^2\,\mathrm{sen}\,\alpha(1-\cos\alpha)$	$\rho^2(1-\cos\alpha)^2$	0	0	0
$x\theta$	0	0	0	$-\frac{1}{2\rho}\,\mathrm{sen}\,2\alpha$	$\cos 2\alpha$	$\mathrm{sen}\,\alpha-\mathrm{sen}\,2\alpha$	$\frac{\rho}{2}\,\mathrm{sen}\,2\alpha$	$\rho(\cos\alpha-\cos 2\alpha)$	$\rho\,\mathrm{sen}\,\alpha(1-\cos\alpha)$	0	0	0
$x\delta$	0	0	0	0	0	$\cos\alpha$	0	$\rho\,\mathrm{sen}\,\alpha$	$\rho(1-\cos\alpha)$	0	0	0
θ^2	0	0	0	$\frac{1}{\rho^2}\,\mathrm{sen}^2\alpha$	$-\frac{1}{\rho}\,\mathrm{sen}\,2\alpha$	$-\frac{2}{\rho}\,\mathrm{sen}^2\alpha$	$\cos^2\alpha$	$\mathrm{sen}\,2\alpha$	$\mathrm{sen}^2\alpha$	0	0	0
$\theta\delta$	0	0	0	0	0	$-\frac{1}{\rho}\,\mathrm{sen}\,\alpha$	0	$\cos\alpha$	$\mathrm{sen}\,\alpha$	0	0	0
δ^2	0	0	0	0	0	0	0	0	1	0	0	0
y^2	0	0	0	0	0	0	0	0	0	1	$2\rho\alpha$	$\rho^2\alpha^2$
$y\phi$	0	0	0	0	0	0	0	0	0	0	1	[illegible]
ϕ^2	0	0	0	0	0	0	0	0	0	0	0	1

Homogeneous Matrix (x)

	y_o	ϕ_o	x_oy_o	$x_o\phi_o$	θ_oy_o	$\theta_o\phi_o$	δy_o	$\delta\phi_o$
y	1		0	$\operatorname{sen}\alpha$	0	$\rho(1-\cos\alpha)$	0	$\rho(\alpha-\operatorname{sen}\alpha)$
ϕ	0	1	0	0	0	0	0	0
xy	0	0	$\cos\alpha$	$\rho\alpha\cos\alpha$	$\rho\operatorname{sen}\alpha$	$\rho^2\alpha\operatorname{sen}\alpha$	$\rho(1-\cos\alpha)$	$\rho^2\alpha(1-\cos\alpha)$
$x\phi$	0	0	0	$\cos\alpha$	0	$\rho\operatorname{sen}\alpha$	0	$\rho(1-\cos\alpha)$
θy	0	0	$-\frac{1}{\rho}\operatorname{sen}\alpha$	$-\alpha\operatorname{sen}\alpha$	$\cos\alpha$	$\rho\alpha\cos\alpha$	$\operatorname{sen}\alpha$	$\rho\alpha\operatorname{sen}\alpha$
$\theta\phi$	0	0	0	$-\frac{1}{\rho}\operatorname{sen}\alpha$	0	$\cos\alpha$	0	$\operatorname{sen}\alpha$
δy	0	0	0	0	0	0	1	0
$\delta\phi$	0	0	0	0	0	0	0	1

Homogeneous Matrix (y)

[illegible]	[illegible]	[illegible]	[illegible]	$x_o\theta_o$	$x_o\delta$	θ_o^2	$\theta_o\delta$	δ^2	y_o^2	$y_o\phi_o$	[illegible]
1	$\rho\alpha$	$\frac{\rho\alpha^2}{2}$	0	ρ	$\frac{\alpha^2}{2}$	$\frac{\rho\alpha^2}{4}$	$\frac{1}{3}\rho\alpha^3$	$\frac{\rho\alpha^2}{2}(\frac{1}{6}\alpha^2-1)$	$\frac{1}{4\rho}\,\mathrm{sen}^2\alpha$	$\frac{\alpha}{2}\ \frac{1}{4}\,\mathrm{sen}2\alpha$	$\frac{\rho}{4}\,\mathrm{sen}^2\alpha$
0	1	α	0	0	0	$-\frac{\alpha}{2}$	$-\frac{\alpha^2}{2}$	$-\alpha(1+\frac{\alpha^2}{3})$	$\frac{1}{4\rho^2}\,\mathrm{sen}2\alpha$	$\frac{1}{\rho}\,\mathrm{sen}^2\alpha$	$\frac{1}{4}\,\mathrm{sen}2\alpha$
0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	$2\rho\alpha$	$\rho\alpha^2$	$\rho^2\alpha^2$	$\rho^2\alpha^3$	$\frac{\rho^2\alpha^4}{[illegible]}$	[illegible]	[illegible]	[illegible]
0	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$\frac{\rho\alpha^3}{2}$	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	0	$\rho\alpha$	$\frac{\rho\alpha^2}{2}$	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2α	[illegible]	[illegible]	[illegible]	[illegible]
0	0	0	0	0	0	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	0	0	0	0	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0	0	0	0	0	0	0	0	0	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	0	0	0	0	0	0	$-\frac{1}{2\rho}\,\mathrm{sen}2\alpha$	$\cos2\alpha$	$\frac{\rho}{2}\,\mathrm{sen}2\alpha$
[illegible]	[illegible]	[illegible]	0	0	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Inhomogeneous Matrix (n = 1) (x)

	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
y	$\cos\alpha$	$\rho\,\text{sen}\,\alpha$	0	$\text{sen}\,\alpha$	$\frac{1}{2}(\alpha\cos\alpha-\text{sen}\,\alpha)$	$\frac{\rho\alpha}{2}\,\text{sen}\,\alpha$	$\frac{\alpha}{4}(\text{sen}\,\alpha+\alpha\cos\alpha)$	$\frac{\rho}{4}\left[\text{sen}\,\alpha+\alpha^2\text{sen}\,\alpha-\alpha\cos\alpha\right]$
ϕ	$-\frac{1}{\rho}\,\text{sen}\,\alpha$	$\cos\alpha$	0	$\frac{1}{\rho}\cos\alpha$	$-\frac{\alpha}{2\rho}\,\text{sen}\,\alpha$	$\frac{1}{2}(\text{sen}\,\alpha+\alpha\cos\alpha)$	$\frac{1}{4\rho}\left[\text{sen}\,\alpha+3\alpha\cos\alpha-\alpha^2\text{sen}\,\alpha\right]$	$\frac{1}{4}\left[3\alpha\,\text{sen}\,\alpha+\alpha^2\cos\alpha\right]$
xy	0	0	$\cos\alpha$	$\rho\,\text{sen}\,\alpha$	$\rho\alpha\cos\alpha$	$\rho^2\alpha\,\text{sen}\,\alpha$	$\frac{\rho\alpha^2}{2}\cos\alpha$	$\frac{\rho^2\alpha^2}{2}\,\text{sen}\,\alpha$
$x\phi$	0	0	$-\frac{1}{\rho}\,\text{sen}\,\alpha$	$\cos\alpha$	$-\alpha\,\text{sen}\,\alpha$	$\rho\alpha\cos\alpha$	$-\frac{\alpha^2}{2}\,\text{sen}\,\alpha$	$\frac{\rho\alpha^2}{2}\cos\alpha$
θy	0	0	0	0	$\cos\alpha$	$\rho\,\text{sen}\,\alpha$	$\alpha\cos\alpha$	$\rho\alpha\,\text{sen}\,\alpha$
$\theta\phi$	0	0	0	0	$-\frac{1}{\rho}\,\text{sen}\,\alpha$	$\cos\alpha$	$-\frac{\alpha}{\rho}\,\text{sen}\,\alpha$	$\alpha\cos\alpha$
$y\delta$	0	0	0	0	0	0	$\cos\alpha$	$\rho\,\text{sen}\,\alpha$
$\phi\delta$	0	0	0	0	0	0	$-\frac{1}{\rho}\,\text{sen}\,\alpha$	$\cos\alpha$

Inhomogeneous Matrix (n = 1) (y)

ROTATION MATRIX

Let be figure 3 the situation for an entrance particle in a magnetic field with the magnet pole edges rotated by an angle β_1.

To get the rotation matrix in the median plane we should correlate the coordinates x_1, Θ_1 without pole rotation, with the coordinates x_o, Θ_o of the rotated pole edges. This can be done by getting first the transfer matrix from x_o, Θ_o to x_s, Θ_s and then from x_s, Θ_s to x_1, Θ_1.

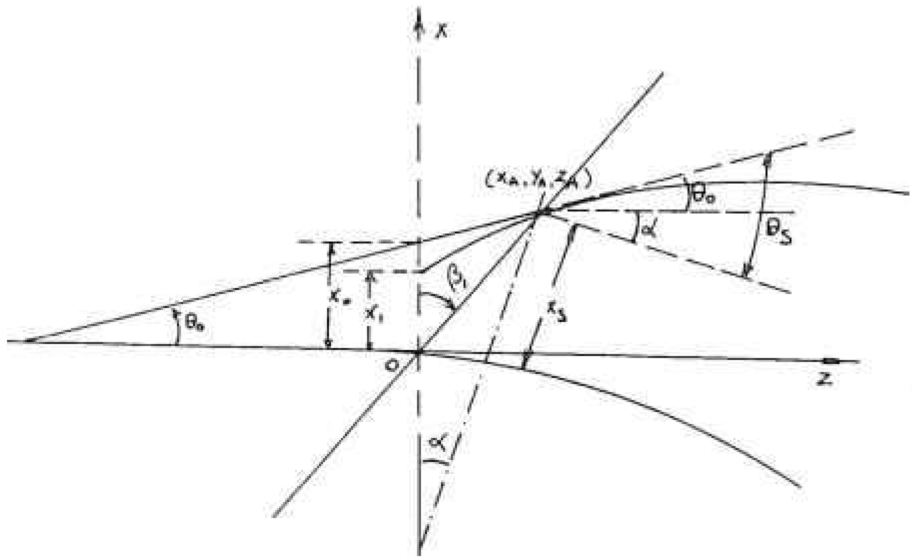

Figure 3 — Entrance pole edge

The calculation of the matrix elements from x_o, Θ_o to x_s, Θ_s can be accomplished by using the following relations:

(6) $\left| \begin{array}{l} x_A = x_o + Z_A \Theta_o \\ y_A = y_o + Z_A \phi_o \end{array} \right.$ equations of the particle trajectory in the field free region

(7) $x_A = Z_A / \text{tg}\, \beta_1$ equation of the rotated pole edge.

(8). $\begin{array}{l} (\rho + x_s) \text{ sen } \alpha = Z_A \\ (\rho + x_s) \cos \alpha = x_A \\ y_A = y_s \end{array}$ equations relating x_A, y_A, Z_A measured in the x, z coordinate system with x_s, y_s, z_s measured in the coordinate system given by figure 2.

Supposing that $\Theta_o \text{ tg } \beta_1 << 1$ and $x_o < \rho$ we get:

(9)
$$\begin{aligned} x_s &= x_o + 1/2\, \rho\, \text{tg}^2 \beta_1 x_o^2 + \text{tg}\, \beta_1 x_o \Theta_o \\ \Theta_s &= \Theta_o + \text{tg}\, \beta_1/\rho\, x_o - \text{tg}\, \beta_1/\rho^2\, x_o^2 + \text{tg}^2 \beta_1/\rho\, x_o \Theta_o \\ y_s &= y_o + \text{tg}\, \beta_1 x_o \phi_o \quad \text{and} \\ \phi_s &= \phi_o . \end{aligned}$$

These expressions give the entrance transfer matrix from x_o, Θ_o to x_s, Θ_s on the median plane, which is shown in the following page.

Note that this matrix is independent of the magnetic field.

To get the transfer matrix from x_o, Θ_o to x_1, Θ_1 one should only multiply the matrix $B(\beta_1)$ by the inverse transfer matrix of a small magnet which goes from x_1, Θ_1 to x_s, Θ_s. One should remember that the bending angle of this small magnet must be substituted by the appropriate aproximation obtained using the relation

$$\alpha \cong \frac{\mathrm{tg}\,\beta_1}{\rho}\, x_o - \frac{\mathrm{tg}\,\beta_1}{\rho^2}\, x_o^2 + \frac{\mathrm{tg}^2\,\beta}{\rho}\, x_o\, \Theta_o$$

The resulting entrance rotation matrix for an homogeneous magnetic field is given by Entrance Rotation Matrix (x) (Homogeneous) — see on page 139.

	x_o	θ_o	δ	x_o^2	$x_o\theta_o$	$x_o\delta$	θ_o^2	$\theta_o\delta$	δ^2	y_ρ^2	$y_\rho\phi_\rho$	ϕ_ρ^2
x	1	0	0	$\frac{tg^2\beta}{2\rho}$	$tg\beta$	0	0	0	0	0	0	0
θ	$\frac{tg\beta}{\rho}$	1	0	$-\frac{tg\beta}{\rho^2}$	$\frac{tg^2\beta}{\rho}$	0	0	0	0	0	0	0
δ	0	0	1	0	0	0	0	0	0	0	0	0
x^2	0	0	0	0	0	0	0	0	0	0	0	0
$x\theta$	0	0	0	$\frac{tg\beta}{\rho}$	1	0	0	0	0	0	0	0
$x\delta$	0	0	0	0	0	1	0	0	0	0	0	0
θ^2	0	0	0	$\frac{tg^2\beta}{\rho^2}$	$\frac{2tg\beta}{\rho}$	0	1	0	0	0	0	0
$\theta\delta$	0	0	0	0	0	$\frac{tg\beta}{\rho}$	0	1	0	0	0	0
δ^2	0	0	0	0	0	0	0	0	1	0	0	0
y^2	0	0	0	0	0	0	0	0	0	1	0	0
$y\phi$	0	0	0	0	0	0	0	0	0	0	1	0
ϕ^2	0	0	0	0	0	0	0	0	0	0	0	1

Entrance Matrix $B(\beta_1)$

	x_o	θ_o	δ	x_o^2	$x_o\theta$	$x_o\delta$	θ_o^2	$\theta_o\delta$	δ^2	y_o^2	$y_o\phi_o$	ϕ_o^2
x	1	0	0	$-\frac{1}{2}\frac{tg^2\beta}{\rho}$	0	0	0	0	0	0	0	0
θ	$\frac{tg\beta}{\rho}$	1	0	0	$\frac{tg^2\beta}{\rho}$	$-\frac{tg\beta}{\rho}$	0	0	0	0	0	0
δ	0	0	1	0	0	0	0	0	0	0	0	0
x^2	0	0	0	1	0	0	0	0	0	0	0	0
$x\theta$	0	0	0	$\frac{tg\beta}{\rho}$	1	0	0	0	0	0	0	0
$x\delta$	0	0	0	0	0	1	0	0	0	0	0	0
θ^2	0	0	0	$\frac{tg^2\beta}{\rho^2}$	$\frac{2\,tg\beta}{\rho}$	0	1	0	0	0	0	0
$\theta\delta$	0	0	0	0	0	$\frac{tg\,\beta}{\rho}$	0	1	0	0	0	0
δ^2	0	0	0	0	0	0	0	0	1	0	0	0
y^2	0	0	0	0	0	0	0	0	0	1	0	0
$y\phi$	0	0	0	0	0	0	0	0	0	0	1	0
ϕ^2	0	0	0	0	0	0	0	0	0	0	0	1

Entrance Rotation Matrix (x) (Homogeneous)

The exit rotation matrix in the median plane can be obtained in a similar way, giving for an homogeneous magnetic field.

	x_o	θ_o	δ	x_o^2	$x_o\theta_o$	$x_o\delta$	θ_o^2	$\theta_o\delta$	δ^2	y_o^2	$y_o\phi_o$	ϕ_o^2
x	1	0	0	$\frac{tg^2\beta}{2\rho}$	0	0	0	0	0	0	0	0
θ	$\frac{tg\beta}{\rho}$	1	0	$-\frac{tg^3\beta}{2\rho^2}$	$-\frac{tg^2\beta}{\rho}$	$-\frac{tg\beta}{\rho}$	0	0	0	0	0	0
δ	0	0	1	0	0	0	0	0	0	0	0	0
x^2	0	0	0	1	0	0	0	0	0	0	0	0
$x\theta$	0	0	0	$\frac{tg\beta}{\rho}$	1	0	0	0	0	0	0	0
$x\delta$	0	0	0	0	0	1	0	0	0	0	0	0
θ^2	0	0	0	$\frac{tg^2\beta}{\rho^2}$	$\frac{2tg\beta}{\rho}$	0	1	0	0	0	0	0
$\theta\delta$	0	0	0	0	0	$\frac{tg\beta}{\rho}$	0	1	0	0	0	0
δ^2	0	0	0	0	0	0	0	0	1	0	0	0
y^2	0	0	0	0	0	0	0	0	0	1	0	0
$y\phi$	0	0	0	0	0	0	0	0	0	0	1	0
ϕ^2	0	0	0	0	0	0	0	0	0	0	0	1

Exit Rotation Matrix (x) (Homogeneous)

For the axial motion one gets the rotation matrix at the entrance and exit of the magnet by using the same relations (9) to get the B (β_1) and B (β_2) matrices and then by multiplying these by the proper transfer matrices of the small magnets.

The results are:

	y_0	ϕ_0	x_0y_0	$x_0\phi_0$	θ_0y_0	$\theta_0\phi_0$	$y_0\delta$	$\phi_0\delta$
y	1	0	0	0	0	0	0	0
ϕ	0	1	0	0	0	0	0	0
xy	0	0	1	0	0	0	0	0
$x\phi$	0	0	0	1	0	0	0	0
θy	0	0	$\text{tg}\,\frac{\theta}{\rho}$	0	1	0	0	0
$\theta\phi$	0	0	0	$\text{tg}\,\frac{\theta}{\rho}$	0	1	0	0
$y\delta$	0	0	0	0	0	0	1	0
$\phi\delta$	0	0	0	0	0	0	0	1

Entrance Rotation Marix (y) (Homogeneous)

	y_0	ϕ_0	x_0y	$x_0\phi$	θ_0y	$\theta_0\phi$	$y_0\delta$	$\phi_0\delta$
y	1	0	0	0	0	0	0	0
ϕ	0	1	0	0	0	0	0	0
xy	0	0	1	0	0	0	0	0
$x\phi$	0	0	0	1	0	0	0	0
θy	0	0	$\text{tg}\,\frac{\theta}{\rho}$	0	1	0	0	0
$\theta\phi$	0	0	0	$\text{tg}\,\frac{\theta}{\rho}$	0	1	0	0
$y\delta$	0	0	0	0	0	0	1	0
$\phi\delta$	0	0	0	0	0	0	0	1

Exit Rotation Matrix (y) (Homogeneous)

ACKNOWLEDGEMENTS

The authors are very much indebted to PROFESSOR MARCELLO DAMY DE SOUZA SANTOS for the incentive and support during the period of execution of the paper.

RESUMO

A teoria geral de óptica de partículas descrita por K. L. BROWN [1] foi utilizada pelos autores no cálculo dos elementos de matriz correspondendo ao movimento de uma partícula em campos magnéticos homogêneos e inomogêneos (n = 1). Esta teoria foi estendida para a rotação da borda do magneto.

RÉSUMÉ

La théorie générale des faisceaux optiques, de K. L. Brown [1] a été utilisée par les auteurs pour trouver les éléments de matrice dus aux mouvements d'une particule dans un champ magnétique homogène et inhomogène (n = 1). Cette théorie a été developpée pour une reorientation du champ de fuite au moyen d'un bord ajustable.

SUMMARY

The general theory of beam transport optics, as described by K. L. Brown [1] was used by the authors to find out the matrix elements corresponding to the motion of a particle in homogeneous and inhomogeneous (n = 1) magnetic fields. This theory was extended for the rotation of the field boundary.

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