"THE ANALYSIS OF CRACKED STRUCTURES"

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THE ANALYSIS OF CRACKED STRUCTURES

Ian Davidson

SUMMARY

The paper is addressed to all structural engineers for there are many classes of structure in which stable crack systems may exist, notably reinforced concrete structures. The paper presents a brief review of the general problem and derives very simple methods of analysis. Some elaborations are described, together with methods of optimising the calculations. Analytical results are compared with experiments.

INTRODUCTION

1. The purpose of this paper is to extend the well-known mesh-wise analysis of stresses and deflections, wich can be provided by the digital computer, from the linear state into a particular non linear elastic state, as stable cracks extend into the structure. The intention is to maintain close equilibrium of internal stresses with external forces, and to maintain a good compatibility of deflections with stresses. It is also intended that this should not involve appreciably more time or expense than conventional methods. Only an elementary knowledge of mathematics and computer programming is necessary.

2. Stable crack systems may arise when bending moments rather than direct loads have the predominating effect on the structure, provided that it can mobilise an increasing moment of resistance as the cracks extend. A notable case is reinforced concrete in which stability depends on the loads induced in the reinforcing bars, as they are stretched across the cracks. A second circumstance in which stable crack systems can exist is when loads are produced by strain incompatibilities, such as result from temperature gradients; extending cracks can relieve the incompatibilities, thus removing the loads.

3. One general observation which should be made at this stage is that the phenomena under discussion involve crack systems which extend slowly as the loads are applied and which become stable, or at least only continue to grow to a limited extent , if the loading system becomes constant, at any stage short of ultimate. This is because the extent of stable cracking is mainly controlled by the changing topology of the structure, rather than by the exact rupture behaviour of the material. The phenomena involved in the onset of rapid or brittle fracture are different, and are not considered in this paper.

GENERAL CONSIDERATIONS

4. It is required to analyse the behaviour of the structure as it is loaded beyond the linear elastic regime, when cracks gradually develop, or if already present, gradually open as the loading increases. With the exception of localised areas near the crack tips, the material is still elastic, so that well-tried elastic analysis computer programmes can be used, provided that the cracks can be represented properly, and provided that the non-linear behaviour of the material near the crack tips can be treated suitably. There are various ways of simulating a crack, but the only accurate one is to introduce pairs of new boundaries to the structure, which form the sides

of each crack.

5. It is obvious that each major crack has its effect on the behaviour of the whole structure. This means that in any theoretical analysis the loads must be increased in stages, as they would in practice, and at each stage the compatible crack system must be established. This forms an essential part of the input data of the analysis at each succeeding load increment. The result is that the calculations must necessarily be repeated for a number of incremental loads. It therefore becomes essential to develop a method of analysis which is economical in computer time, without losing too much in accuracy and adaptability.

CHOICE OF ANALYTICAL METHOD

6. To meet the requirements of the last two paragraphs, finite difference formulations have been chosen, using dynamic relaxation for solution. The reasons are as follows: - additional boundaries can be introduced and extended as often as required, without having to calculate a new stiffness matrix, which would make a finite element analysis extremely costly. The conditions at the boundaries can also be fully specified without any trouble at all, eg at an open orthogonal crack the shear can be set to zero, while at an interlocking crack it can easily be set to the appropriate value, as will be explained later. Moreover the process of dynamic relaxation has such a simple physical analogy, that any design engineer can readily write a computer programme to suit his particular requirements, and this is the prime purpose of this paper.

DYNAMIC RELAXATION

7. The method of dynamic relaxation has been fully described in the literature[2,3], so that only a very brief outline is required here. The structure is divided into an array of rectangular blocks and the loads are applied instantaneously. The resulting acceleration of each block is calculated, and converted into a deflection after a short time interval. The stresses resulting from the deflections are calculated and are used, with the loads, to calculate the accelerations and deflections after a second time interval, and so on.

Critical viscous damping is applied so that the system converges to the correct values of deflections and stresses for each block after a sufficient number of time intervals or iterations. The mesh lines on which the deflections are calculated are displaced half a mesh from those of the stresses so that a typical block of unit thickness in plane stress is treated as shown in figure 1.

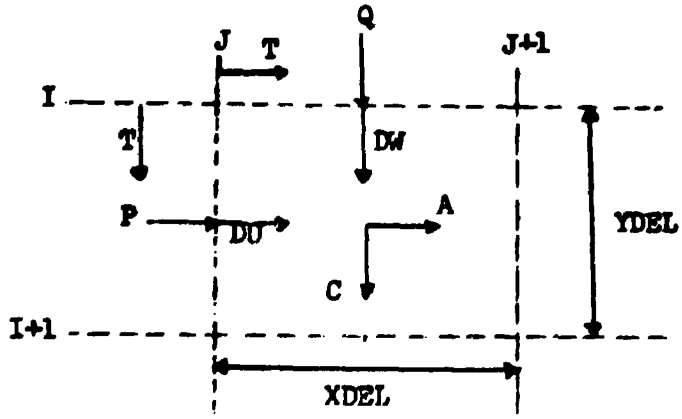


Figure 1

8. The block shown has co-ordinates (I,J) with horizontal stress A, vertical stress C and shear stress T. The stresses are assumed to be constant across each block. The horizontal and vertical deflections are DU & DW respectively, calculated at the points shown. P & Q are pressures which represente the loads which are applied to the structure. The sketch shows the positive direction of all parameters. It will be seen that if it is required to calculate the deflections of the right-hand edge and bottom edge of any structure a row of special blocks may be used outside the right hand edge and another row below the bottom edge.

9. It is convenient to use the velocities U & W, rather than the deflections, in the calculations and it will be found that the basic finite difference equations applicable to the simple block in Fig. 1 are as follows:

(The names, symbols and dimensions are given in Appendix 1, and the derivation of the formulae in Appendix 2).

$$U_a(I,J) = \frac{1 - K/2}{1 + K/2} U_b(I,J) + \frac{1}{1 + K/2} \times \frac{\Delta t}{\rho} \left[\frac{P(I,J) + A(I,J-1) - A(I,J)}{XDEL} + \frac{T(I,J) - T(I+1,J)}{YDEL} \right]$$

$$W_a(I,J) = \frac{1 - K/2}{1 + K/2} W_b(I,J) + \frac{1}{1 + K/2} \times \frac{\Delta t}{\rho} \left[\frac{Q(I,J) + C(I-1,J) - C(I,J)}{YDEL} + \frac{T(I,J) - T(I,J+1)}{XDEL} \right]$$

$$A_a(I,J) = A_b(I,J) + \frac{E \Delta t}{1 - \nu^2} \left[\frac{U(I,J) - U(I,J+1)}{XDEL} + \frac{\nu\{W(I,J) - W(I+1,J)\}}{YDEL} \right]$$

$$C_a(I,J) = C_b(I,J) + \frac{E \Delta t}{1 - \nu^2} \left[\frac{W(I,J) - W(I+1,J)}{YDEL} + \frac{\nu\{U(I,J) - U(I,J+1)\}}{XDEL} \right]$$

$$T_a(I,J) = T_b(I,J) + \frac{E \Delta t}{2(1 - \nu)} \left[\frac{U(I-1,J) - U(I,J)}{YDEL} + \frac{W(I,J-1) - W(I,J)}{XDEL} \right]$$

10. It is only necessary to calculate U & W first, then A, C & T, followed by U & W, and so alternately until satisfactory convergence has been achieved, when U & W will tend to zero. At each iteration DU & DW are calculated by the simple relationship:

$$DU_a(I,J) = DU_b(I,J) + U(I,J) \quad \Delta t.$$

In order to secure compatibility all the deflections, velocities and stresses are normally set to zero before the calculation starts.

11. The above basic equations must be modified at boundaries, and some simple assumptions are made in deriving the appropriate equations. The formulae are explained in paragraph 13, and some typical examples are shown in appendix 3. At an orthogonal boundary,

it is assumed that the shear stress varies linearly from the mid-point of the side of the adjacent block to zero at the boundary: an example of this is shown in the calculation of U in code 15, appendix 3. Re-entrant corners have been treated by calculating the shear in the ordinary manner, as shown in appendix 2, but the shear modulus is multiplied by 0.5, because calculations have shown that this is an appropriate figure. This shear is then assumed to apply in full inside the structure, but is taken as zero along the boundaries which form the corner. This is shown in the calculation of T,WT & UL in code 15, appendix 3. Note that the equations in appendix 3 have been written for the case XDEL = YDEL = 1.

12. Because there are some forty different boundary conditions it has been found convenient to list them by code numbers. In order to describe to the computer the structure and cracks systems to be analysed, it is only necessary to input the code number applicable to each block, so that complete freedom is readily available. A typical code array is shown in figure 2.

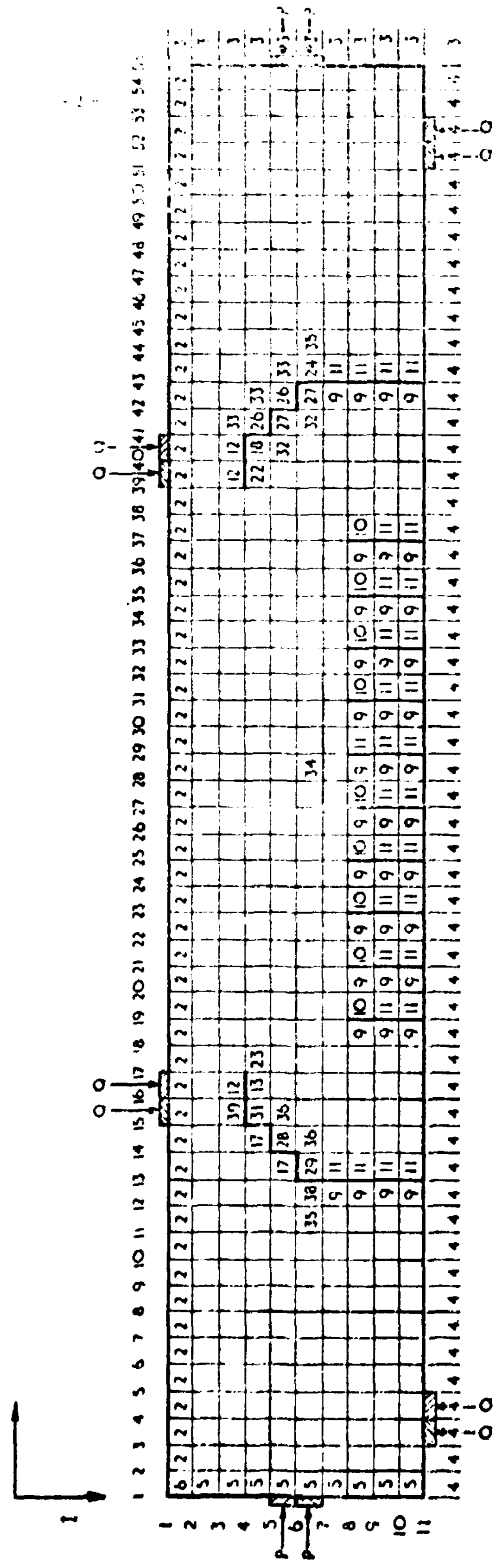
CRACKS

13. The folllowing conventions have been adopted in regard to cracks. All cracks are assumed to occur along mesh lines, and the right hand sides of vertical cracks and the bottom sides of horizontal cracks are regarded as being the sides of the contained block, and their deflections are referred to as DU(I,J) and DW(I,J) respectively. The left-hand sides of vertical cracks and the top sides of horizontal cracks are calculated specially, and their deflections are referred to as DUL(I,J) and DWT(I,J) respectively. The widths of the vertical and horizontal cracks are obviously (DU – DUL) and (DW – DWT) respectively, and these widths are calculated and printed out by the programme.

CRACK STABILITY

14. For each load increment it is required to find the compatible crack system, ie the direction and extent of each crack. As regards the latter it is of little use to examine the computed stresses in the region of the assumed crack tip, because the presence of the singularity prevents any meaningful figures being calculated, however small the mesh, and because the real material exhibits non-linear stress-strain relationships at the crack tip. It has been found to be advisable to examine the crack-opening displacement angle[1], which seems to provide a reliable criterion for crack extent. For the conditions of the test beams and with 1 inch meshes it has been established that if the apparent width of crack, half a mesh from the tip, is between 5×10^{-4} and 13×10^{-4} inches, the crack is stable. If above this range, the crack should be allowed to extend, and if below, the crack does not exist. It may be noted that the crack opening quoted automatically includes a correction factor to allow for the fact that the region of the crack tip has been treated as linearly elastic.

15. The programme carries out the following procedure. At each stage of loading vertical and horizontal cracks are inserted by the design engineer, to a smaller extent than is likely to be compatible with the loads actually applied. After a certain number of iterations the crack width at the tip is automatically examined, and if it is more than the appropriate value, the code numbers are automatically altered to move the tip forward one mesh. The new tip is re-examined after a further number of iterations, and so on. It is, therefore, possible to establish the extent of the compatible cracks, for each load increment, in one computer run. Note that it is quite permissible to change the parameters during the convergence, if due



N.B. UNMARKED MESHES HAVE CODE Nº 1

FIG 2

precautions are taken.

CRACK DIRECTION

16. The control of crack direction rests with the design engineer, who chooses the code numbers as seem appropriate. The choice may be verified at each load increment in the following simple manner. Consider a zig-zag crack (figure 3a) and let δh and δv represent the widths of the vertical and horizontal cracks respectively.

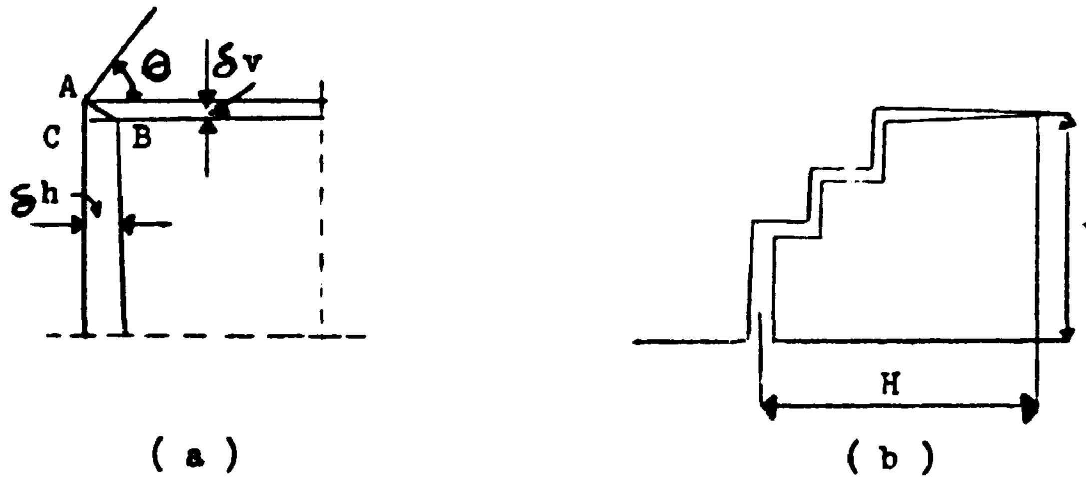


Figure 3

Evidently the preferred direction of the real crack, wich is simulated by the zig-zag crack, is determined by the ratio δh/δv. Assume that the real crack is inserted, when the zig-zag cracks can close. Then AB represents the width of real crack, and its direction must be normal to AB. Assume the real crack makes an angle θ with the horizontal. Then the angle CAB = θ, CB $\approx \delta h$ and AC $\approx \delta v$

$\therefore \tan\theta \approx \frac{\delta h}{\delta v}$, or making an approximate allowance for the convergence of the sides of the crack, $\tan\theta = 0.9\,\frac{\delta h}{\delta v}$. This argument can be extended to an assembly of zig-zag and straight cracks (figure 3b), in which V and H represent the vertical and horizontal coordinates of the crack tip, with the start of the crack as origin. These coordinates can be measured conveniently by counting the number of blocks.

$$\text{Then } \frac{V}{H} = 0.9\,\frac{\Sigma\,\delta h}{\Sigma\,\delta v}$$

17. This simple test can be applied after each load increment, and if it is not approximately true, the next increment of crack may be adjusted accordingly. It is not generally necessary to rerun the last load increment, because minor errors in the crack tip position will have a negligible effect on the bahaviour of the structure as a whole, and because intermediate errors in the crack tip position will not effect the validity of this verification procedure when it is applied to subsequent load increments.

BONDED REINFORCEMENT

18. In general, the strength of a reinforced concrete structure depends on the loads induced in the reinforcement by the tensile strain it undergoes at the cracks which it spans. The numerical value of the strain will be the same if the cracks are assumed to occur at every other mesh line, rather than at the unknown intervals which might occur in practice.

19. Such cracks are, therefore, introduced into the structure being analysed, where bonded reinforcement occurs, as shown in Figure 2, and the forces in the bar are calculated at each iteration as the product of the elastic modulus of the bar, the cross-section area of the bar, and the width of each crack, divided by two mesh lengths. The calculated forces in the bar are applied at every iteration as loads on each side of each crack, tending to close the cracks. This routine is extremely simple and it provides a solution to a problem which can otherwise only be treated by approximate conventional methods.

INTERLOCKING CRACKS

20. Natural cracks in a material such as concrete do not form plane surfaces, and it has been found by experiment that in fact movement parallel to the crack is limited by interlocking of the rough faces. On the assumption that the interlocking faces have an average inclination of 45° to the mean line of the crack, it follows that free movement parallel to the crack, which will be called slip, cannot exceed the width of the crack. This limitation of slip has a negligible effect on a straight crack, such as occurs in plain bending. But if the crack changes direction, as for instance a vertical bending crack turning into an inclined shear crack as shown in Figure 2, the limitation of slip on the vertical portion can have a profound effect on the extension of the inclined portion of the crack.

21. Provision can readily be made for comparing slip with crack width. If the former is greater than the latter then the formulae are automatically changed, so that the correct degree of restraint is applied. The accuracy of the method has been demonstrated by comparison with several sets of test results, including that described later in this paper.

CONVERGENCE

22. The process of solution by dynamic relaxation is in fact a study of the dynamic behaviour of the structure under the given conditions, and in order to secure sufficiently good convergence the iterations should be continued for a time equal to at least one period of the fundamental oscillation.

CHOICE OF TIME INTERVAL

23. Evidently the required cyclic time may be achieved in fewer iterations if the time interval is increased. But the real waves must not be allowed to outrun the computed waves, or the process will become unstable and very large and fictitious values will be calculated. Methods for determining appropriate values of time interval have been published[4, 5] and for a structure in plane stress

$$\Delta t < \sqrt{\frac{\rho}{(\lambda + 2\mu)} / \left(\frac{1}{XDEL^2} + \frac{1}{YDEL^2}\right)}$$

where λ and μ are the Lame constants. It has been found that the above value can be used safely, if the following precaution is taken. In effect $\Delta t \sim\sqrt{\rho/E}$ and if this ratio is kept constant a single value of Δt throughout the beam can also be the optimum value The introduction of ERATIO disturbs the ratio ρ/E, unless a fictitious density ρ x ERATIO is used.

CHOICE OF DAMPING

The damping factor should be a little less than critical and critical damping = $2\,\omega_o\,\Delta t$, where ω_o is the fundamental angular frequency of the structure, and Δt is the time interval, which has been determined in accordance with the previous paragraph. Correct convergence may be observed in Figure 4, in which horizontal and vertical velocities at selected points have been plotted by the computer at regular intervals during the convergence. This is a very useful technique which is recommended in all cases. If convergence has not been achieved, the shape of the plot will show immediately whether the damping factor has been too large or too small, or whether more iterations are required.

TOTAL NUMBER OF ITERATIONS

25. The required number of iterations works out at 12 divided by the damping factor, if the latter is about critical, as determined from the last paragraph.

SUPPORTS

26. It would be sufficient to fix the support points in the same way as in the actual structure, but something can be gained by a little consideration. Firstly, if the reactions are obvious, it will save computational time to feed them in as applied loads. Secondly, it has been seen that the required number of iterations is directly related to the fundamental period of the structure, and this may be minimised by treating it in the "free-free" mode.

DYNAMIC CALCULATIONS

27. Because the programme can provide a study of the dynamic behaviour of the structure, it may be used to follow in great detail the response of the structure to shock loading, or to forcing oscillations as in seismic studies [6] . Some obvious precautions must be taken, but the additions which must be made for this type of analysis are very small.

OVERLAPPING CRACKS

28. During simple calculations of cracked beams, including the crack extension routine, it is immaterial whether the sides of the crack overlap during the iterations. Provided that the cracks have a positive width at convergence, the calculations will be correct. But negative cracks cause instability during the convergence if the reinforced concrete or interlocking crack routines are being used. Overlapping cracks are also inadmissable in the case of the dynamic calculations mentioned in the last paragraph. Statements have, therefore, been added to the programme to prevent negative cracks occuring.

SERIES OF INCREMENTAL LOADS

29. As already mentioned, the calculations are normally started with zero stresses and

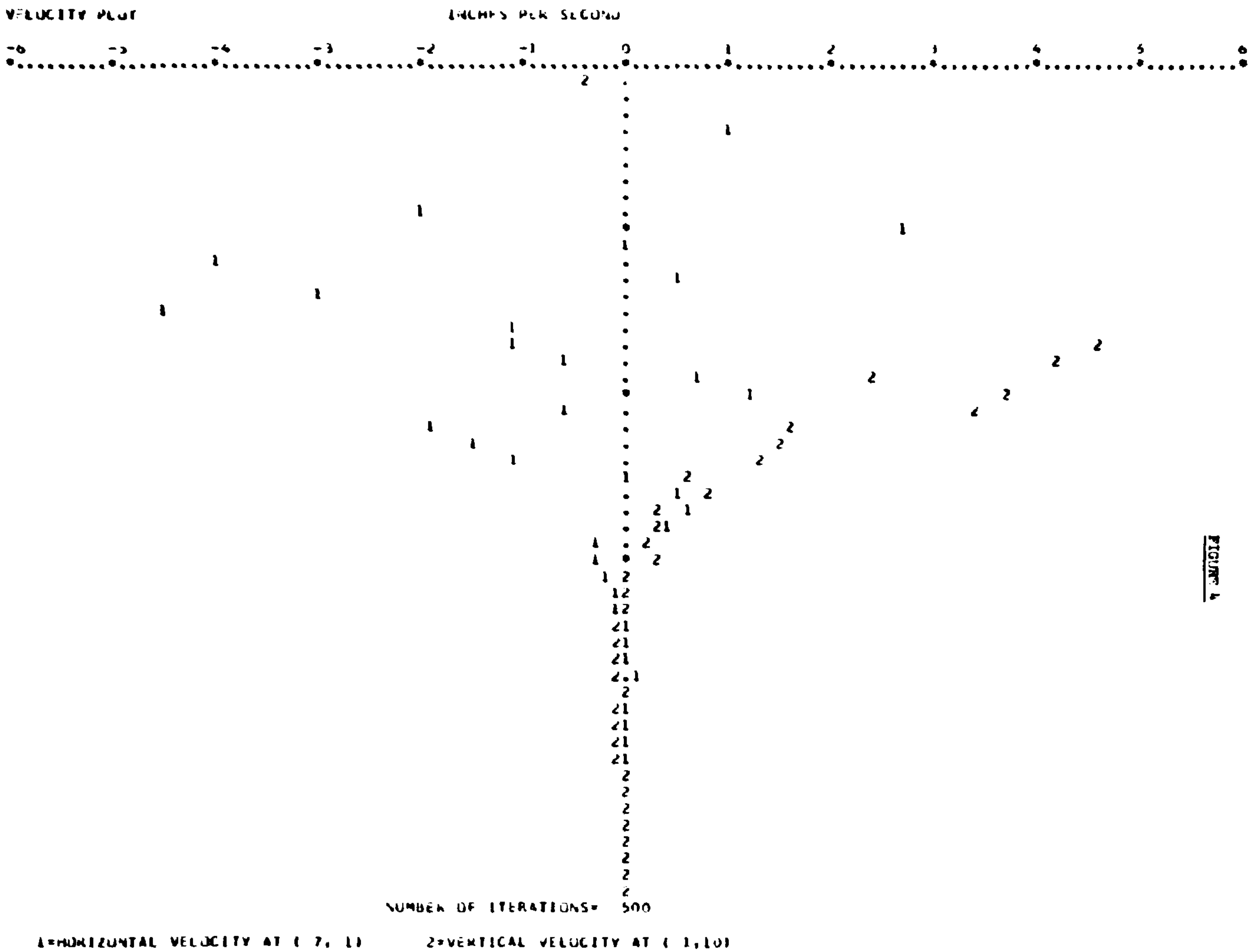

VELOCITY PLOT
INCHES PER SECOND
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
NUMBER OF ITERATIONS= 500
1=HORIZONTAL VELOCITY AT (7, 1)
2=VERTICAL VELOCITY AT (1,10)

FIGURE 4

deflections, but any other compatible set would do. In the case where a series of incremental loads are being applied to one structure, it is obviously convenient and economical to start each run with the converged set of stresses and deflections reached at the end of the previous run. This can easily be done by storing them on magnetic tape, and the code numbers are also recorded, so that cracks which have extended in one run become part of the input for the next run.

PROGRAMME

30. The programme is exceedingly simple and requires a minimal knowledge of digital computer programming. The iterations are in two nested DO loops. The calculations of deflections and stresses for each block are in inset DO loops, and the correct formulae are selected by COMPUTED GO TO's, controlled by KODE (I,J). Simple IF statements handle the various elaborations which have been described.

INPUT

31. The input data consists of 13 constants and 6 arrays involving about 20 cards. But only about 4 of these will need to be changed for different loadings on the same structure.

OUTPUT

32. Because the structure is treated as an array, and the output is also printed in arrays, the output represents a picture of the structure, and the stresses and deflections of all the blocks are printed in their correct relative positions, which is extremely convenient. Running time for one load increment and 605 blocks is about 10 minutes when starting from zero, including extending cracks to compatibility, and about 6 minutes when starting from a previously converged set of figures. It is thought that the ease of input and shortness of running time, together with the simplicity of the programme, show that the proposed methods comply with the intention that they should not involve appreciably more time or expense than conventional methods.

RESULTS

33. The verification of the proposed methods of analysis is of course a wide-ranging matter which will require more time and more space than is now available. As an illustration, Table 1 shows selected experimental results (Ex) from the testing of beam 55/72/1 adjusted to 1 inch width of beam. The analytical results (An) are also given, with the percentage error of the latter. In this series of tests unsymmetrical vertical deflections were applied at the points P_A & P_B. By mistake, the axial load at the right hand end was applied 0.12 ins. above the axis, and this has been reproduced in the analysis. The reactions at the supports R_A & R_B were measured, together with the vertical deflection at the centre top of the beam, and the strains in the reinforcement bars. It will be seen that the reactions and deflections as calculated have a maximum error of 18%, excluding station 44, when the beam was on the verge of collapse. The error in the mean load in the reinforcing bar is 32% at station 16, diminishing to 1% at station 42. The former error is probably due to the relatively small stress in the bar, which is 6650 psi at station 16, and 10300 psi at station 42: there is also the experimental difficulty of measuring the strains.

34. Figure 5 shows a plot of the observed cracking, with the apparent tip indicated at the various loading stations. The corresponding points are shown for the analytical results, and it will be seen that the agreement is good. Similar agreement as regards loads, deflections and crack patterns has been obtained for several other test assemblies.

AXI-SYMMETRIC STRUCTURES

35. The application of the methods outlined in this paper to axi-symmetric structures will be obvious, but there is one point worth mentioning. Considerations of solid geometry make it evident that cracks in horizontal and tangential planes, of the types already discussed, cannot exist without simultaneous cracks in radial planes [1] The consideration of any particular radial crack is impossible so long as we are concerned with an axi-symmetrical structure. However it is possible to consider the axi-symmetric consequences if the structure is in a state of having radial cracks, and these are:

1) The hoop stresses B become zero.
2) Therefore the expressions for the stresses A & C must be modified.
3) It can be assumed that the radial cracks embrace a sector wider than that which is being analysed, so that plane strain can still be assumed.
4) The total width of all the radial cracks at the block (I,J) is equal to 2π x DU (I,J).

In order that the above conclusions can be implemented conveniently, an additional array, KRAK, has been introduced. This programme has been checked against tests on a model prestressed concrete pressure vessel for a nuclear reactor.

THREE DIMENSIONAL STRUCTURES.

36. The methods of this paper can easily be extended to three dimensional structures, and a programme has been written in cylindrical coordinates, which is now being checked.

CONCLUSIONS

37. It has been shown that the finite difference equations connecting stresses with deflections are extremely simple, and that the method of solving these equations by dynamic relaxation requires nothing more than elementary logic. The various developments and applications of these simple basic ideas are equally straight-forward, yet the combined result can provide every structural engineer with a most powerful method of analysis. The author is aware that limitations of space may have made the paper difficult to follow, in some respects, and he will therefore be very pleased to elaborat on any points.

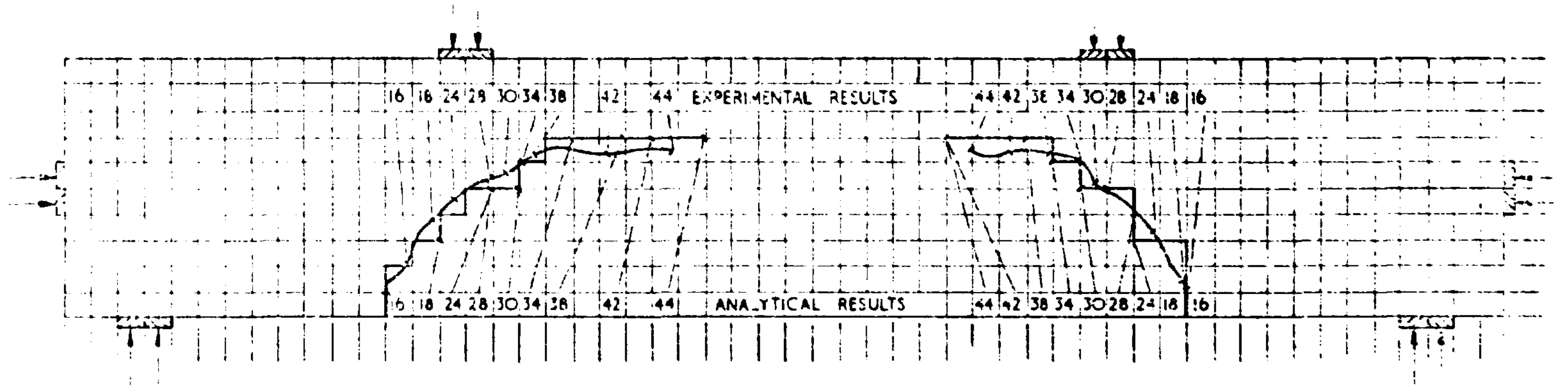

16 18 24 28 30 34 38 42 44 EXPERIMENTAL RESULTS 44 42 38 34 30 28 24 18 16
16 18 24 28 30 34 38 42 44 ANALYTICAL RESULTS 44 42 38 34 30 28 24 18 16

FIGURE 5

TABLE 1

Station Number	Deflection x 10^4		End Load	Reaction at Supports						Mean load in Reinforcement			Central Deflection x 10^4		
	PA	PB		RA			RB								
	ins.	ins.	lbs.	Ex. lbs.	An. lbs.	Error %	Ex. lbs.	An. lbs.	Error %	Ex. lbs.	An. lbs.	Error %	Ex. ins.	An. ins.	Error %
16	230	220	9025	2300	2320	+ 1	1990	2010	+ 1	650	445	- 32	303	294	- 3
18	262	252	9025	2300	2360	+ 3	2080	2290	+ 10	829	560	- 32	354	353	0
24	294	284	9175	2390	2400	0	2210	2360	+ 7	897	627	- 30	379	372	- 2
28	326	316	9050	2360	2500	+ 6	2180	2360	+ 8	912	665	- 27	415	412	- 1
30	358	345	9250	2460	2600	+ 6	2220	2420	+ 9	915	735	- 19	450	450	0
34	430	404	9100	2490	2700	+ 8	2210	2580	+ 16	929	835	- 10	523	538	+ 3
38	526	500	9075	2520	2820	+ 12	2220	2600	+ 17	953	908	- 5	[illegible]	[illegible]	+ 5
42	686	660	9000	2600	2570	+ 14	2310	2740	+ 18	1006	992	- 1	834	[illegible]	+ [illegible]
44	[illegible]	820	9000	2580	3080	+ 20	2390	3010	+ 28	911	973	+ 6	1058	1116	+ 5

13

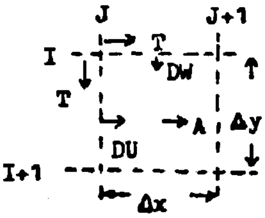
APPENDIX 1

NAMES, SYMBOLS AND DIMENSIONS.

A	Horizontal stress
ASTEEL	Cross-section area of bonded steel
C	Vertical Stress
DAMP= k	Damping co-efficient
DU	Horizontal deflection
DUL	Horizontal deflection left-side of crack
DW	Vertical deflection
DWT	Vertical deflection top of crack
ELAST = E	Young's modulus
ERATIO	Young's modulus ratio
I	Mesh co-ordinate
J	Mesh co-ordinate
KODE	Describes boundary conditions of mesh
P	Horizontal body force
POISS = ν	Poissons ratio
Q	Vertical body force
RHO = ρ	Mass density
SLIP	Slip control
T	Shear stress
TDEL = Δt	Time interval
U	Horizontal velocity
UL	Horizontal velocity left of crack
W	Vertical velocity
WT	Vertical velocity top of crack
XDEL	Horizontal dimension of block
YDEL	Vertical dimension of block
Subscript$_b$	Before iteration
Subscript$_a$	After iteration
Note	All dimensions are in inches, pounds and seconds, but any self-consistent system could be used, including S.I.
Note	Positive stresses are compressive.

APPENDIX 2

FINITE DIFFERENCE EQUATIONS



$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)$ for plane stress

$$\therefore A = \frac{E}{1-\nu^2}\left[\left\{\frac{DU-DU(J+1)}{\Delta x}\right\} + \nu\left\{\frac{DW-DW(I+1)}{\Delta y}\right\}\right]$$

Differentiate with respect to time

$$\frac{\Delta A}{\Delta t} = \frac{E}{1-\nu^2}\left[\left\{\frac{U-U(J+1)}{\Delta x}\right\} + \nu\left\{\frac{W-W(I+1)}{\Delta y}\right\}\right]$$

But $\Delta A = A_a - A_b$

$$\therefore A_a = A_b + \frac{E\Delta t}{1-\nu^2}\left[\left\{\frac{U-U(J+1)}{\Delta x}\right\} + \nu\left\{\frac{W-W(I+1)}{\Delta y}\right\}\right]$$

——— " ———

$T = \frac{E}{2(1+\nu)}\phi$ for plane stress

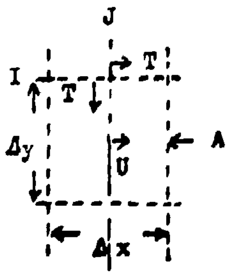
$$= \frac{E}{2(1+\nu)}\left[\frac{DU(I-1) - DU}{\Delta y} + \frac{DW(J-1) - DW}{\Delta x}\right]$$

$$\therefore \frac{\Delta T}{\Delta t} = \frac{E}{2(1+\nu)}\left[\frac{U(I-1) - U}{\Delta y} + \frac{W(J-1) - W}{\Delta x}\right]$$

But $\Delta T = T_a - T_b$

$$\therefore T_a = T_b + \frac{E\Delta t}{2(1+\nu)}\left[\frac{U(I-1) - U}{\Delta y} + \frac{W(J-1) - W}{\Delta x}\right]$$

——— " ———



$$P = Mf$$

$$= \frac{M}{\Delta t}[\Delta U + \text{viscous damping } kU]$$

$$\therefore \Delta y\{A(J-1) - A\} + \Delta x\{T-T(I+1)\} = \frac{\rho\Delta x\Delta y}{\Delta t}[\Delta U + kU]$$

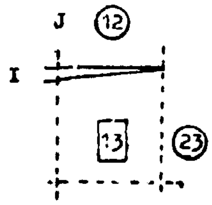
But $U = \frac{U_a + U_b}{2}$ & $\Delta U = U_a - U_b$

$$\therefore U_a - U_b + \frac{k}{2}(U_a + U_b) = \frac{\Delta t}{\rho\Delta x\Delta y}[\Delta y\{A(J-1)-A\} + \Delta x\{T-T(I+1)\}]$$

$$\therefore U_a = \frac{1-k/2}{1+k/2}U_b + \frac{1}{1+k/2}\cdot\frac{\Delta t}{\rho}\left[\frac{A(J-1) - A}{\Delta x} + \frac{T-T(I+1)}{\Delta y}\right]$$

FOUBE 2 CODE NUMBERS

LEFT HAND CRACKS



13. Bottom side of horizontal crack at tip.

A_a = normal

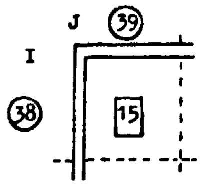
C_a = normal

$T_a = 0$

$$W_a = \frac{1-k/2}{1+k/2} W_b + \frac{2}{1+k/2} \cdot \frac{\Delta t}{\rho}\left[Q-C + \frac{T(I+1)}{4} - \frac{T(J+1)}{2}\right]$$

$$WT_a = \frac{1-k/2}{1+k/2} WT_b + \frac{2}{1+k/2} \cdot \frac{\Delta t}{\rho}\left[Q+C(I-1) + \frac{T(I-1)}{4} - \frac{T(J+1)}{2}\right]$$

U_a = normal



15. Cracks on left hand side and top

A_a = normal

C_a = normal

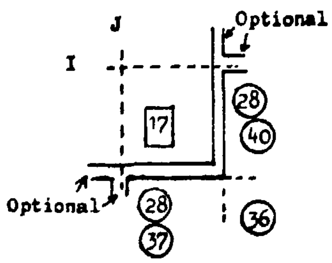
$$T_a = T_b + \frac{E\Delta t}{2(1+\nu)}\left[U(I-1) - UL + W(J-1) - WT\right] \times 0.3$$

$$W_a = \frac{1-k/2}{1+k/2} W_b + \frac{2}{1+k/2} \cdot \frac{\Delta t}{\rho}\left[Q-C - \frac{T(I+1,\ J+1)}{4}\right]$$

$$WT_a = \frac{1-k/2}{1+k/2} WT_b + \frac{2}{1+k/2} \cdot \frac{\Delta t}{\rho}\left[Q+C(I-1) + \frac{T}{2} - \frac{T(I-1,\ J+1)}{4}\right]$$

$$U_a = \frac{1-k/2}{1+k/2} U_b + \frac{2}{1+k/2} \cdot \frac{\Delta t}{\rho}\left[P-A - \frac{T(I+1,\ J+1)}{4}\right]$$

$$UL_a = \frac{1-k/2}{1+k/2} UL_b + \frac{2}{1+k/2} \cdot \frac{\Delta t}{\rho}\left[P+A(J-1) + \frac{T}{2} - \frac{T(I+1,\ J-1)}{4}\right]$$



17. Cracks on right hand side and bottom

$$A_a = A_b + \frac{E\Delta t}{1-\nu^2}\left[U - UL(J+1)\right] + \frac{\nu E\Delta t}{1-\nu^2}\left[W - WT(I+1)\right]$$

$$C_a = C_b + \frac{E\Delta t}{1-\nu^2}\left[W - WT(I+1)\right] + \frac{\nu E\Delta t}{1-\nu^2}\left[U - UL(J+1)\right]$$

T_a = normal

$$W_a = \frac{1-k/2}{1+k/2} W_b + \frac{1}{1+k/2} \cdot \frac{\Delta t}{\rho}\left[Q + C(I-1) - C + T - \frac{T(J+1)}{2}\right]$$

$$U_a = \frac{1-k/2}{1+k/2} U_b + \frac{1}{1+k/2} \cdot \frac{\Delta t}{\rho}\left[P + A(J-1) - A + T - \frac{T(I+1)}{2}\right]$$

SUMÁRIO

Este relatório é dedicado a todos os engenheiros civis que se preocupam com certos tipos de estruturas que podem apresentar sistemas estáveis de fissuras, o que ocorre notadamente, em estruturas de concreto armado. Apresenta-se uma breve revisão do problema geral e a dedução de métodos muito simples de análise. Alguns detalhes são descritos, juntamente com métodos de otimização de cálculos, e os resultados analíticos são comparados com os experimentais.

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