TWO-GROUP MILNE PROBLEM: A NUMERICAL STUDY OF THE EFFECT OF SCATTERING ANISOTROPY

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ABSTRACT

The Milne problem in two group neutron transport theory for linearly anisotropic scattering is solved in terms of $\underset{\sim}{H}$ matrices using half range orthogonality relations of the eigenfunctions

The effect of scattering anisotropy is studied numerically using Metcalf and Zweifel s isotropic scattering data set for light water and considering the anisotropic part of the transfer matrix given by $\underset{\sim}{B} = \underset{\sim}{C}\,\underset{\sim}{P}$ where $\underset{\sim}{C}$ is the isotropic part and $\underset{\sim}{P}$ is a 2 x 2 matrix with elements $p_j\ \delta_{ij}$

INTRODUCTION

As an energy dependent model in transpot theory, the multigroup equation has been extensively investigated in recent years One of the earliest applications of the Case method to the multigroup model was made by Siewert and Zweifel,[20 21] to a model of radiative transfer. Inherent to the method, they obtained rigorous general solutions to the transport equation, showing their full and half range completeness and full range orthogonality Further they obtained half range orthogonality relations of the eigenfunctions in a special case. Two-group neutron-transport problems were studied by, among others, Zelazny and Kuszell[25], Metcalf and Zweifel[9 10], and Siewert and Shieh[19] within the time independent, plane parallel medium and isotropic scattering model Siewert and Shieh, in particular, obtained concise solutions and analysed the discrete spectrum General solutions of the N group neutron-transport equation have also been obtained by Yoshimura and Katsuragi[24]

Most of these works, however, were restricted to isotropic scattering and their applications to transport problems were some what limited in that the half-range completeness and orthogonality of the eigenfunctions were not shown.

Extentions to anisotropic scattering[8 14 15 23] and to half-range problems[11 13 15] have been studied by several authors Pahor[11 13], with co-workers, has solved half-space problems, applying the principle of invariance, in the energy dependent as well as one-group model. However, Siewert's work[16] suggested the possibility of another approach to half-range problems in the multigroup model His work was followed by Siewert and Ishiguro[18] who obtained the half-range orthogonality relation in the two-group model with isotropic scattering Though the work was based on invariance principles, which are restricted to nonmultiplying media, the works of Siewert, Burniston, and Kriese[17] and Burniston, Mullikin, and Siewert[4] have shown that the obtained theorem can be placed on a firm mathematical basis in the multiplying as well as nonmultiplying case The technique has also been applied to the case of linearly anisotropic scattering by Ishiguro[5] to derive half range orthogonality relations of the eigenfunctions

Though several problems have been solved for isotropic scattering and numerical results reported[10,18,7], numerical calculations for the case of anisotropic scattering have been reported only for a few cases[5,1]. Further, to our knowledge, no study has been made on the behavior of the neutron population when anisotropy is taken into consideration compared to isotropic scattering. Therefore we would like to study here the effect of scattering anisotropy considering the half space Milne problem. The expansion coefficients are obtained in terms of matrix functions, similar to Chandrasekhar's H-function, which can be calculated from regular integral equations.

In section 2, we list the results of Reith and Siewert[14] to establish our notation and the half range orthogonality relation previously obtained[5]. Though the analysis of the half-range completeness or the existence and uniqueness of the fundamental matrices has not been completed, we base our study on the principle of invariance and numerical checks of the results.

In section 3, we report our numerical results for several sets of parameters and also discuss numerical checks we have employed.

BASIC ANALYSIS

We consider here the two-group equation written as

$$\mu \frac{\partial}{\partial x} \underset{\sim}{\Psi}(x,\mu) + \underset{\sim}{\Sigma}\underset{\sim}{\Psi}(x,\mu) = \underset{\sim}{C} \int_{-1}^{1} \underset{\sim}{\Psi}(x,\mu')d\mu' + \mu \underset{\sim}{B} \int_{-1}^{1} \underset{\sim}{\Psi}(x,\mu')\mu' d\mu' \tag{1}$$

where all symbols are the same as in Reith and Siewert[14] (hereafter refered to as RS). The general solution of Eq. (1) has been obtained and thus the Milne problem solution can be written as

$$\begin{aligned}\underset{\sim}{\Psi}(x,\mu) &= A(-\nu_1)\underset{\sim}{\Phi}(-\nu_1,\mu)\exp(x/\nu_1) + \sum_{i=1}^{\kappa} A(\nu_i)\underset{\sim}{\Phi}(\nu_i,\mu)\exp(-x/\nu_i) \\ &+ \int_0^{1/\sigma} [A_1^{(1)}(\nu)\underset{\sim}{\Phi}_1^{(1)}(\nu,\mu) + A_2^{(1)}(\nu)\underset{\sim}{\Phi}_2^{(1)}(\nu,\mu)]\exp(-x/\nu)d\nu \\ &+ \int_{1/\sigma}^{1} A^{(2)}(\nu)\underset{\sim}{\Phi}^{(2)}(\nu,\mu)\exp(-x/\nu)d\nu, \quad x \geqslant 0, \ \mu \in (-1,1),\end{aligned} \tag{2}$$

where the eigenfunctions $\underset{\sim}{\Phi}$ are also given in RS, and κ is the number of pairs of the discrete eigenvalues $\pm \nu_i$. Since all our cases correspond to $\kappa = 1$, we consider hereafter only that case. With the normalization $A(-\nu_1) = 1$, the expansion coefficients must be determined from the boundary condition

$$\begin{aligned}-\underset{\sim}{\Phi}(-\nu_1,\mu) &= A(\nu_1)\underset{\sim}{\Phi}(\nu_1,\mu) + \int_0^{1/\sigma} [A_1^{(1)}(\nu)\underset{\sim}{\Phi}_1^{(1)}(\nu,\mu) + A_2^{(1)}(\nu)\underset{\sim}{\Phi}_2^{(1)}(\nu,\mu)]d\nu \\ &+ \int_{1/\sigma}^{1} A^{(2)}(\nu)\underset{\sim}{\Phi}^{(2)}(\nu,\mu)d\nu, \quad \mu \in (0,1).\end{aligned} \tag{3}$$

We would now like to summarize the half-range orthogonality relation of the eigenfunctions[5] which enables us to solve Eq. (3) in a concise manner. Since we use different notations from those of RS, we would first like to list some of their results in our notation.

We write Eq. (1) as

$$\mu \frac{\partial}{\partial x} \underset{\sim}{\Psi}(x,\mu) + \underset{\sim}{\Sigma}\underset{\sim}{\Psi}(x,\mu) = \underset{\sim}{Q}(\mu)\underset{\sim}{D} \int_{-1}^{1} \underset{\sim}{\tilde{Q}}(\mu')\underset{\sim}{\Psi}(x,\mu')d\mu', \tag{4}$$

where $\underset{\sim}{Q}(\mu)$ is a 2 x 4 matrix and $\underset{\sim}{D}$ a 4 x 4 matrix defined as

$$\underset{\sim}{Q}(\mu) = \left[\underset{\sim}{I} \quad \mu\underset{\sim}{I} \right], \quad \underset{\sim}{I} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \tag{5}$$

$$\underset{\sim}{D} = \begin{vmatrix} \underset{\sim}{C} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B} \end{vmatrix} \tag{6}$$

and we have used a superscript tilde to denote the transpose operation. In the usual manner, proposing solutions of the form

$$\underset{\sim}{\Psi}(x,\mu) = \underset{\sim}{F}(\nu,\mu) \exp(-x/\nu), \tag{7}$$

Eq. (4) reduces to

$$[\underset{\sim}{\Sigma} - \frac{\mu}{\nu} \underset{\sim}{I}] \underset{\sim}{F}(\nu,\mu) = \underset{\sim}{Q}(\mu)\underset{\sim}{D}\underset{\sim}{\Gamma}(\nu) \underset{\sim}{M}(\nu), \tag{8}$$

where we have defined

$$\underset{\sim}{M}(\nu) = \int_{-1}^{1} \underset{\sim}{F}(\nu,\mu)d\mu \tag{9}$$

and

$$\underset{\sim}{\Gamma}(\nu) = \begin{bmatrix} \underset{\sim}{I} \\ \nu[\underset{\sim}{\Sigma} - 2\underset{\sim}{C}] \end{bmatrix} \tag{10}$$

We write the solution of Eq. (8) as

$$\underset{\sim}{F}(\nu,\mu) = \underset{\sim}{E}(\nu,\mu) \underset{\sim}{Q}(\mu) \underset{\sim}{D} \underset{\sim}{\Gamma}(\nu) \underset{\sim}{M}(\nu), \tag{11}$$

where

$$\underset{\sim}{E}(\pm\nu_1,\mu) = \begin{bmatrix} \dfrac{\nu_1}{\sigma\nu_1 \mp \mu} & 0 \\ 0 & \dfrac{\nu_1}{\nu_1 \mp \mu} \end{bmatrix}, \tag{12a}$$

$$\underset{\sim}{E}_\alpha^{(1)}(\nu,\mu) = \nu\underset{\sim}{K}(\nu,\mu) + \omega_\alpha^{(1)}(\nu)\underset{\sim}{\delta}(\nu,\mu) \quad \nu\epsilon(-\frac{1}{\sigma},\frac{1}{\sigma}), \alpha = 1,2, \tag{12b}$$

$$\underset{\sim}{E}^{(2)}(\nu,\mu) = \nu\underset{\sim}{K}(\nu,\mu) + \omega^{(2)}(\nu)\underset{\sim}{\delta}(\nu,\mu), \quad \nu\epsilon(-1,-\frac{1}{\sigma}) \cup (\frac{1}{\sigma},1), \tag{12c}$$

$$K(\nu,\mu) = \begin{vmatrix} \frac{P}{\sigma\nu - \mu} & 0 \\ 0 & \frac{P}{\nu - \mu} \end{vmatrix}, \qquad \delta(\nu,\mu) = \begin{vmatrix} \delta(\sigma\nu - \mu) & 0 \\ 0 & \delta(\nu - \mu) \end{vmatrix}, \tag{12d,e}$$

and we have used the symbol P to denote that the integral is to be evaluated in the Cauchy principal value sense. The discrete eigenvalue ν_1 is the positive zero of the dispersion function $\Lambda(z) = \det \Lambda(z)$ with

$$\Lambda(z) = I + z \int_{-1}^{1} \Theta(\mu)\, Q^*(\mu) \frac{d\mu}{\mu - z} D\Gamma(z), \tag{13}$$

and the normalization vector $M(\pm\nu_1)$ must satisfy

$$\Lambda(\pm\nu_1)\, M(\pm\nu_1) = 0. \tag{14}$$

Here we have defined

$$\Theta(\mu) = \begin{vmatrix} \Theta(\mu) & 0 \\ 0 & 1 \end{vmatrix} \tag{15}$$

with $\Theta(\mu) = 1, \mu\epsilon(-\frac{1}{\sigma}, \frac{1}{\sigma})$, and $\Theta(\mu) = 0$, otherwise, and a superscript * is used to denote the operation $\mu \to \sigma\mu$ in the top row, hence

$$Q^*(\mu) = [\, I \quad \mu\Sigma \,] \tag{16}$$

For the continuum eigenvalue $\nu\epsilon(-1, 1)$, the functions $\omega_\alpha^{(1)}(\nu)$ and $\omega^{(2)}(\nu)$ are determined by

$$\det \left| \Lambda(\nu) - \omega(\nu)\Theta(\nu) Q^*(\nu)\, D\, \Gamma(\nu) \right| = 0, \tag{17}$$

where

$$\Lambda(\nu) = I + \nu P \int_{-1}^{1} \Theta(\mu)\, Q^*(\mu) \frac{d\mu}{\mu - \nu} D\Gamma(\nu), \tag{18}$$

and corresponding vectors $M_\alpha^{(1)}(\nu)$ and $M^{(2)}(\nu)$ can be determined from

$$\left| \Lambda(\nu) - \omega_\alpha^{(1)}(\nu)\, \Theta(\nu)\, Q^*(\nu)\, D\, \Gamma(\nu) \right| M_\alpha^{(1)}(\nu) = 0 \tag{19a}$$

and

$$\left| \Lambda(\nu) - \omega^{(2)}(\nu)\, \Theta(\nu)\, Q^*(\nu)\, D\, \Gamma(\nu) \right| M^{(2)}(\nu) = 0 \tag{19b}$$

The adjoint transport equation defined by replacing D in Eq. (4) by $\tilde{D}$ can be solved in the same manner. In particular, it has been shown[22] that the eigenvalue spectrum of the adjoint equation is identical to that of Eq. (4). Hereafter we use a subscript a to denote functions related to the adjoint equation.

The half range orthogonality relation was derived using the invariant imbedding technique. Thus if we define 2 x 2 matrices $\underset{\sim}{s}(\mu,\mu')$, $\underset{\sim}{h}(\mu)$, and $\underset{\sim}{\ell}(\mu)$ by

$$\underset{\sim}{\Psi}(0,-\mu) = \frac{1}{2\mu}\int_0^1 \underset{\sim}{s}(\mu,\mu')\underset{\sim}{\Psi}(0,\mu')d\mu', \quad \mu \in (0,1), \tag{20}$$

$$\int_{-1}^1 \underset{\sim}{\Psi}(x,\mu)d\mu = \int_0^1 \tilde{\underset{\sim}{h}}(\mu)\,\underset{\sim}{\Psi}(x,\mu)d\mu, \tag{21}$$

and

$$\int_{-1}^1 \underset{\sim}{\Psi}(x,\mu)\mu d\mu = \int_0^1 \tilde{\underset{\sim}{\ell}}(\mu)\underset{\sim}{\Psi}(x,\mu)\,d\mu, \tag{22}$$

where $\underset{\sim}{\Psi}(x,\mu)$ is a half-space solution of Eq. (1), following relations can be derived using the principle of invariance and the reciprocity of the $\underset{\sim}{s}$-matrix :

$$\tilde{\underset{\sim}{h}}(\mu) = \underset{\sim}{I} + \frac{1}{2}\int_0^1 \underset{\sim}{s}(\mu',\mu)\frac{d\mu'}{\mu'}, \tag{23}$$

$$\tilde{\underset{\sim}{\ell}}(\mu) = \mu\underset{\sim}{I} - \frac{1}{2}\int_0^1 \underset{\sim}{s}(\mu',\mu)d\mu', \tag{24}$$

and

$$\frac{1}{\mu}\underset{\sim}{\Sigma}\underset{\sim}{s}(\mu,\mu_0) + \frac{1}{\mu_0}\underset{\sim}{s}(\mu,\mu_0)\underset{\sim}{\Sigma} = 2\underset{\sim}{\Phi}(\mu)\underset{\sim}{D}\tilde{\underset{\sim}{\Psi}}(\mu_0), \tag{25}$$

where we have defined 2 x 4 matrices $\underset{\sim}{\Psi}(\mu)$ and $\underset{\sim}{\Phi}(\mu)$ by

$$\underset{\sim}{\Psi}(\mu) = \left[\underset{\sim}{h}(\mu)\;\underset{\sim}{\ell}(\mu)\right] \tag{26}$$

and

$$\underset{\sim}{\Phi}(\mu) = \left[\underset{\sim}{h}_0(\mu) \; - \underset{\sim}{\ell}_0(\mu)\right]. \tag{27}$$

If we now define

$$\underset{\sim}{S}(\mu,\mu_0) = \begin{bmatrix} s_{11}(\sigma\mu,\sigma\mu_0) & s_{12}(\sigma\mu,\mu_0) \\ s_{21}(\mu\;,\sigma\mu_0) & s_{22}(\mu\;,\mu_0) \end{bmatrix}, \tag{28}$$

$$\underset{\sim}{H}(\mu) = \underset{\sim}{h}^*(\mu) = \begin{bmatrix} h_{11}(\sigma\mu) & h_{12}(\sigma\mu) \\ h_{21}(\mu\;) & h_{22}(\mu\;) \end{bmatrix}, \tag{29}$$

and

$$\underset{\sim}{L}(\mu) = \underset{\sim}{\ell}^*(\mu) = \begin{bmatrix} \ell_{11}(\sigma\mu) & \ell_{12}(\sigma\mu) \\ \ell_{21}(\mu\;) & \ell_{22}(\mu\;) \end{bmatrix}, \tag{30}$$

the $\underset{\sim}{S}$-matrix can be expressed as

$$\underset{\sim}{S}(\mu,\mu_0) = \frac{2\mu\mu_0}{\mu+\mu_0}\,\underset{\sim}{\Phi}^*(\mu)\underset{\sim}{D}\widetilde{\underset{\sim}{\Psi}}^*(\mu_0)\,, \tag{31}$$

and from Eq. (23) we obtain an integral equation

$$\underset{\sim}{H}(\mu) = \underset{\sim}{I} + \mu\underset{\sim}{H}(\mu)\,\widetilde{\underset{\sim}{C}}\int_0^1 \widetilde{\underset{\sim}{H}}_a(\mu')\underset{\sim}{\Theta}(\mu')\frac{d\mu'}{\mu'+\mu} - \mu\underset{\sim}{L}(\mu)\,\widetilde{\underset{\sim}{B}}\int_0^1 \widetilde{\underset{\sim}{L}}_a(\mu')\underset{\sim}{\Theta}(\mu')\frac{d\mu'}{\mu'+\mu} \tag{32a}$$

and, similarly, its adjoint

$$\underset{\sim}{H}_a(\mu) = \underset{\sim}{I} + \mu\underset{\sim}{H}_a(\mu)\,\underset{\sim}{C}\int_0^1 \widetilde{\underset{\sim}{H}}(\mu')\underset{\sim}{\Theta}(\mu')\frac{d\mu'}{\mu'+\mu} - \mu\underset{\sim}{L}_a(\mu)\,\underset{\sim}{B}\int_0^1 \widetilde{\underset{\sim}{L}}(\mu')\underset{\sim}{\Theta}(\mu')\frac{d\mu'}{\mu'+\mu}. \tag{32b}$$

Similar integral equations can be derived for $\underset{\sim}{L}$ and $\underset{\sim}{L}_a$ matrices. It can easily be shown, however, that these matrices are related by

$$\mu\underset{\sim}{H}(\mu)\,[\,\underset{\sim}{I} - \widetilde{\underset{\sim}{C}}\widetilde{\underset{\sim}{H}}_{ao}\,] = \underset{\sim}{L}(\mu)\,[\,\underset{\sim}{\Sigma}^{-1} - \mu\widetilde{\underset{\sim}{B}}\widetilde{\underset{\sim}{L}}_{ao}\,]\,, \tag{32c}$$

and

$$\mu\underset{\sim}{H}_a(\mu)\,[\,\underset{\sim}{I} - \underset{\sim}{C}\widetilde{\underset{\sim}{H}}_o\,] = \underset{\sim}{L}_a(\mu)\,[\,\underset{\sim}{\Sigma}^{-1} - \mu\underset{\sim}{B}\widetilde{\underset{\sim}{L}}_o\,]\,, \tag{32d}$$

where we have defined the moment of $\underset{\sim}{H}$ of order α by

$$\underset{\sim}{H}_\alpha = \int_0^1 \underset{\sim}{\Theta}(\mu)\,\underset{\sim}{H}(\mu)\,\mu^\alpha d\mu\,, \tag{33}$$

and similarly of other matrices.

From Eq. (21), we can also derive a singular integral equation of the $\underset{\sim}{H}$-matrix,

$$\widetilde{\underset{\sim}{H}}(\nu)\,\underset{\sim}{\lambda}(\nu) = \underset{\sim}{I} + \nu P\int_0^1 \widetilde{\underset{\sim}{H}}(\mu)\underset{\sim}{\Theta}(\mu)\underset{\sim}{Q}^*(\mu)\frac{d\mu}{\mu-\nu}\,\underset{\sim}{D}\,\underset{\sim}{\Gamma}(\nu), \quad \nu\epsilon(0,1)\,, \tag{34a}$$

and a discrete constraint on $\underset{\sim}{H}$,

$$[\,\underset{\sim}{I} + \nu_1\int_0^1 \widetilde{\underset{\sim}{H}}(\mu)\underset{\sim}{\Theta}(\mu)\underset{\sim}{Q}^*(\mu)\frac{d\mu}{\mu-\nu_1}\,\underset{\sim}{D}\,\underset{\sim}{\Gamma}(\nu_1)\,]\,\underset{\sim}{M}(\nu_1) = \underset{\sim}{0}\,. \tag{34b}$$

Considering the $\underset{\sim}{H}$-matrix as a function of the complex variable z, it can be shown that $\underset{\sim}{H}(z)$ (and other matrices) is analytic everywhere in the complex plane cut from - 1 to 0 along the real axis except at $z = -\nu_1$, where it has a simple pole.

Based on these matrices, Ishiguro[5] has shown that the eigenfunction $\underset{\sim}{F}(\xi,\mu)$, $\xi = \nu_1$ or $\epsilon(0,1)$, is orthogonal on the half-range $\mu\epsilon(0,1)$ to the related se; $\underset{\sim}{G}(\xi,\mu)$ in the sense that

$$\int_0^1 \widetilde{\underset{\sim}{G}}(\xi',\mu)\underset{\sim}{F}(\xi,\mu)\mu d\mu = 0, \quad \xi \neq \xi'; \quad \xi,\xi' = \nu_1 \quad \text{or} \quad \epsilon(0,1), \tag{35}$$

where

$$\underset{\sim}{G}(\xi,\mu) = \underset{\sim}{E}(\xi,\mu)\underset{\sim}{\Psi}(\mu)\widetilde{\underset{\sim}{D}}\underset{\sim}{\Omega}_a(\xi)\,\underset{\sim}{M}_a(\xi), \tag{36}$$

with the 4 x 2 matrix $\underset{\sim}{\Omega}_a(\xi)$ being defined by

/

$$\underset{\sim}{\Omega}_a(\xi) = \underset{\sim}{\Gamma}_a(\xi) - \xi\int_0^1 \underset{\sim}{\tilde{\Psi}}^*(\mu)\underset{\sim}{\Theta}(\mu)\underset{\sim}{Q}^*(\mu)\frac{d\mu}{\mu+\xi}\underset{\sim}{D}\underset{\sim}{\Gamma}_a(-\xi) \tag{37}$$

For the explicit eigenfunction $\underset{\sim}{\Phi}$ derived in RS, Eq (35) can be modified as

$$\int_0^1 \underset{\sim}{\tilde{\Theta}}(\xi',\mu)\underset{\sim}{\Phi}(\xi,\mu)\mu d\mu = 0, \xi \neq \xi'; \quad \xi,\xi' = \nu_1 \text{ or } \epsilon(0,1), \tag{38a}$$

where

$$\underset{\sim}{\Theta}(\xi,\mu) = [\xi\underset{\sim}{K}(\xi,\mu)\underset{\sim}{\Psi}(\mu)\underset{\sim}{\tilde{D}}\underset{\sim}{\Omega}_a(\xi) + \underset{\sim}{\delta}(\xi,\mu)\underset{\sim}{\lambda}_a(\xi)]\underset{\sim}{V}(\xi) \tag{38b}$$

Here the two vector $\underset{\sim}{V}(\xi)$ is defined by

$$\underset{\sim}{V}(\xi) = \begin{bmatrix} -\Lambda_{a12}(\xi) \\ \Lambda_{a11}(\xi) \end{bmatrix}, \quad \xi = \nu_1 \text{ or } \epsilon(\tfrac{1}{0},1), \tag{39a}$$

and

$$\underset{\sim}{V}_1^{(1)}(\xi) = \begin{bmatrix} N_{22}(\xi) \\ -N_{12}(\xi) \end{bmatrix}, \quad \underset{\sim}{V}_2^{(1)}(\xi) = \begin{bmatrix} -N_{21}(\xi) \\ N_{11}(\xi) \end{bmatrix}, \quad \xi\epsilon(0,\tfrac{1}{0}), \tag{39b,c}$$

where the functions $N_{\alpha\beta}(\xi)$ are given in RS For eventual applications, the following integral has been evaluated[5]:

$$\int_0^1 \underset{\sim}{\tilde{\Theta}}(\xi',\mu)\underset{\sim}{\Phi}(\xi,\mu)\mu d\mu = \frac{\xi\xi'}{\xi+\xi'}\underset{\sim}{V}(\xi')\underset{\sim}{\tilde{\Omega}}_a(\xi')\underset{\sim}{D}\underset{\sim}{\Omega}(\xi)\underset{\sim}{U}(\xi), \quad \xi,\xi' = \nu_1 \text{ or } \epsilon(0,1), \tag{40}$$

where the vector $\underset{\sim}{U}(\xi)$ is also given in RS and the 4 x 2 matrix $\underset{\sim}{\Omega}(\xi)$ is defined by

$$\underset{\sim}{\Omega}(\xi) = \underset{\sim}{\Gamma}(-\xi) - \xi\int_0^1 \underset{\sim}{\tilde{\Psi}}^*(\mu)\underset{\sim}{\Theta}(\mu)\underset{\sim}{Q}^*(\mu)\frac{d\mu}{\mu+\xi}\underset{\sim}{D}\underset{\sim}{\Gamma}(\xi) \tag{41}$$

(Comparing with Eq (37) the definition of $\underset{\sim}{\Omega}_a(\xi)$ is one (and only) exception to the convention of subscript a)

The Milne problem, Eq (3), can now be solved using Eqs (38) and (40) to give

$$A(\nu_1) = \frac{\nu_1}{2}\frac{1}{N(\nu_1)}\underset{\sim}{\tilde{V}}(\nu_1)\underset{\sim}{\tilde{\Omega}}_a(\nu_1)\underset{\sim}{D}\underset{\sim}{\Omega}(\nu_1)\underset{\sim}{U}(\nu_1), \tag{42a}$$

$$A_\alpha^{(1)}(\nu) = \frac{\nu\nu_1}{\nu+\nu_1}\frac{1}{N^{(1)}(\nu)}\underset{\sim}{\tilde{V}}_\alpha^{(1)}(\nu)\underset{\sim}{\tilde{\Omega}}_a(\nu)\underset{\sim}{D}\underset{\sim}{\Omega}(\nu_1)\underset{\sim}{U}(\nu_1), \tag{42b}$$

and

$$A^{(2)}(\nu) = \frac{\nu\nu_1}{\nu+\nu_1}\frac{1}{N^{(2)}(\nu)}\underset{\sim}{\tilde{V}}^{(2)}(\nu)\underset{\sim}{\tilde{\Omega}}_a(\nu)\underset{\sim}{D}\underset{\sim}{\Omega}_a(\nu_1)\underset{\sim}{U}(\nu_1). \tag{42c}$$

where

$$\underset{\sim}{U}(\nu_1) = \begin{vmatrix} -\Lambda_{1j}(\nu_1) \\ \Lambda_{11}(\nu_1) \end{vmatrix} \tag{43}$$

and the functions $N(\nu_i)$, $N^{(1)}(\nu)$, and $N^{(2)}(\nu)$ are the full range normalization integrals derived in RS

At this point, three mathematical problems arise[3], namely,

1) the half range completeness of the expansion in Eq (3),
2) the equivalence of the nonlinear Eqs (32) and the singular integral Eqs (34) (with related equations), and
3) the existence of a unique solution of either set of equations

We have not been able to resolve any of these problems However, since we believe that, mathematical proofs aside, the half range orthogonality theorem is valid, we proceed to study the effect of scattering anisotropy, relying on numerical checks of the above problems

NUMERICAL RESULTS

We consider the data set, given in Table I, used by Metcalf and Zweifel[10], and later also by Siewert and Ishiguro[18], to describe light water medium The elements of the matrices $\underset{\sim}{\Sigma}$ and $\underset{\sim}{C}$ are given by

$$\sigma = \frac{\sigma_1}{\sigma_{\prime}}, \quad c_{ij} = \frac{1}{2\sigma_i}\sigma_{ij}, \tag{44a,b}$$

and we define the matrix $\underset{\sim}{B}$ by

$$\underset{\sim}{B} = \underset{\sim}{C}\begin{vmatrix} p_1 & 0 \\ 0 & p_{\prime} \end{vmatrix}, \tag{45}$$

where we consider various combinations of p_1 and $p_{\prime}$ within the range $0 \leqslant p_i < 1$

The discrete eigenvalue ν_1 was calculated first using the analytical solution[5,2] of $\Lambda(z) = 0$ and then refined by iteration to at least twelve significant figures (Table II)

To calculate the fundamental matrices, $\underset{\sim}{H}$, $\underset{\sim}{H}_a$, $\underset{\sim}{L}$, and $\underset{\sim}{L}_a$, we used a set of rapidly converging equations[5], rather than Eqs (32), which can be derived using the discrete constraint Eq (34b) and similar equations[6] The matrices were solved at discrete points using Gauss quadrature sets to evaluate integrals All calculations were perfomed in double precision on an IBM 370/155 computer In Table III we report a sample calculation To check the accuracy of the converged solutions we used, for the $\underset{\sim}{H}$ matrix, (1) the discrete constraint Eq (34b), (2) $\det \underset{\sim}{H}^{-1}(-\nu_1) = 0$, and similar equations for other matrices, and (3) the moment relation of the matrices,

$$[\underset{\sim}{\Psi}_0 - \underset{\sim}{I}_4]\, \underset{\sim}{\widetilde{D}}\, [\underset{\sim}{\widetilde{\Phi}}_0 - \underset{\sim}{I}_4] = \underset{\sim}{\widetilde{D}}\, [\underset{\sim}{I}_4 - 2\underset{\sim}{Q}_0\underset{\sim}{\widetilde{D}}], \tag{46}$$

where

$$\underset{\sim}{\Psi}_0 = \underset{\sim}{\widetilde{D}} \int_0^1 \underset{\sim}{\widetilde{Q}}^*(\mu)\underset{\sim}{\Theta}(\mu)\underset{\sim}{\Psi}^*(\mu)d\mu, \tag{47a}$$

$$\underset{\sim}{\Phi}_0 = \underset{\sim}{D} \int_0^1 \underset{\sim}{\widetilde{Q}}^*(-\mu)\underset{\sim}{\Theta}(\mu)\underset{\sim}{\Phi}^*(\mu)d\mu, \tag{47b}$$

$$\underset{\sim}{Q}_0 = \begin{bmatrix} \underset{\sim}{\Sigma}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \frac{1}{3}\underset{\sim}{\Sigma}^{-1} \end{bmatrix}, \tag{47c}$$

and $\underset{\sim}{I}_4$ is the 4 x 4 unit matrix. In all these checks we found an accuracy of at least twelve significant figures (after thirteen iterations). We also calculated Eq. (34a) at various points (not equal to nodal points) by regularizing the singular integral and analytically evaluating the principal-value integral and, further, compared moments of order up to ten of each side of Eq. (34a), namely, writing the equation symbolically as $\underset{\sim}{L} = \underset{\sim}{R}$, we calculated

$$\int_0^1 \underset{\sim}{L}\nu^{\alpha}d\nu = \int_0^1 \underset{\sim}{R}\nu^{\alpha}d\nu, \quad \alpha = 0,1, \ldots, 10,$$

to confirm the equivalence of the two sets of equations.

Having obtained the fundamental matrices, the expansion coefficients, Eqs. (42), can be evaluated easily at arbitrary points (Table IV).

The essential question (related to the half-range completeness) is how accurately the obtained coefficients satisfy Eq. (3). For this, we employed the same checks as for the singular integral Eq. (34a), namely, the pointwise check and the comparison of moments. We report some of the results in Table V.

Finally we repeated calculations using higher order quadratures and confirmed that the results were unchanged and that the accuracy of various checks increased.

Based on these checks, we believe that our solutions are indeed valid and that all values reported in the following tables are accurate within the usual round-off convention.

We would now like to define and make some comments on the quantities reported in the tables.

Table I. The neutron energies considered are

group 1: $0 < E < 0.0253$ eV
group 2: $0.0253 < E < 0.532$ eV.

The set represents pure light water (Set 1 of Metcalf and Zweifel[10]).

Table IV The values of the coefficients at $\nu = 0$, $1/\sigma$, and 1 are evaluated taking the limit analytically in Eqs (42)

Table V The Δ in the table is the absolute value of the difference between the two sides divided by the left hand side, $\Delta(\mu)$ refering to the pointwise check at μ and $\Delta(\alpha)$ to the comparison of moments of order α. The notation $Q = N_1 + N_2$ refers to the use of two quadrature sets, a N_1-point set in $(0, 1/\sigma)$ and a N_2 point set in $(1/\sigma, 1)$ The accuracy for group 2 is of the same order.

Table VIII The total flux is defined by

$$\underset{\sim}{\phi}(x) = \int_{-1}^{1} \underset{\sim}{\Psi}(x,\mu)d\mu$$

and the asymptotic flux, $\underset{\sim}{\phi}_{asy}(x)$, is the total flux corresponding to only the discrete terms.

Table IX The current is defined by

$$\underset{\sim}{J}(x) = \int_{-1}^{1} \underset{\sim}{\Psi}(x,\mu)\mu d\mu$$

Table X The extrapolated endpoint, z_0, is defined by $\underset{\sim}{\phi}_{asy}(-z_0) = \underset{\sim}{0}$ and is given by

$$z_0 = -\frac{\nu_0}{2} \ln [-A(\nu_1)]$$

Table XI. Although all values reported in Tables IV - IX are based on the normalization $A(-\nu_1) = 1$, numerical comparisons of various cases should be based on the same source condition We have calculated the coefficient $A(+\nu_1)$ for the half space with an incident flux $\underset{\sim}{\Psi}(0,\mu) = \underset{\sim}{F}, \mu\epsilon(0,1)$ The Q in the table is the ratio of $A(\nu_1)$ for the case of anisotropic scattering to that for isotropic scattering, Q_1 corresponding to $\underset{\sim}{F} = [1\ 0]^T$ and Q_2 to $\underset{\sim}{F} = [0\ 1]^T$.

CONCLUSIONS

We have investigated the effect of scattering anisotropy based on propositions that transfer functions can be represented by two-term series in Legendre polynomials and that the ratio of coefficients in the series is dependent only on the initial speed of neutrons ($b_{ij} / c_{ij} = p_j$) Although the second proposition may not be realistic in general, and though we have considered only one medium we believe we have gained some insight into the behavior of the neutron population when we consider two terms as compared to one term.

We also believe our results show that the half-range orthogonality theorem, though lacking some mathematical proofs to be considered established, can be used successfully, combined with numerical checks of the results, to solve half-space and slab problems, at least in some cases.

TABLE I

Data Set for Isotropic Scattering

σ_1	4.8822	σ_2	3.2343
σ_{11}	3.8180	σ_{12}	0.3524
σ_{21}	1.0326	σ_{22}	2.8669

TABLE II

Discrete Eigenvalues

p_1	0.0	0.0	0.0	0.0
p_2	0.0	0.1	0.3	0.5
ν_1	7.190978	7.287445	7.494821	7.[illegible]24217
p_1	0.1	0.1	0.3	
p_2	0.3	0.5	0.5	
ν_1	7.519677	7.749547	7.804291	

TABLE III

Fundamental Matrices, $p_1 = 0.1$, $p_2 = 0.3$

μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	1.0	0.0	0.0	1.0
0.1	1.196762	0.137860	0.034539	1.193600
0.2	1.326058	0.274349	0.065446	1.342018
0.3	1.431995	0.413361	0.095386	1.476304
0.4	1.523488	0.553615	0.124591	1.601823
0.5	1.604756	0.694044	0.153116	1.720848
0.6	1.678271	0.833876	0.180981	1.834609
0.7	1.745642	0.972563	0.208197	1.943867
0.8	1.807989	1.109719	0.234774	2.049144
0.9	1.866132	1.245072	0.260723	2.150817
1.0	1.920687	1.378434	0.286058	2.249178
μ	$H_{a11}(\mu)$	$H_{a12}(\mu)$	$H_{a21}(\mu)$	$H_{a22}(\mu)$
0.0	1.0	0.0	0.0	1.0
0.1	1.196771	0.046987	0.101324	1.193590
0.2	1.326070	0.093469	0.192050	1.342002
0.3	1.432001	0.140774	0.279986	1.476287
0.4	1.523478	0.188465	0.365811	1.601811
0.5	1.604721	0.236180	0.449685	1.720847
0.6	1.678199	0.283656	0.531661	1.834623
0.7	1.745523	0.330706	0.611770	1.943903
0.8	1.807813	0.377199	0.690043	2.049207
0.9	1.865887	0.423044	0.766511	2.150911
1.0	1.920365	0.468178	0.841212	2.249309
Matrix	Element 11	12	21	22
$\underset{\sim}{H}_o$	0.950316	0.303922	0.150217	1.695183
$\underset{\sim}{H}_{ao}$	0.950305	0.103447	0.441347	1.695204
$\underset{\sim}{L}_o$	0.198366	−0.152128	−0.078128	0.162992
$\underset{\sim}{L}_{ao}$	0.198366	−0.051757	−0.229638	0.162992

TABLE IV

Expansion Coefficients, $p_1 = 0.1$, $p_2 = 0.3$

ν	$A_1^{(1)}(\nu)$	$A_2^{(1)}(\nu)$	ν	$A^{(2)}(\nu)$
0.02	−0.0060679	−0.0143569	0.70	0.008293
0.10	−0.0049270	−0.0121060	0.76	0.012472
0.20	−0.0040180	−0.0101885	0.82	−0.009207
0.30	−0.0033289	−0.0086964	0.86	−0.211537
0.40	−0.0027294	−0.0074366	0.88	−0.383416
0.50	−0.0021195	−0.0063181	0.90	−0.269097
0.60	−0.0012826	−0.0053109	0.94	−0.094693
0.66	0.0000332	0.0049402	0.98	−0.037670

$A(\nu_1) = -0.823386$	$A_2^{(1)}(0) = -0.0152801$
$A_1^{(1)}(0) = -0.0065707$	$A_2^{(1)}(1/\sigma) = -0.0052221$
$A_1^{(1)}(1/\sigma) = 0.0004480$	$A^{(2)}(1/\sigma) = A^{(2)}(1) = 0$

TABLE V

Checks of the Boundary Condition, $p_1 = 0.1$, $p_2 = 0.3$

μ	Q = 20 + 20	20 + 40	40 + 40
	$\Delta(\mu)$*		
0.05	6.0 (-6)	8.3 (-7)	3.2 (-7)
0.10	6.1 (-6)	8.9 (-7)	3.4 (-7)
0.20	6.4 (-6)	1.0 (-6)	3.7 (-7)
0.30	6.7 (-6)	1.1 (-6)	4.2 (-7)
0.40	6.9 (-6)	1.3 (-6)	4.7 (-7)
0.50	7.0 (-6)	1.6 (-6)	5.4 (-7)
0.60	6.8 (-6)	1.9 (-6)	6.3 (-7)
0.70	6.1 (-6)	2.5 (-6)	7.5 (-7)
0.80	4.0 (-6)	3.6 (-6)	9.6 (-7)
0.90	2.2 (-6)	6.8 (-6)	1.4 (-6)
0.95	1.1 (-5)	1.3 (-5)	2.2 (-6)
0.99	4.7 (-5)	5.5 (-5)	7.6 (-6)
α	$\Delta(\alpha)$*		
0	5.4 (-6)	1.5 (-6)	5.9 (-7)
1	4.3 (-6)	1.7 (-6)	7.1 (-7)
2	3.7 (-6)	1.5 (-6)	7.5 (-7)
3	3.2 (-6)	1.1 (-6)	7.4 (-7)
4	3.0 (-6)	4.3 (-7)	7.1 (-7)
5	2.9 (-6)	3.3 (-7)	6.6 (-7)
6	3.0 (-6)	1.2 (-6)	5.9 (-7)
7	3.3 (-6)	2.2 (-6)	5.0 (-7)
8	3.7 (-6)	3.2 (-6)	4.0 (-7)
9	4.1 (-6)	4.4 (-6)	2.9 (-7)
10	4.7 (-6)	5.6 (-6)	1.8 (-7)

* The number $A \times 10^B$ is written as A(B)

TABLE VI
Angular Fluxes inside the Medium

μ	$\rho_1 = 0.0$	$\rho_2 = 0.0$	$\rho_1 = 0.1$	$\rho_2 = 0.3$
	$\Psi_1(x,\mu)$	$\Psi_2(x,\mu)$	$\Psi_1(x,\mu)$	$\Psi_2(x,\mu)$
		x = 1		
−1.0	0.021322	0.074932	0.020777	0.074308
−0.8	0.019960	0.068739	0.019422	0.067982
−0.6	0.018609	0.062726	0.018076	0.061826
−0.4	0.017263	0.056860	0.016734	0.055811
−0.2	0.015916	0.051106	0.015386	0.049902
0.0	0.014555	0.045422	0.014021	0.044061
0.2	0.013156	0.039644	0.012614	0.038122
0.4	0.011642	0.033514	0.011084	0.031825
0.6	0.010074	0.028274	0.009491	0.026447
0.8	0.008529	0.024216	0.007911	0.022282
0.9	0.007732	0.022554	0.007092	0.020576
		x = 2		
−1.0	0.031256	0.105182	0.030043	0.102593
−0.8	0.029813	0.098553	0.028614	0.095851
−0.6	0.028390	0.092159	0.027201	0.089332
−0.4	0.026980	0.085966	0.025799	0.083003
−0.2	0.025577	0.079942	0.024400	0.076837
0.0	0.024169	0.074057	0.022993	0.070802
0.2	0.022743	0.068275	0.021562	0.064867
0.4	0.021271	0.062506	0.020078	0.058941
0.6	0.019701	0.056827	0.018486	0.053109
0.8	0.017960	0.051588	0.016709	0.047728
0.9	0.016980	0.049195	0.015702	0.045271
		x = 5		
−1.0	0.065529	0.210509	0.061745	0.200288
−0.8	0.063668	0.201724	0.059934	0.191516
−0.6	0.061851	0.193363	0.058162	0.183139
−0.4	0.060073	0.185380	0.056422	0.175116
−0.2	0.058324	0.177733	0.054706	0.167409
0.0	0.056595	0.170385	0.053003	0.159984
0.2	0.054870	0.163301	0.051297	0.152808
0.4	0.053124	0.156447	0.049562	0.145851
0.6	0.051309	0.149793	0.047745	0.139083
0.8	0.049318	0.143316	0.045735	0.132486
0.9	0.048188	0.140147	0.044586	0.129254
		x = 10		
−1.0	0.152418	0.480657	0.140108	0.444900
−0.8	0.149082	0.464519	0.136977	0.429331
−0.6	0.145858	0.449306	0.133941	0.414605
−0.4	0.142734	0.434932	0.130993	0.400647
−0.2	0.139701	0.421320	0.128120	0.387388
0.0	0.136744	0.408403	0.125311	0.374769
0.2	0.133845	0.396117	0.122546	0.362734
0.4	0.130977	0.384408	0.119798	0.351233
0.6	0.128091	0.373225	0.117013	0.340221
0.8	0.125077	0.362522	0.114076	0.329656
0.9	0.123452	0.357337	0.112479	0.324528

TABLE VII

Exit Angular Fluxes

μ	$\rho_1 = 0.0$ $\rho_2 = 0.0$	0.0 0.1	0.0 0.3	0.0 0.5	0.1 0.3	0.1 0.5	0.3 0.5
				$\Psi_1(0, -\mu)$			
0.05	0.004841	0.004885	0.004980	0.005084	0.004903	0.005006	0.004848
0.10	0.005277	0.005323	0.005424	0.005534	0.005342	0.005451	0.005283
0.20	0.006066	0.006116	0.006224	0.006343	0.006135	0.006253	0.006070
0.30	0.006803	0.006857	0.006971	0.007097	0.006876	0.007002	0.006806
0.40	0.007515	0.007570	0.007690	0.007822	0.007590	0.007721	0.007515
0.50	0.008210	0.008268	0.008391	0.008528	0.008286	0.008422	0.008207
0.60	0.008896	0.008954	0.009081	0.009221	0.008972	0.009112	0.008887
0.70	0.009574	0.009634	0.009763	0.009907	0.009651	0.009793	0.00956[illegible]
0.80	0.010249	0.010309	0.010440	0.010586	0.010324	0.010470	0.010230
0.90	0.010921	0.010982	0.011115	0.011263	0.010995	0.011142	0.010895
1.00	0.011592	0.011654	0.011787	0.011937	0.011664	0.011813	0.011559

μ	$\rho_1 = 0.0$ $\rho_2 = 0.0$	0.0 0.1	0.0 0.3	0.0 0.5	0.1 0.3	0.1 0.5	0.3 0.5
				$\Psi_2(0, -\mu)$			
0.05	0.017021	0.017295	0.017876	0.018511	0.017435	0.018057	0.017149
0.10	0.018736	0.019039	0.019685	0.020388	0.019195	0.019885	0.018380
0.20	0.021900	0.022259	0.023024	0.023858	0.022446	0.023263	0.022074
0.30	0.024920	0.025333	0.026213	0.027172	0.025550	0.026488	0.025124
0.40	0.027879	0.028346	0.029338	0.030420	0.028591	0.029650	0.028112
0.50	0.030813	0.031332	0.032436	0.033639	0.031605	0.032783	0.031072
0.60	0.033739	0.034311	0.035526	0.036849	0.034611	0.035907	0.034024
0.70	0.036672	0.037295	0.038620	0.040064	0.037621	0.039035	0.036979
0.80	0.039620	0.040294	0.041729	0.043292	0.040645	0.042176	0.039947
0.90	0.042590	0.043316	0.044860	0.046542	0.043690	0.045337	0.042932
1.00	0.045588	0.046366	0.048018	0.049818	0.046762	0.048525	0.045943

TABLE VIII

Total Fluxes

x	$\rho_1 = 0.0$ $\rho_2 = 0.0$	0.0 0.1	0.0 0.3	0.0 0.5	0.1 0.3	0.1 0.5	0.3 0.5
				$\phi_1(x)$			
0.0	0.008151	0.008206	0.008326	0.008459	0.008223	0.008355	0.008144
0.5	0.020050	0.020067	0.020108	0.020163	0.019624	0.019580	0.018718
1.0	0.030107	0.030042	0.029912	0.029788	0.029115	0.028999	0.027427
2.0	0.049828	0.049546	0.048968	0.048371	0.047591	0.047019	0.044323
5.0	0.115069	0.113824	0.111260	0.108592	0.108008	0.105441	0.099152
10.0	0.275830	0.270950	0.261057	0.250981	0.253032	0.243337	0.228099
				$\phi_{1\,asy}(x) / \phi_1(x)$			
0.0	1.50131	1.51559	1.54545	1.57714	1.52536	1.55648	1.51507
0.5	1.07179	1.07530	1.08279	1.09094	1.07975	1.08785	1.08130
1.0	1.02452	1.02589	1.02885	1.03211	1.02790	1.03116	1.02908
2.0	1.00449	1.00477	1.00538	1.00607	1.00526	1.00595	1.00567
5.0	1.00007	1.00007	1.00008	1.00009	1.00008	1.00009	1.00009
10.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
				$\phi_2(x)$			
0.0	0.030729	0.031246	0.032345	0.033542	0.031515	0.032657	0.030980
0.5	0.063085	0.063345	0.063910	0.064550	0.062185	0.062819	0.059367
1.0	0.091643	0.091577	0.091464	0.091386	0.088929	0.088870	0.083853
2.0	0.149148	0.148375	0.146789	0.145149	0.142610	0.141046	0.132861
5.0	0.342995	0.339356	0.331851	0.324030	0.322070	0.314560	0.295638
10.0	0.822133	0.807752	0.778580	0.748840	0.754466	0.725866	0.680045
				$\phi_{2\,asy}(x) / \phi_2(x)$			
0.0	1.18692	1.18666	1.18645	1.18666	1.18679	1.18676	1.18740
0.5	1.01529	1.01550	1.01604	1.01672	1.01597	1.01659	1.01644
1.0	1.00321	1.00330	1.00350	1.00377	1.00343	1.00367	1.00352
2.0	1.00023	1.00025	1.00028	1.00033	1.00026	1.00030	1.00024
5.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
10.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

TABLE IX

Currents

x	p_1 = 0.0 p_2 = 0.0	0.0 0.1	0.0 0.3	0.0 0.5	0.1 0.3	0.1 0.5	0.3 0.5
	$-J_1(x)$						
0.0	0.004664	0.004693	0.004756	0.004827	0.004701	0.004771	0.004657
0.5	0.004447	0.004461	0.004493	0.004529	0.004459	0.004495	0.004424
1.0	0.004362	0.004367	0.004380	0.004396	0.004356	0.004372	0.004322
2.0	0.004407	0.004402	0.004392	0.004385	0.004375	0.004368	0.004333
5.0	0.005391	0.005358	0.005290	0.005225	0.005267	0.005203	0.005156
10.0	0.009411	0.009257	0.008951	0.008650	0.008889	0.008593	0.008473
	$-J_2(x)$						
0.0	0.017852	0.018154	0.018797	0.019498	0.018311	0.018998	0.017998
0.5	0.018251	0.018570	0.019247	0.019984	0.018736	0.019458	0.018406
1.0	0.018638	0.018966	0.019663	0.020421	0.019133	0.019876	0.018787
2.0	0.019541	0.019876	0.020587	0.021362	0.020025	0.020784	0.019629
5.0	0.024316	0.024629	0.025295	0.026020	0.024576	0.025289	0.023830
10.0	0.042467	0.042578	0.042823	0.043106	0.041503	0.041795	0.039185

TABLE X

Extrapolated Endpoints

p_1	0.0	0.0	0.0	0.0	0.1	0.1	0.3
p_2	0.0	0.1	0.3	0.5	0.3	0.5	0.5
z_0	0.665826	0.685091	0.727570	0.776237	0.730648	0.779354	0.786477

TABLE XI

Source Normalization Constants

p_1	0.0	0.0	0.0	0.0	0.1	0.1	0.3
p_2	0.0	0.1	0.3	0.5	0.3	0.5	0.5
Q_1	1.0	0.981801	0.945395	0.908966	0.988185	0.950117	1.041100
Q_2	1.0	0.999785	0.999179	0.998303	1.021450	1.020562	1.068313

RESUMO

O problema de Milne é solucionado utilizando-se a teoria de transporte de neutrons para espalhamento linearmente anisotrópico em termos das matrizes $\underset{\sim}{H}$ utilizando-se relações de ortogonalidade de autofunções em semi-intervalos.

O efeito de espalhamento anisotrópico é estudado numericamente utilizando o conjunto de dados de espalhamento isotrópico para água leve de Metcalf e Zweifel e considerando a parte anisotrópica da matriz de transferência dada por $\underset{\sim}{B} = \underset{\sim}{C}\,\underset{\sim}{P}$ onde $\underset{\sim}{C}$ é a parte isotrópica e $\underset{\sim}{P}$ uma matriz 2 x 2 de elementos $\rho_j\,\delta_{ij}$

RÉSUMÉ

Le problème de Milne est résolu dans la théorie de transport des neutrons à deux groupes pour la ralentissement linéaire anisotrope en termes des matrices $\underset{\sim}{H}$ en utilisant les relations d'orthogonalité à demi portée entre fonctions propres.

L'effet de ralentissement anisotrope est étudié nuomeriquement eu se servant des données de la ralentissement isotrope de Metcalf et Zweifel pour l'eau légère et en considerant la partie anisotrope de la matrice de transfert donnée par $\underset{\sim}{B} = \underset{\sim}{C}\,\underset{\sim}{P}$ ou $\underset{\sim}{C}$ est la partie isotrope et $\underset{\sim}{P}$ une matrice 2 x 2 d'éléments $\rho_j\,\delta_{ij}$

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