MATHEMATICAL FOUNDATIONS OF TRANSPORT THEORY

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ABSTRACT

Among various methods of solution of the linear transport equation, Case's method of singular eigenfunction expansion is considered the most elegant exact analytical solution. Although the technique has been applied by many researchers to various problems and the solution is amenable to numerical evaluation to a high degree of accuracy, it requires rather unfamiliar mathematics.

In this report fundamental concepts and theorems of the required mathematics are presented.

A short review of the theory of functions of a complex variable is followed by the definition of the Cauchy principal-value integral The theory of singular integral equations of Cauchy-type kernel and related Riemann boundary-value problems for a function are summarized. As an application of the developed theory, analytical solutions of a class of transcendental equations are found. Further, systems of singular integral equations and the matrix Riemann problems, as required in the multi-group model of transport theory, are discussed

Preâmbulo

Este relatório contém as notas de aula sobre as Bases Matemáticas da Teoria de Transporte que o Professor Ernest E. Burniston da Universidade Estadual da Carolina do Norte, EUA, compilou durante a sua estadia na Coordenadoria de Engenharia Nuclear do Instituto de Energia Atômica no mês de maio de 1975.

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CHAPTER I

Review of Complex Variable Theory

In this chapter we give a brief summary of the complex variable theory required for the methods described in the following chapters. Proofs of the results which we will quote may be found in any standard text, such as L.V. Ahlfors[1], and E.T. Copson[2]. To establish our terminology we first give some definitions relating to the complex plane.

A **neighborhood** of a point z_0 is the set of points z satisfying the relation

$$|z - z_0| < \epsilon,$$

where ϵ is a positive constant.

Let S denote a set of points in the complex plane. A point z_0 of S is an **interior point** of S if there exists a neighborhood of z_0 containing only points of S. If a set contains only interior points, then the set is said to be **open**.

We adopt the simple minded approach to connectivity in that we say a set S is connected if each pair of points of S can be joined by a polygonal arc which consists only of points of S.

A **domain** will be an open, connected set.

If to each point z of a set S, we are given a rule, which we denote by f, which associates with that point a unique complex number f(z) say, then we say f **determines a function*** of a complex variable on S. In some texts, this is defined as a **single-valued function**. If we may associate two or more complex numbers with the point z we say that f **determines a multivalued function** on S. A function is differentiable at a point z, if

$$f'(z) = \lim_{|h| \to 0} \frac{f(z+h) - f(z)}{h}$$

exists independently of the path along which $|h| \to 0$. f is said to be **analytic** at the point z if it is differentiable in a neighborhood of z. A function is analytic on a set S if it is analytic at each point of S.

Consequently, we should think of analytic functions being defined on domains. A point at which f is not analytic is said to be a **singular-point** or a **singularity** of f.

An arc or contour is said to be **smooth** if(1) it is simple, i.e., does not intersect itself, and(2) it possesses a continuously turning tangent. Thus if it is given parametrically by

$$x = x(t),\ y = y(t),\ t_a \leq t \leq t_b$$

then x(t) and y(t) are continuously differentiable functions satisfying

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 > 0.$$

We may now state, without proofs, some theorems which we will need. **CAUCHY'S THEOREM:** Let C denote a smooth contour and D its interior. If f is analytic in D and continuous on D + C, then

$$\int_C f(z)dz = 0$$

CAUCHY'S INTEGRAL FORMULAE: Let D^+ denote the interior of smooth countor C, and D^+ the exterior. If f is analytic in D^+ and continuous on $D^+ + C$, then

$$\int_C \frac{f(t)}{t-z}\, dt = 2\pi i\, f(z),\ z \in D^+,$$

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* At this stage we are making the distinction between a function f and its value f(z). Later, however, we shall speak of "the function f(z)".

$$\int_C \frac{f(t)}{t-z}\,dt = 0, \quad z \in D^-$$

If f is analytic in D^- and continuous in $D^- + C$, then

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt = f(\infty), \qquad z \in D^+,$$

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt = f(\infty) - f(z), \quad z \in D^-$$

A function that is analytic in every finite domain is called an **integral function,** e g polynomials are integral functions. An important theorem concerning integral functions is **Lioville's Theorem: If f(z) is an integral function satisfying the inequality that**

$$|f(z)| \leqslant M, \text{ for all } z,$$

where M is a constant, then f(z) is a constant, i e , the only bounded integral function is a constant An extension of this result is that if f(z) is an integral function satisfying

$$|f(z)| \leqslant M\,|z|^{\alpha}, \text{ for all } z$$

where α is a real, positive constant, then f(z) is a polynomial of degree $[\alpha]$, where $[\alpha]$ denotes the largest integer, which does not exceed α

TAYLOR'S THEOREM: If f(z) is analytic in a domain D containing the point z_0, then the infinite series representation

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}\,f'(z_0) + \frac{(z-z_0)^2}{2!}\,f''(z_0) + \ldots,$$

is valid in the largest neighborhood of z_0 contained in D

LARUENT'S THEOREM: If f(z) is analytic in the annulus $R_1 < |z - z_0| < R_2$, then f(z) has the representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}\,.$$

in that annulus. The second series is usually referred to as the **principal part** of f(z) at z_0

If at least one of b_n's is not zero, then we say that f has an **isolated singularity** at z_0 If the principal part terminates, i e., $b_n = 0$, $n = N+1, N+2, \ldots$, then f has a pole of order N at z_0, with residue b_1. If the principal part does not terminate the f has an **isolated essential singularity** at z_0 If all the b_n's are zero then either f is analytic at z_0 or has a **removable singularity**, i.e., a singularity which may be removed by suitably defining $f(z_0)$

IDENTITY THEOREM: Let f_1 and f_2 be two functions, analytic in a domain D If $f_1(z) = f_2(z)$ for all z on some arc within D, then $f_1(z) = f_2(z)$ for all z in D (There are more general versions of this theorem but the one given here is sufficient for our purposes).

MORERA'S THEOREM: This is the converse of Cauchy's Theorem. If f is continuous in a domain D and is such that

$$\int_C f(z)dz = 0,$$

for any contour in D, then f is analytic in D

These last two results may be used to establish two important analytic continuation results

ANALYTIC CONTINUATION THEOREM: Let D_1 and D_2 be two disjoint domains, whose boundaries intersect in an arc C. If f_1 is analytic in D_1 and continuous in $D_1 + C$, f_2 is analytic in D_2 and continuous in $D_2 + C$, while $f_1(z) = f_2(z)$ for all z on C then f_1 and f_2 are analytic continuations of each other, and define a unique analytic function in $D_1 + D_2 + C$.

SCHWARZ REFLECTION PRINCIPLE: Let C denote a part of the real axis, and f a function analytic for $y > 0$ (S^+) continuous onto C such that f (x) is real for all x on C. Then f can be analytically continued into $y < 0$ (S^-) by defining for $y < 0$,

$$f(z) = \overline{f(\bar{z})},$$

and so

$$g(z) = f(z),\ y > 0,$$

$$= \overline{f(\bar{z})},\ y < 0,$$

defines a unique analytic function in $S^+ + S^- + C$

ARGUMENT PRINCIPLE: Let f be analytic within and on a smooth contour C, except for at most a finite number of poles within C, (f is meromorphic within C). In addition let f have only a finite number of zeros within C and no zeros on C. Then if N is the number of zeros and P the number of poles within C, we have that

$$\frac{1}{2\pi}[\text{Arg } f(z)]_C = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P,$$

where multiple zeros and poles are counted as to their multiplicity. The notation $[\text{Arg } f(z)]_C$ denotes the change in the argument of f as the contour C is traversed one time in the positive sense.

We now turn our attention to the question of multivalued functions. The multivalued function which is basic to the study of this class, is the argument function. The argument of a complex number has infinitely many values which differ by multiples of 2π, i.e., if we set $z = re^{i\theta}$ then $\arg z = \theta + 2n\pi$, $\alpha < \theta < \alpha + 2\pi$ where α is a constant and $n = 0, \pm 1, \pm 2, \ldots$. Thus the logarithm function

$$\log z = \ln |z| + i \arg z,$$

is clearly multivalued, with its values differing by multiples of $2\pi i$. Let C be any contour enclosing the origin and z_0 a point on C. If we let z traverse C starting at z_0 and returning to z_0, then clearly if we compute logz continuously, starting with any particular initial value, as z traverses C, then the initial and final values of $\log z_0$ will differ by $2\pi i$. Consequently, to construct analytic functions from the log function we must restrict the domain so as not to include contours enclosing the origin. This may be done by removing all the points on the ray $re^{i\alpha}$, where $0 \leqslant r$ and α is an arbitrary constant, from the complex plane. It is now a straight forward matter to show that, for each given n, the functions

$$f_n(z) = \ln r + i(\theta + 2n\pi),\ n = 0, \pm 1, \pm 2 \ldots ,$$

are analytic in the domain $r > 0$, $\alpha < \theta < \alpha + 2\pi$. These analytic functions are called **branches** of the log function in the domain $r > 0$, $\alpha < \theta < \alpha + 2\pi$. The ray $re^{i\alpha}$, $r \geqslant 0$ is the **branch cut** for these branches with the origin and the point at infinity being **branch points**. The particular case of $\alpha = -\pi$ and $n = 0$ defines the **principal branch** of the log function which we will denote by Log z, i.e.

$$\text{Log}\, z = \ln |z| + i \arg z,\ |z| > 0,\ -\pi < \arg z < \pi.$$

We shall determine branches of other multivalued functions by means of the log function. For instance, for the square root function we can write

$$z^{1/2} = \exp(1/2 \log z)$$

$$= \exp\left(\frac{\ln |z|}{2} + \frac{i}{2}[\arg z + 2n\pi i]\right),$$

where $\alpha < \arg z < \alpha + 2\pi$, and $n = 0, \pm 1, \pm 2, \ldots$, thus,

$$z^{1/2} = r^{1/2} e^{i(\theta/2 + n\pi)},$$

$$= r^{1/2} e^{i\theta/2},\ n \text{ even},$$

$$= -r^{1/2} e^{i\theta/2},\ n \text{ odd}.$$

Therefore, for any given α, there are two branches of the square root function.

The principal branch of $z^{1/2}$, however, is determined by using the principle branch of the log function, i.e.,

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2},\ r > 0,\ -\pi < \theta < \pi$$

More complicated functions may be handled by reducing them to simple functions by means of transformations, compositions, etc. For example, consider $(z^2 - 1)^{1/2}$. In terms of logarithms we have

$$(z^2 - 1)^{1/2} = \exp\ \{ \tfrac{1}{2} \log(z^2 - 1) \}$$

$$= \exp\ \{ \tfrac{1}{2} \log(z - 1) + \tfrac{1}{2} \log(z + 1) \}$$

$$= \exp\ \{ \tfrac{1}{2} \ln|z - 1| + \tfrac{1}{2} \ln|z + 1| + i \arg(z - 1) + i \arg(z + 1) \}$$

On taking the principal branch of each of these log functions we find that the principal branch of $(z^2 - 1)^{1/2}$ can be expressed as

$$(z^2 - 1)^{1/2} = \sqrt{|z^2 - 1|}\ \exp\left[\frac{i}{2}(\phi_1 + \phi_2)\right], -\pi < \phi_1, \phi_2 < \pi,$$

where $\phi_1 = \arg\ (z - 1)$ and $\phi_2 = \arg(z + 1)$. It appears that the branch cut here is the union of the two branch cuts corresponding to the two log functions, i.e., the half line $x \leqslant 1$, $y = 0$. However, this is not the case. Let us consider the limiting values or $(x^2 - 1)^{1/2}$ on the real axis to the left of $z = -1$, i.e., for $x < -1$, $y = 0$.

$$\lim_{y \to 0^+} (z^2 - 1)^{1/2} = \sqrt{x^2 - 1}\ \exp\left[\frac{i}{2}(\pi + \pi)\right],$$

$$= -\sqrt{x^2 - 1}.$$

$$\lim_{y \to 0^-} (z^2 - 1)^{1/2} = \sqrt{x^2 - 1}\ \exp\left[\frac{i}{2}(-\pi - \pi)\right],$$

$$= -\sqrt{x^2 - 1}.$$

and so by the analytic continuation theorem $(z_2 - 1)^{1/2}$ is analytic for $x < -1$, and so the branch cut is $-1 \leqslant x \leqslant 1$, $y = 0$.

Another interesting example is the arcsin function. We pose the problem of determining all analytic functions (for some suitable domain) such that

$$\sin\omega = z.$$

Replacing $\sin\omega$ by its exponential form gives

$$e^{2i\omega} - 2ize^{i\omega} - 1 = 0,$$

which yields

$$e^{i\omega} = i(z \pm \sqrt{z^2 - 1}).$$

To make this equation meaningful we assign the branch cut $(-\infty - 1]$ and $[+1, +\infty)$, and so $e^{i\omega}$ may be either of the two analytical functions;

$$i(z + \sqrt{z^2 - 1}) \text{ or } i(z - \sqrt{z^2 - 1})$$

where $\sqrt{z^2 - 1}$ represents that branch of the square root function determined by assigning $\sqrt{-1}$ the value i. In other words

$$\sqrt{z^2 - 1} = \exp\left\{ \tfrac{1}{2} \log(z - 1) + \tfrac{1}{2} \operatorname{Log}(z + 1) \right\},$$

where

$$0 < \arg(z - 1) < 2\pi \text{ and } -\pi < \arg(z + 1) < \pi.$$

If we now make the observation neither $z + \sqrt{z^2 - 1}$ nor $z - \sqrt{z^2 - 1}$ can vanish in the specified domain, then it makes sense to take logarithms and so we may write

$$\omega = \frac{\pi}{2} - i \log(z + \sqrt{z^2 - 1}),$$

or

$$\omega = \frac{\pi}{2} - i \log(z - \sqrt{z^2 - 1}).$$

Before specifying a branch of the log function we make use of the fact

$$(z + \sqrt{z^2 - 1})(z - \sqrt{z^2 - 1}) = 1,$$

and so ω may be represented by

$$\omega = \frac{\pi}{2} \pm i \log(z + \sqrt{z^2 - 1}).$$

Now noting that the function $z+\sqrt{z^2+1}$ can never assume a negative real value for any z in our domain, we may specify the principal branch of the log function and finally deduce that all the analytic functions satisfying $\sin\omega = z$, may be represented by

$$\omega_k(z) = k\pi + (-1)^k \left\{ \frac{\pi}{2} - i\,\mathrm{Log}(z+\sqrt{z^2-1}) \right\} \quad k = 0, \pm 1, \pm 2,$$

CHAPTER II

Cauchy Integrals and Riemann Problems

Our purpose in this chapter will be to present the basic properties of Cauchy integrals and to give an introduction to the so called Riemann boundary value problem. We will make no attempt to be exhaustive and in a simpler vein we will not give a theorem-proof type of development. There are several excellent texts on these topics, N. I. Muskhelishvili[3] and F. D. Gakhov [4], for instance, which we recommend for those who require a more detailed study. We will include only those proofs which we feel useful in themselves in understanding the basic concepts. We begin by defining a Cauchy integral:

DEFINITION (2.1): Let C be a smoth arc or contour and $\phi(t)$ a given function, integrable on C, then the integral

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(t)dt}{t-z}, \tag{2.1}$$

is called a **Cauchy-integral** (sometimes referred to as a Cauchy-type integral) The function $\phi(t)$ is called the **density function** and $(t-z)^{-1}$ the **kernel.** A general theorem concerning integrals which define analytic functions is the following.

THEOREM (2.1): Let C be a smooth arc or contour Let f(t,z) be a continuous function of t on C, and also be analytic in some domain D for all t on C Then the function

$$F(z) = \int_C f(t,z)dt$$

is analytic in D. The theorem is easily proved by showing that F'(z) exists in D The points at which f(t,z) ceases to be analytic are singular points of F(z), and so Cauchy integrals are singular at each point of C. Thus if C is an arc, $\Phi(z)$ given by equation (2.1) will be analytic in the plane cut along C. If C is a contour, the integral of equation (2.1) will define in general, two functions, one analytic in the interior of C and one analytic in the exterior of C

Example: $\phi(t) = 1, C = \{ t : -1 \leqslant t \leqslant 1 \}$

$$\Phi(z) = \frac{1}{2\pi i} \int_{-1}^{+1} \frac{dt}{t-z} = \frac{1}{2\pi i} [\log(t-z)]_{-1}^{+1}$$

To compute this for a given z we must specify a branch of log(t − z) in the t-plane with a cut from the point t = z to the point at infinity, not passing through [− 1,1]. Thus we can now write

$$\Phi(z) = \frac{1}{2\pi i}\{\log(z-1) - \log(z+1)\}.$$

If we take a branch of log(t − z) with the branch always to the left (right) then $\Phi(z)$ will be analytic in the plane cut from −1 to 1.

Example:

$$\phi(t) = \frac{3}{t(t-3)}, C = \{t: |t| = 1\}.$$

Let D^+ denote the interior of the unit circle and D^- the exterior of the unit circle. Rewrite the integral as

$$\frac{1}{2\pi i}\int_{|t|=1} \frac{3}{t(t-3)} \cdot \frac{dt}{t-z} = \frac{1}{2\pi i}\int_{|t|=1} \left(\frac{1}{t-3} - \frac{1}{t}\right)\frac{dt}{t-z}.$$

Now, by the Cauchy integral theorem

$$\frac{1}{2\pi i}\int_{|t|=1} \frac{1}{t-3}\cdot\frac{dt}{t-z} = \frac{1}{z-3}, \; z \in D^+,$$

$$= 0, \; z \in D^-.$$

while

$$\frac{1}{2\pi i}\int_{|t|=1} \frac{1}{t}\cdot\frac{dt}{t-z} = 0, \quad z \in D^+,$$

$$= -\frac{1}{z}, \; z \in D^-$$

Thus we have that

$$\Phi(z) = \frac{1}{z-3}, \; z \in D^+,$$

$$= \frac{1}{z}, \; z \in D^-.$$

The largest class of functions for the density function $\phi(t)$ in equation (2.1), which admits a fairly simple treatment is the class of Hölder functions.

DEFINITION (2.2): Let C be a smooth arc or contour and $\phi(t)$ a given function defined on C. Then $\phi(t)$ is said to satisfy a **Hölder condition** on C if for every pair of points t_1 and t_2 on C

$$|\phi(t_1) - \phi(t_2)| \leq A|t_2 - t_1|^\lambda, \tag{2.2}$$

where A and λ are positive constants. The constant λ is called the **Hölder index**. For brevity, we say $\phi(t) \in H(\lambda)$ on C. If $\lambda = 1$, the condition (2.2) becomes the **Lipschitz condition**. Clearly if $\lambda > 1$, then $\phi(t)$ is a constant on C. We thus will assume that $0 < \lambda \leq 1$.

Example: Show that if $\phi_1(t) \in H(\lambda_1)$ and $\phi_2(t) \in H(\lambda_2)$ then

(a) $\phi_1(t) + \phi_2(t)$

(b) $\phi_1(t) \, . \, \phi_2(t)$

(c) $\dfrac{\phi_1(t)}{\phi_2(t)}$, $[\phi_2(t) \neq 0]$

are $H(\lambda)$ where $\lambda = \min(\lambda_1, \lambda_2)$.

Example: Show that

$$\phi(t) = \frac{1}{\ln t}, \quad 0 < t \leq \frac{1}{2}$$
$$= 0 \qquad t = 0$$

is continuous but not Hölder in $[0, \frac{1}{2}]$

Example: Show that $\phi(t) = |t|$ is Lipschitz but not differentiable in $[-1, 1]$.

DEFINITION (2.3): The Cauchy-principal value of

$$\int_a^b \frac{dx}{x-c}, \quad a < c < b,$$

is defined as

$$\lim_{\epsilon \to 0} \left[-\int_a^{c-\epsilon} \frac{dx}{c-x} + \int_{c+\epsilon}^b \frac{dx}{x-c} \right],$$

its value is, of course, given by

$$P \int_a^b \frac{dx}{x-c} = \ln\left(\frac{b-c}{c-a}\right), \tag{2.3}$$

where we have introduced the symbol P to denote the Cauchy principal value. We now consider the Cauchy principal value of

$$\int_a^b \frac{\phi(x)dx}{x-c}, \quad a < c < b.$$

THEOREM (2.2): If $\phi(x)$ satisfies a Hölder condition on (a,b), then

$\int_a^b \frac{\phi(x)dx}{x-c}$, $a < c < b$, exists as a Cauchy principal value.

PROOF: Formally we may write

$$\int_a^b \frac{\phi(x)dx}{x-c} = \int_a^b \frac{\phi(x)-\phi(c)}{x-c}\,dx + \phi(c)\int_a^b \frac{dx}{x-c}\,.$$

Now the first integral exists in the ordinary sense, since

$$|\phi(x) - \phi(c)| < A|x-c|^{\lambda},\ 0 < \lambda \leq 1,$$

and so

$$\left|\frac{\phi(x)-\phi(c)}{x-c}\right| < \frac{A}{|x-c|^{1-\lambda}}\,.$$

Consequently if we take the Cauchy principal value for the second integral we have

$$P\int_a^b \frac{\phi(x)\,dx}{x-c} = \int_a^b \frac{\phi(x)-\phi(c)}{x-c}\,dx + \phi(c)\ln\left(\frac{b-c}{c-a}\right),\ a < c < b, \qquad (2.4)$$

We now generalize this to a smooth arc C and so consider

$$P\int_C \frac{\phi(\tau)d\tau}{\tau - t},\ \text{t not an end point,}$$

where τ and t are points on C. We draw a circle of radius ϵ, center at t such that the circle intersects C at two points t_1 and t_2 only (this can always be done if ϵ is sufficiently small). Denote that part of C within the circle by C_ϵ.

DEFINITION (2.4): The Cauchy principal value of the integral

$$\int_C \frac{\phi(\tau)d\tau}{\tau - t},$$

is defined as

$$\lim_{\epsilon \to 0} \int_{C-C_\epsilon} \frac{\phi(\tau)d\tau}{\tau - t},$$

provided that this limit exists. In order to state a general theorem concerning such integrals we first consider the case of $\phi(\tau) = 1$ on C, i.e.

$$P \int_C \frac{d\tau}{\tau - t}.$$

Now the primitive here is $\log(\tau - t)$. So we must first give a rule for computing this function for τ on C. This can be done in a variety of ways, but a convenient one is to let $\log(\tau - t)$ be the value of a branch of $\log(z - t)$ on C, with a branch cut from $z = t$ to the point at infinity. For definiteness we will adopt the convention that the cut is to the right of t, not intersecting C, of course, see Figure 1.

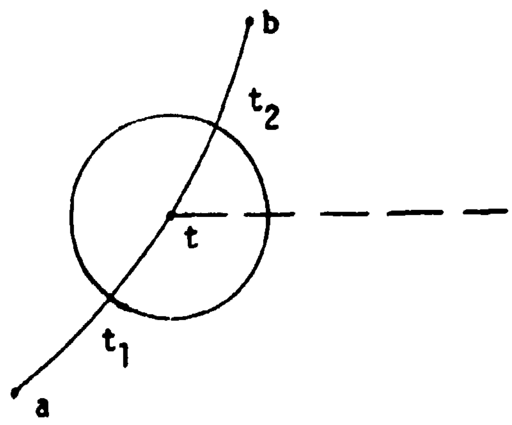


Figure 1

Thus having chosen a definite branch we can write

$$\int_{C - C_\epsilon} \frac{d\tau}{\tau - t} = [\log(\tau - t)]_a^{t_1} + [\log(\tau - t)]_{t_2}^{b},$$

$$= -\log\left(\frac{a - t}{b - t}\right) + \log\left(\frac{t_1 - t}{t_2 - t}\right).$$

where a and b are the end points of C where by $\log\left(\frac{a - t}{b - t}\right)$ we mean $\log(a - t) - \log(b - t)$. The term $\log\left(\frac{t_1 - t}{t_2 - t}\right)$ is interpreted in the same way. We shall always use the notation that as C is traversed in the positive direction, the first or initial end point will be denoted by a and the second or final point by b. Now

$$\log\left(\frac{t_1 - t}{t_2 - t}\right) = \ln\left|\frac{t_1 - t}{t_2 - t}\right| + i(\theta_1 - \theta_2),$$

where $\theta_1 = \arg(t_1 - t)$ and $\theta_2 = \arg(t_2 - t)$. The first term here is zero as t_1 and t_2 are equidistant from t. As the curve at t is smooth it is also clear that

$$\lim_{\epsilon \to 0}(\theta_1 - \theta_2) = \pi$$

so that

$$P\int_C \frac{d\tau}{\tau - t} = -\log\left(\frac{a - t}{b - t}\right) + i\pi \qquad (2.5)$$

This result enables us to establish the following theorem:

THEOREM (2.3): If $\phi(\tau)$ satisfies a Hölder condition on a smooth arc C then the following principal value integral

$$\int_C \frac{\phi(\tau)\,d\tau}{\tau - t} \text{ exists, (t not an end point) and}$$

is given by

$$P\int_C \frac{\phi(\tau)\,d\tau}{\tau - t} = \int_C \frac{\phi(\tau) - \phi(t)}{\tau - t}\,d\tau + \phi(t)\left[-\log\left(\frac{a - t}{b - t}\right) + i\pi\right] \qquad (2.6)$$

where again $\log(\frac{a-t}{b-t})$ is interpreted as $\log(a-t) - \log(b-t)$.

Example: Indicate the simplifications in formulas (2.5) and (2.6) if C is a contour.

We now turn our attention to the limiting values of Cauchy integrals as z approaches points on the contour. We will need the following Lemma

LEMMA (2.4): If the density function $\phi(\tau)$ of a Cauchy integral satisfies a Hölder condition on a smooth arc C, and if the point t is not an end point, then the function

$$\Psi_t(z) = \int_C \frac{\phi(\tau) - \phi(t)}{\tau - z} d\tau,$$

is continuous at t from the left and from the right of C. In other words

$$\lim_{z \to t} \Psi_t(z) = \int_C \frac{\phi(\tau) - \phi(t)}{\tau - t} d\tau = \Psi_t(t),$$

as $z \to t$ along any path, where $\Psi_t(t)$ is continuous on C.

We omit the proof of this Lemma and refer the readers to N. I. Mushkelishvili[3], or F. D Gakhov[4] for the details. The importance of this lemma is in establishing the following basic theorem

THEOREM (2.5): Let C be a smooth closed contour and $\phi(\tau)$ a $H(\lambda)$ function on C, then the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\tau) d\tau}{\tau - z},$$

is continuous on C from the left and also from the right.

PROOF: As usual we denote the interior of C by D^+ and the exterior of C by D^-. We also denote the limit process as z approaches t along a path in D^+ by $z \to t^-$. Likewise if the path is in D^-, we write $z \to t^+$. Similarly we define

$$\Phi^+(t) = \lim_{z \to t^+} \Phi(z),$$

$$\Phi^-(t) = \lim_{z \to t^-} \Phi(z).$$

We shall also use the following results, which we derived earlier,

$$\frac{1}{2\pi i}\int_c \frac{dt}{t-z} = 1,\ z \in D^+,$$

$$= 0,\ z \in D^-,$$

while

$$\frac{P}{2\pi i}\int_c \frac{d\tau}{\tau - t} = ½,\ t \in C.$$

Consider now the function

$$\Psi_t(z) = \frac{1}{2\pi i}\int_c \frac{\phi(\tau)-\phi(t)}{\tau - z}d\tau.$$

On writing

$$\Psi_t(z) = \frac{1}{2\pi i}\int_c \frac{\phi(\tau)d\tau}{\tau - z} - \frac{\phi(t)}{2\pi i}\int_c \frac{d\tau}{\tau - z},$$

we have

$$\Psi_t^+(t) = \lim_{z\to t^+} \Psi_t(z) = \phi^+(t) - \phi(t),$$

and

$$\Psi_t^-(t) = \lim_{z\to t^-} \Psi_t(z) = \Psi_t(z) = \phi^-(t),$$

and in addition

$$\Psi_t(t) = \frac{P}{2\pi i}\int_c \frac{\phi(\tau)\,d\tau}{\tau - t} - \frac{\phi(t)}{2}.$$

However by Lemma (2.4)

$$\Psi_t^+(t) = \Psi_t^-(t) = \Psi_t(t),$$

and so we have

$$\Phi^+(t) = \tfrac{1}{2}\,\phi(t) + \frac{P}{2\pi i}\int_C \frac{\phi(\tau)d\tau}{\tau - t} \qquad (2.7a)$$

$$\Phi^-(t) = -\tfrac{1}{2}\,\phi(t) + \frac{P}{2\pi i}\int_C \frac{\phi(\tau)d\tau}{\tau - t} \qquad (2.7b)$$

Formulas (2.7) are generally referred to as the Plemelj formulas, although the name Sokhotski is also used.

THEOREM (2.6): If C is a smooth arc and $\phi(\tau)$ is Hölder on C, such that $\phi(a) = \phi(b) = 0$, the formulas (2.7) still apply.

COROLLARY (2.7): If $\phi(a)$ and $\phi(b)$ are both nonzero, then the formulas (2.7) still hold with exception of these ends.

Notice that an equivalent form of these formulas, which we will be using is

$$\Phi^+(t) - \Phi^-(t) = \phi(t) \qquad (2.8a)$$

$$\Phi^+(t) + \Phi^-(t) = \frac{1}{\pi i}\int_C \frac{\phi(\tau)d\tau}{\tau - t} \qquad (2.8b)$$

Consider now the behavior of a Cauchy integral near an end point. As before we suppose C is a smooth arc, $\phi(t)$ a Hölder function on C, including the end points. Now if $\phi(a) \neq 0$ we have

$$\Phi(z) = \frac{1}{2\pi i}\int_C \frac{\phi(t)dt}{t - z}$$

$$= \frac{\phi(a)}{2\pi i} \int_C \frac{dt}{t-z} + \frac{1}{2\pi i} \int_C \frac{\phi(t) - \phi(a)}{t-z} dt,$$

$$= \frac{\phi(a)}{2\pi i} \log(\frac{z-b}{z-a}) - \Psi_a(z),$$

where by log $(\frac{z-b}{z-a})$ we mean $\log(z-b) - \log(z-a)$ and

$$\Psi_a(z) = \frac{1}{2\pi i} \int_C \frac{\phi(t) - \phi(a)}{t-z} dt.$$

Now, the density function of this last Cauchy integral vanishes at $t = a$ and so by Theorem (2 6) the Cauchy integral is continuous at a and hence bounded. For the multi-valued log function we mean a branch analytic in the plane cut along C. Thus near $z = a$ we have

$$\Phi(z) = -\frac{\phi(a)}{2\pi i} \log(z-a) + \Phi^*(z) \qquad (2.9a)$$

where $\Phi^*(z)$ is analytic near a in the cut plane and continuous on C at a. We can effect the splitting off of the term $\log(z-b)$ taking a branch of $\log(z-a)$ analytic in the plane cut along C + C′, where C′ is a line joining $z = b$ and the point at infinity, not intersecting C, and a branch of $\log(z-b)$ analytic in the plane cut along C′

Note that while the branch of $\log(z-a)$ may be chosen arbitrarily the branch of $\log(z-b)$ must be chosen so that log $(\frac{z-b}{z-a})$is the original log function. The corresponding formula for $z = b$;

$$\Phi(z) = \frac{\phi(b)}{2\pi i} \log(z-b) + \Phi^{**}(z) \qquad (2.9b)$$

where $\Phi^{**}(z)$ is analytic near b in the cut plane.

Example: Determine the end point behavior when

$$C = \{t : 0 \leqslant t \leqslant 1\} \text{ near } z = 0, \text{ if}$$

(i) $\phi(t) = 1$, (ii) $\phi(t) = t$, (iii) $\phi(t) = \ln t$ and (iv) $\phi(t) = t^\alpha$, $-1 < \alpha < 0$.

Now, we come to our development of the Riemann problem for arcs. We begin by establishing some definitions and terminology (We will generally adhere to that used in N. I. Muskhelishvili's book[3]) As before we will let C denote a smooth arc and S will denote the plane cut along C.

DEFINITION (2.5): A function $\Phi(z)$ is said to be **sectionally analytic** in S if (i) it is analytic in S, except possibly at the point at infinity, (ii) it is continuous on C from the left and from the right, with the possible exception of the end points and (iii) if near the end points it satisfies the inequality

$$|\Phi(z)| < \frac{M}{|z-c|^{\alpha}}, \quad 0 \leqslant \alpha < 1,$$

where M is a constant, and c denotes either end.

Now let $\phi(t)$ be defined on C, then we have the following definitions:

DEFINITION (2.6): We say $\phi(t)$ is Hölder on C only if it is Hölder at each point of C, including the end points.

DEFINITION (2.7): $\phi(t)$ is Hölder at each point of C except possibly at the end points where it is such that we can write

$$(t-c)^{\alpha}\phi(t) = \phi^*(t), \quad 0 \leqslant \alpha < 1,$$

where $\phi^*(t)$ is Hölder on C, then $\phi(t)$ is found to be of the class H^* on C.

DEFINITION (2.8): If $\phi(t)$ is H^* on C for arbitrarily small $\alpha = \epsilon > 0$, i.e.

$$(t-c)^{\epsilon}\,\phi(t),$$

is Hölder on C for every positive ϵ, then $\phi(t)$ is said to be of the class H_ϵ^* on C.

Example: $(t-c)^{i\beta}$, β real, and $\log(t-c)$ are H_ϵ^* functions.

DEFINITION (2.9): The **Riemann problem** for a smooth arc C will be to determine a sectionally analytic function $\Phi(z)$ in S; of finite degree at infinity whose boundary values satisfy

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in C, \qquad (2.10)$$

where G(t) and g(t) are H(λ) functions on C, and G(t) is non-vanishing on C. In general we exclude the end points in equation (2.10). The function G(t) is referred to as the **coefficient** of the Riemann problem.

We first consider the auxiliary problem

$$\Phi^+(t) = \Phi^-(t) + \phi(t),\ t \in C,$$

where $\phi(t)$ is Hölder on C. By the Plemelj formula (2.8a) a solution to this which vanishes at infinity is

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\tau)d\tau}{\tau - z}.$$

Clearly, if we require a solution of degree k at infinity then

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\tau)d\tau}{\tau - z} + P_k(z),$$

where $P_k(z)$ is a polynomial of degree k. In fact, we can easily show that this is the only solution. Consider $\Psi(z) = \Phi - \Phi_1(z)$ where $\Phi_1(z)$ is some other solution. Now $\Psi(z)$ will be analytic in the plane except possibly at the end points, which are now isolated singularities. However $\Psi(z)$ is sectionally analytic and hence satisfies an inequality of the kind

$$|\Psi(z)| < \frac{M}{|z-c|^\alpha},\ 0 \leqslant \alpha < 1,$$

near each end point and so these singularities must be removable singularities. Thus by Liouville's theorem $\Psi(z)$ is a polynomial of degree k.

Example: Show that if $\phi(t)$ is H^* on C the above solution is still valid. This uses the result that if $\phi(t)$ is H^* on C then near the end points

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(t)dt}{t-z}$$

$$= \pm \frac{e^{\pm\alpha\pi i}}{2i \sin\alpha\pi} \cdot \frac{\phi^*(c)}{(z-c)^\alpha} + \Phi^{**}(z),$$

where we take the upper signs for c = a, and the lower signs for c = b. In addition the function $\Phi^{**}(z)$ satisfies

$$|\Phi^{**}(z)| < \frac{C}{|z-c|^{\alpha_0}},\ \alpha_c < \alpha$$

near z = c (See N. I. Mushelishvili[3]).

We may now proceed to the homogeneous Riemann problem, namely

$$\Phi^+(t) = G(t)\,\Phi^-(t),\ t \in C \tag{2.11}$$

Now as G(t) does not vanish on C, we may choose a value of log G(t), which is continuous on C. Having chosen a particular value we set

$$\Gamma(z) = \frac{1}{2\pi i}\int_C \frac{\log G(t)}{t-z}\,dt\cdot \tag{2.12}$$

Now by the Plemelj formula

$$\Gamma^+(t) = \Gamma^-(t) + \log G(t),\ t \in C,$$

and so

$$e^{\Gamma^+(t)} = G(t)e^{\Gamma^-(t)},\ t \in C$$

which shows that exp $\Gamma(z)$ satisfies the homogeneous condition (2.10). However, it may not be sectionally analytic, because it may not have the correct end point behavior. $\Gamma(z)$ certainly has the correct end point behavior but exponentiating it may destroy it.

Now by equation (2.9), near an end point

$$\Gamma(z) = \mp\ \frac{\log G(c_k)}{2\pi i}\log(z - c_k) + \Gamma_z^*(z)$$

and so

$$e^{\Gamma(z)} = (z - c_k)^{\alpha_k + i\beta_k}e^{\Gamma_k^*(z)},\ k = 1,2 \tag{2.13a}$$

where

$$\alpha_k + i\beta_k = \mp\ \frac{\log G(c_k)}{2\pi i},\ k = 1,2 \tag{2.13b}$$

with the upper signs for $c_1 \equiv a$ and the lower signs for $c_2 \equiv b$. In addition $\Gamma_1^*(z)$ is analytis in S and bounded at $z = a$, while $\Gamma_2^*(z)$ is analytic in S and bounded at $z = b$. Thus if $\alpha_k \leqslant -1$ some modification of $\exp \Gamma(z)$ is necessary. We select integers λ_K so that

$$-1 < \alpha_k + \lambda_k \leqslant 0, \ k = 1,2,$$

and now examine

$$X(z) = (z-a)^{\lambda_1}(z-b)^{\lambda_2} e^{\Gamma(z)} \tag{2.14}$$

Clearly $X(z)$ is sectionally analytic in S, and the fact that the expression $(z-a)^{\lambda_1}(z-b)^{\lambda_2}$ multiplying $\exp \Gamma(z)$ is a rational function means that $X(z)$ satisfies the boundary condition (2.11). Consequently, $X(z)$ is a solution of the homogeneous Riemann problem, in addition however, it has, because of the right hand side of the inequality for λ_k i.e., $\alpha_k + \lambda_k \leqslant 0$, some further important properties. Taking limiting values on the cut gives

$$X^{\pm}(t) = (t-a)^{\lambda_1}(t-b)^{\lambda_2} \exp[\pm \tfrac{1}{2} \log G(t) + \Gamma(t)],$$

where

$$\Gamma(t) = \frac{P}{2\pi i} \int_c \frac{\log G(\tau)}{\tau - t} d\tau,$$

and so if we set

$$X(t) = (t-a)^{\lambda_1}(t-b)^{\lambda_2} \exp \Gamma(t),$$

then

$$X^+(t) = X(t)\sqrt{G(t)},$$

and

$$X^-(t) = \frac{X(t)}{\sqrt{G(t)}}.$$

Now from the manner in which the λ_k were chosen it is evident that $X(t)$ does not vanish on C, and so $X^{\pm}(t)$ do not vanish on C. As $X(z)$ does not vanish for $z \notin C$ (except perhaps at infinity) it follows that the solution $X(z)$ is nonvanishing in the finite plane. Such a solution is referred to as a canonical solution (clearly $cX(z)$ where c is a non-zero constant is also a canonical solution). A further property of a canonical solution is that every other solution can be expressed in terms of it.

Let $\Phi(z)$ be any other solution then we have

$$\Phi^+(t) = G(t)\,\Phi^-(t),\; t \in C,$$

and

$$X^+(t) = G(t)\,X^-(t),\; t \in C,$$

dividing the first of these equations by the second, which is permissible since $X^{\pm}(t)$ and $G(t)$ do not vanish, yields

$$\frac{\Phi^+(t)}{X^+(t)} = \frac{\Phi^-(t)}{X^-(t)},\; t \in C.$$

This implies however, that the function $\Phi(z)/X(z)$ is analytic in the entire plane and so by Liouville's theorem is a polynomial. We may now state the following important theorem:

THEOREM (2.8): If $X(z)$ is a canonical solution of the homogeneous Riemann problem (2.11) then any other solution $\Phi(z)$, of finite degree at infinity can be written as

$$\Phi(z) = X(z)P(z), \tag{2.15}$$

where $P(z)$ is a polynomial

Example: Show that $X(t)$ and $X^{\pm}(t)$ are H^* on C.

The non-homogeneous Riemann problem (2.10) is readily solved once a canonical solution of the corresponding homogeneous problem is known. We have on replacing $G(t)$ by $X^+(t)/X^-(t)$ that the equation (2.10) can be written as

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)},\; t \in C.$$

Thus, on using a familiar argument the solution of finite degree at infinity is seen to be

$$\Phi(z) = X(z)\left\{\frac{1}{2\pi i}\int_C \frac{g(t)}{X^+(t)}\frac{dt}{t-z} + P(z)\right\}.$$

where P(z) is a polynomial.

As an example of these results let us consider the problem

$$\Phi^{+}(t) = -\Phi^{-}(t) + g(t), \; t \in C,$$

where C is any smooth arc. As $G(t) = -1$, we take $\log G(t) = i\pi$ (any other value is permissible). Thus,

$$\Gamma(z) = \frac{1}{2\pi i} \int_C \frac{\log G(t)}{t-z}\, dt$$

$$= \frac{1}{2\pi i} \int_C \frac{\pi i}{t-z}\, dt$$

$$= \frac{1}{2} \log \left(\frac{z-b}{z-a}\right),$$

where $\log \left(\frac{z-b}{z-a}\right)$ means $\log(z-b) - \log(z-a)$.

Consequently,

$$e^{\Gamma(z)} = \sqrt{(z-b)/(z-a)}.$$

Now recall from equation (2.13b) that

$$\alpha_k + i\beta_k = \mp \frac{\log G(c_k)}{2\pi i}$$
$$= \mp \frac{1}{2}.$$

So from the inequality $-1 < \alpha_k + \lambda_k \leq 0$, it follows that

$$\lambda_1 = 0,$$

$$\lambda_2 = -1.$$

Hence a canonical solution is given by

$$X(z) = \frac{1}{(z-b)}\sqrt{\frac{z-b}{z-a}} = \frac{1}{\sqrt{(z-a)(z-b)}}.$$

Exercise: Complete the solution if $C = \{t: -a \leqslant t \leqslant a\}$.

CHAPTER III

THE TRANSCEDENTAL EQUATIONS $\tan\beta = \omega\beta$ and $\beta\tan\beta = \omega$.[5]

Consider the following Sturm-Liouville problem:

$$y'' + \lambda y = 0,$$

$$y(0) = 0$$

$$\omega y'(1) - y(1) = 0$$

where ω is a real constant, which can arise in a variety of ways, for example in solving partial differential equations using a variable-separable technique. Generally we are required to find all values of λ (eigen-values) which yield non-trivial real solutions (eigen-functions) satisfying both the differential equations and the boundary conditions. We usually begin by determining whether or not there are any negative eigen-values and for this we set $\lambda = -\alpha^2$. For this example it is a simple matter to show that non-trivial solutions exist only if α is a root of the equation

$$\tanh\alpha = \omega\alpha. \tag{3.1}$$

We will show later that this equation has real roots other than $\alpha = 0$, only if $0 < \omega < 1$. The reader may demonstrate this by sketching the graph of $\tanh\alpha$ and $\omega\alpha$. If we now examine the case $\lambda = 0$ we see that this is an eigen-value only if $\omega = 1$. Turning then to the determination of the positive eigen-values we set $\lambda = \beta^2$. In a straightforward manner we see that the positive eigen-values are the roots of

$$\tan\beta = \omega\beta. \tag{3.2}$$

We will show that this equation has infinitely many roots for all ω. Again we suggest that the reader demonstrate this by sketchig the graphs of $\tan\beta$ and $\omega\beta$.

Now equations (3.1) and (3.2) are transcendental equations as they involve transcendental functions of the argument. Alternatively we can say that they are transcendental equations because they are not algebraic equations. This requires, of course, the definition of an algebraic function, which is a function $f(x)$ say, for which there exists a finite number n of polynomials $\alpha_i(x)$, not all zero, $i = 1,2,\ldots n$, such that

$$\sum_{i=1}^{n} \alpha_i(x)(f(x))^i = 0.$$

An algebraic equation then is one which can be written as $f(x) = 0$, where $f(x)$ is an algebraic function.

Returning to equations (3.1) and (3.2) we note that any real solutions of (3.1) give purely imaginary roots of (3.2) and so need only consider the determination of all the roots of (3.2). Our first step in solving equation (3.2) will be to change the problem from that of solving an equation possessing infinitely many roots to that of solving a system of equations each possessing a finite number of roots. One reason for this is that the use of the argument principle in the former case can hardly be expected to yield any useful information. The technique we use for equation (3.2) will be to "split" the inverse tangent function into its branches. We first make the substitution

$$\beta = \frac{i}{\omega z}, \tag{3.3}$$

and deduce that

$$\tan(\pm n\pi + \frac{i}{\omega z}) = \frac{i}{z}, n = 0,1,2 .. \tag{3.4}$$

(Note that (3.3) requires $\omega \neq 0$, however, the case $\omega = 0$ can be solved immediately). On using the identity

$$\tanh^{-1}\zeta = \frac{1}{2}\log(\frac{1+\zeta}{1-\zeta}),$$

We can write equation (3.4) as

$$1 + \frac{1}{2}\omega z \{\mathrm{Log}\ \frac{z-1}{z+1} \pm 2n\pi i\} = 0, \tag{3.5}$$

where the symbol Log denotes the principal branch of the log function. Consequently the cut is the interval $[-1,1]$. Thus we consider the problem of determining the roots, in the cut plane, of equation (3.5). For this purpose we introduce the functions

$$\Lambda_0(z) = 1 + \frac{1}{2}\omega z\ \mathrm{Log}(\frac{z-1}{z+1}), \tag{3.6}$$

and

$$\Lambda_n(z) = \Lambda_o(z) + n\pi i\omega z. \qquad (3.7)$$

We note the following obvious properties of these functions

$$\Lambda_o(z) = \Lambda_o(-z). \qquad (3.8)$$

$$\Lambda_n(-z) = \Lambda_{-n}(z) \text{ or } \Lambda_n(z) = \Lambda_{-n}(-z), \qquad (3.9)$$

and

$$\Lambda_o(\infty) = 1 - \omega. \qquad (3.10)$$

The first two equations mean that if ζ is a root of equation (3.5) then so is $-\zeta$, i.e. the roots of equation (3.2) occur in equal and opposite pairs.

We first consider the case when n = 0, and so we apply the argument principle to $\Lambda_o(z)$ in a domain bounded internally by a contour encircling the cut [– 1,1] and externally by a circle of radius $R >> 1$. The boundary values of $\Lambda_o(z)$ on the cut are

$$\Lambda_o^{\pm}(t) = 1 + \frac{1}{2}\omega t \ln\left(\frac{1-t}{1+t}\right) \pm \frac{1}{2}\omega t\pi i \qquad (3.11)$$

and so $\Lambda_o^+(t)$ and $\Lambda_o^-(t)$ are complex conjugates. Consequently, if we shrink the inner contour into the cut the change in the argument of $\Lambda_o(z)$ around this contour will be twice the change of the arguments of $\Lambda_o^+(t)$ in the interval [– 1,1]. (The positive direction is assumed in each case). Now

$$\arg \Lambda_o^+(t) = \tan^{-1}\left\{\frac{\tfrac{1}{2}\omega t\pi}{1 + \tfrac{1}{2}\omega t \ln\left(\frac{1-t}{1+t}\right)}\right\}.$$

In order to compute the change in argument of $\Lambda_o^+(t)$ we need to know the behavior of

$$D(t) = 1 + \frac{1}{2}\omega t \ln\left(\frac{1-t}{1+t}\right) \text{ on } [-1,1].$$

Now for $\omega > 0$

$$\frac{dD(t)}{dt} = \omega\left[\frac{1}{2} \ln \left(\frac{1-t}{1+t}\right) - \frac{t}{1-t^2}\right]$$

$$\therefore \frac{dD(t)}{dt} \begin{cases} > 0 \text{ when } -1 < t < 0 \\ < 0 \text{ when } 0 < t < 1 \end{cases}$$

Thus for $t < 0$ the slope of $D(t)$ is positive and for $t > 0$, the slope is negative and so a sketch of $D(t)$ for $\omega > 0$ is as in Figure 2.

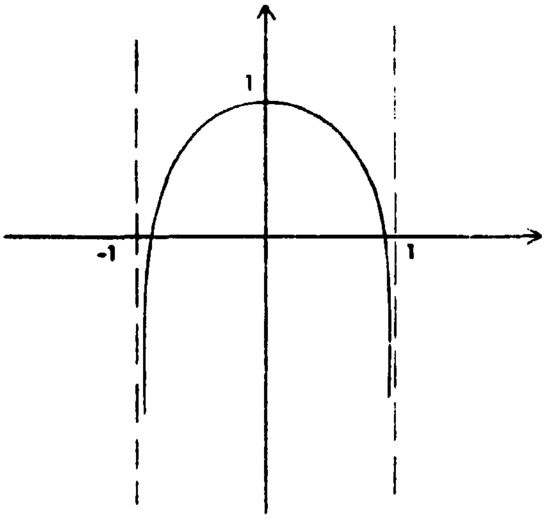


Figure 2 — D(t) for $\omega > 0$

Thus the change in argument for $\Lambda_0^+(t)$, $\omega > 0$, will be 2π as shown in Figure 3.

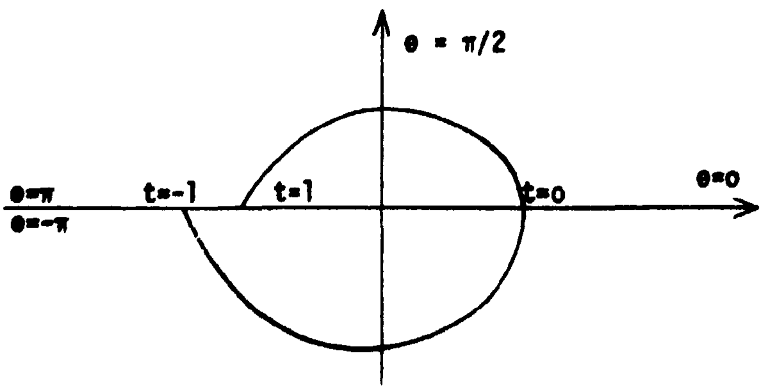


Figure 3

Thus there is a change of 4π in the argument of $\Lambda_0(z)$ around the cut. From equation (3.10) we see that $\Lambda_0(z) \to 1-\omega$, as $|z| \to \infty$ and so the change in argument on the outer circle tends to zero as $R \to \infty$. Thus for $\omega > 0$ $\Lambda_0(z)$ has two zeros in the cut plane. (For $\omega = 1$ these zeros occur at the point at infinity). Now for $\omega < 0$ we observe that $D(t) \geqslant 1$ and so $\Lambda_0(z)$ has no zeros in the cut plane for $\omega < 0$. We can further reason by examining $\Lambda_0(z)$, and showing $\overline{\Lambda_0(z)} = \Lambda_0(\bar{z})$ that for $x > 1$ the zeros are real for $0 < \omega < 1$ and imaginary for $\omega > 1$. We are now in a position to determine these two zeros. Consider the Riemann problem whose coefficient is given by

$$G_0(t) = \frac{\Lambda_0^+(t)}{\Lambda_0^-(t)},\ 0 < t < 1 \tag{3.12}$$

Now from the methods given in Chapter 2 a canonical solution for this problem is

$$X_0(z) = \frac{1}{z-1} \exp \Gamma_0(z),\ \omega \geqslant 0 \tag{3.13a}$$

where

$$\Gamma_0(z) = \frac{1}{\pi} \int_0^1 \arg \Lambda_0^+ \frac{dt}{t-z} \tag{3.13b}$$

and where we have chosen $\arg \Lambda_0^+(0) = 0$.

Consider now the function

$$\Psi(z) = \frac{\Lambda_0(z)}{X_0(-z)} \tag{3.14}$$

We first compute the limiting values of $\Psi(z)$ in the interval $(-1,1)$. For $t < 0$ we have

$$\Psi^+(t) = \frac{\Lambda_0^+(t)}{X_0^-(-t)} = \frac{\Lambda_0^+(t)}{G_0^{-1}(-t)X^+(-t)},$$

$$= \frac{\Lambda_0^+(t)\Lambda_0^+(-t)}{\Lambda_0^-(-t)X_0^+(-t)}.$$

On using the fact that

$$\Lambda^+(-t) = \Lambda^-(t),$$

We can write

$$\Psi^+(t) = \frac{\Lambda_0^+(t)\,\Lambda_0^-(t)}{\Lambda_0^+(t)X_0^+(-t)} = \frac{\Lambda_0^-(t)}{X_0^+(-t)}\,.$$

However

$$\Psi^-(t) = \frac{\Lambda_0^-(t)}{X_0^+(-t)}\,.$$

Consequently by our analytic continuation theorem $\Psi(z)$ is analytic for $Rz < 0$.

For $t > 0$

$$\Psi^\pm(t) = \frac{\Lambda_0^\pm(t)}{X_0(-t)},$$

from which it follows, on using equation (3.12), that

$$\Psi^+(t) = G_0(t)\,\Psi^-(t),$$

i.e., $\Psi(z)$ is a solution of the homogeneous Riemann problem with coefficient $G_0(t)$. Clearly it is of finite degree at infinity, consequently from theorem (2.8) we can write

$$\Psi(z) = X_0(z)P(z)$$

where $P(z)$ is a polynomial. From equations (3.8) and (3.14) we deduce what we shall call a factorization of $\Lambda_0(z)$ in terms of $X_0(z)$:

$$\Lambda_0(z) = X_0(z)X_0(-z)(z_0^2 - z^2)(1-\omega), \qquad (3.15)$$

where $\pm z_0$ are the zeros of $\Lambda_0(z)$, $\omega \neq 1$. (Recall $X_0(z)$ is non-vanishing in the plane cut from 0 to 1 on the real axis, except at infinity). Now equation (3.15) is an identity in z, i.e.,

$$z_0^2 = z^2 + \frac{\Lambda_0(z)}{(1-\omega)X_0(z)X_0(-z)},$$

so we can assign any convenient value to z. An especially simple result follows if we set $z = 0$

$$z_0 = i(\omega - 1)^{-1/2} \exp \left\{ -\frac{1}{\pi} \int_0^1 \mathrm{Arg}\, \Lambda_0^+(t) \frac{dt}{t} \right\}, \ \omega > 0.$$

Finally from equation (3.3) we find

$$\beta_0 = \pm \frac{1}{\pi} (\omega - 1)^{1/2} \exp \left\{ -\frac{1}{\pi} \int_0^1 \mathrm{Arg}\, \Lambda_0^-(t) \frac{dt}{t} \right\}, \ \omega > 0. \qquad (3.16)$$

As the exponential term here is real it is evident that for $0 < \omega < 1$, β_0 is imaginary, while for $\omega > 1$, β_0 is real.

We now turn to the case $n \geqslant 1$, noting from equation (3.9) that this will simultaneously cover the cases $n \leqslant -1$. On applying the argument principle to $\Lambda_n(z)$ around the cut in a manner similar to that used for $\Lambda_0(z)$, we find that arg $\Lambda_n^+(t)$ increases by 2π as t proceeds from -1 to 1 but that arg $\Lambda_n^-(t)$ decreases by 2π as t proceeds from 1 to -1 (Note from equation (3.7) that $\Lambda_n^+(t)$ and $\Lambda_n^-(t)$ are not complex conjugates of each other).

Thus there is no net change in arg $\Lambda_n(z)$ around the cut. There is a change around the large circle however, due to the term [illegible] in equation (3.7). As $R \to \infty$ this term yields a net increase of 2π in the argument of $\Lambda_0(z)$, and so $\Lambda_n(z)$ has precisely one zero in the cut plane, say z_n. This result holds for all real values of ω. From equation (3.9) it is clear that $-z_n$ is a zero of $\Lambda_{-n}(z)$, which suggests that we consider the even function

$$\Omega_n(z) = \Lambda_n(z)\Lambda_{-n}(z). \qquad (3.17)$$

We proceed as before to solve the homogeneous Riemann problem whose coefficient is given by

$$G_n(t) = \frac{\Omega_n^+(t)}{\Omega_n^-(t)}, \ 0 < t < 1. \qquad (3.18)$$

The canonical solution for this problem

$$X_n(z) = \exp \Gamma_n(z) \qquad (3.19a)$$

where

$$\Gamma_n(z) = \frac{1}{\pi} \int_0^1 \arg \Omega_n^+(t) \frac{dt}{t - z} \qquad (3.19b)$$

with arg $\Omega_n^+(0) = 0$. Using the same techniques for $\Omega_n(z)$ as we used for $\Psi(z)$ we show that $\Omega_n(z)/X_n(-z)$ is a solution of the Riemann problem with coefficient $G_n(t)$ given by equation (3.18) and so the factorization of $\Omega_n(z)$ in terms of $X_n(z)$ takes the form

$$\Omega_n(z) = X_n(z)X_n(-z)(z^2 - z_n^2)n^2\pi^2\omega^2, \tag{3.20}$$

which we can immediately solve for z_n. If we again set $z = 0$ in equation (3.20) we deduce that

$$z_n = i[n\pi\omega\, X_n(0)]^{-1},$$

and so from equation (3.3)

$$\beta_n = \pm n\pi \exp\{\frac{1}{\pi}\int_0^1 \arg \Omega_n^+(t)\frac{dt}{t}\},\ n = 1,2,\ldots(-\infty<\omega<\infty), \tag{3.21}$$

where from equations (3.7), (3.11) and (3.17)

$$\Omega_n^+(t) = [\Lambda_0^+(t)]^2 + n^2\pi^2\omega^2 t^2.$$

Clearly the β_n, $n \geqslant 1$, are all real. Figure 4-a gives a graphical display of these zeros for $0 < \omega < 1$ and figure 4-b for $\omega > 1$

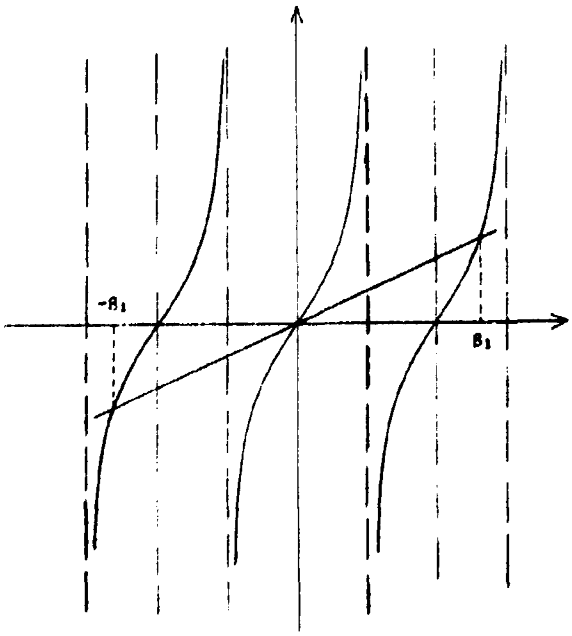


Figure 4a — The β_i for $0 < \omega < 1$

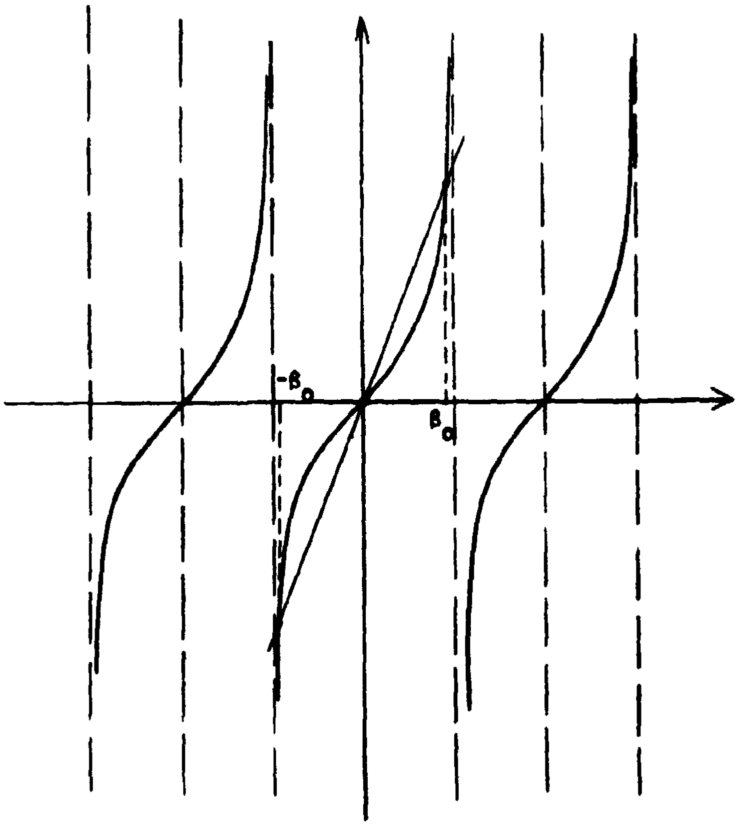


Figure 4b — The β_i for $\omega > 1$.

Example: Are the Riemann coefficients given by equations (3.12) and (3.18) Hölder on [0,1]? If not, can our expressions for the canonical solutions be justified?

Example: Carry out the details on the following method of determining the change in Arg $\Lambda_0^+(t$), which is valid for complex ω. Set $\omega = 1/\zeta$ and consider

$$\zeta\Lambda_0^+(t) = \zeta - \{-\frac{1}{2} t \ln(\frac{1-t}{1+t}) - \frac{\pi i t}{2}\}.$$

Let C derote the curve determined by

$$x:y = -\frac{1}{2} t \ln(\frac{1-t}{1+t}) : \frac{-\pi t}{2}, -1 \leq t \leq 1,$$

see Figure (5) (The arrows denote t increasing).

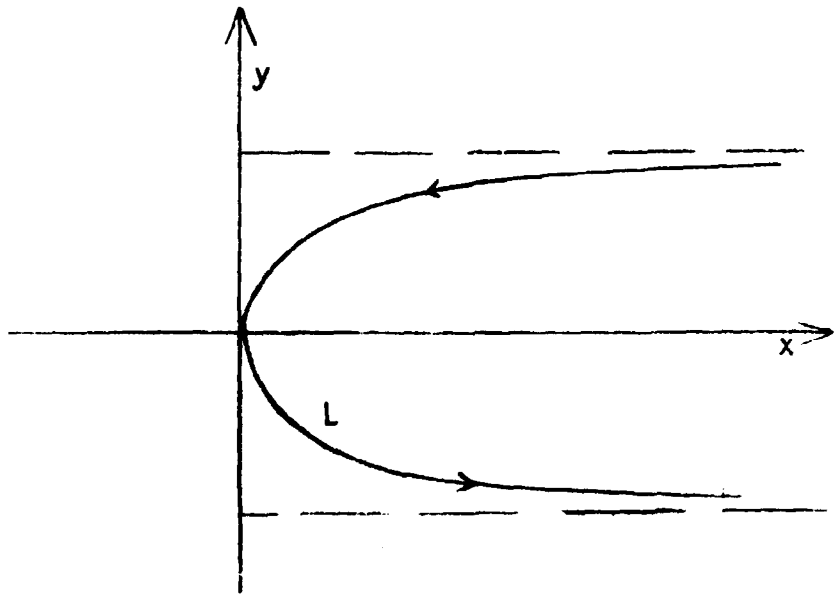


Figure 5 – The $\Lambda_0^+(t) = 0$ curve

Consequentely for $\zeta \in S^+$ the change in arg $\Lambda_0^+(t)$ is 2π, while for $\zeta \in S^-$ the change is zero.

We now turn our attention to the determination of the roots of the equation

$$\beta \tan \beta = \omega, \ \omega \text{ real.} \tag{3.22}$$

We leave the reader to pose a Sturm-Liouville problem for which (3.22) serves to determine the eigen-values. In much the same manner as we derived equation (3.5) the substitution

$$\beta = i\omega z \tag{3.23}$$

leads to

$$z - \frac{1}{2\omega} \{\mathrm{Log}(\frac{z-1}{z+1}) \pm 2in\pi\} = 0, \tag{3.24}$$

from which we can determine the roots. We introduce the functions

$$\hat{\Lambda}_o(z) = z\{z - \frac{1}{2\omega}\,\mathrm{Log}(\frac{z-1}{z+1})\}, \tag{3.25}$$

$$z\hat{\Lambda}_n(z) = \hat{\Lambda}_o(z) - \frac{i}{\omega}\,n\pi z, \tag{3.26}$$

and

$$\hat{\Omega}_n(z) = \hat{\Lambda}_n(z)\hat{\Lambda}_{-n}(z). \tag{3.27}$$

The boundary values of these functions on the cut are given by

$$\Lambda_o^{\pm}(t) = t\{t - \frac{1}{2\omega}\ln(\frac{1-t}{1+t}) \mp \frac{i\pi}{2\omega}\}, \tag{3.28}$$

$$\Lambda_n^{\pm}(t) = \frac{1}{t}\Lambda_o^{\pm}(t) - \frac{in\pi}{\omega}, \tag{3.29}$$

and

$$t^2\hat{\Omega}_n^{\pm}(t) = [\Lambda_o^{\pm}(t)]^2 + \frac{n^2\pi^2 t^2}{\omega^2}. \tag{3.30}$$

We now wish to apply the argument principle in turn to $\hat{\Lambda}_o(z)$ and $\hat{\Lambda}_n(z)$, $n \geqslant 1$. From equation (3.28)

$$\mathrm{Arg}\,\Lambda_o^{\pm}(t) = \mathrm{Arg}\,t + \mathrm{Arg}\,\tilde{\Lambda}^{\pm}(t),$$

where

$$\tilde{\Lambda}^{\pm}(t) = t - \frac{1}{2\omega}\ln(\frac{1-t}{1+t}) \mp \frac{i\pi}{2\omega},$$

and so to determine the change in $\arg\hat{\Lambda}_o(z)$ around the cut we need to examine

$$\arg\tilde{\Lambda}^{+}(t) = \tan^{-1}\{\frac{\frac{-\pi}{2\omega}}{t - \frac{1}{2\omega}\ln\frac{1-t}{1+t}}\},$$

which in turn depends on the behavior of

$$\hat{D}(t) = t - \frac{1}{2\omega} \ln\left(\frac{1-t}{1+t}\right).$$

Now

$$\frac{d\hat{D}(t)}{dt} = 1 + \frac{1}{\omega(1-t^2)},$$

and so for $\omega > 0$, $\hat{D}(t)$ is an increasing function passing through the origin. For $\omega < 0$ howerer, the situation is more complicated. Clearly $\hat{D}(t)$ has stationary points at

$$t = \pm\sqrt{1 \pm \frac{1}{\omega}},$$

so the behavior of $\hat{D}(t)$ will depend on whether $-1 < \omega < 0$, or $\omega < -1$. Figure 6 gives sketches of the various cases.

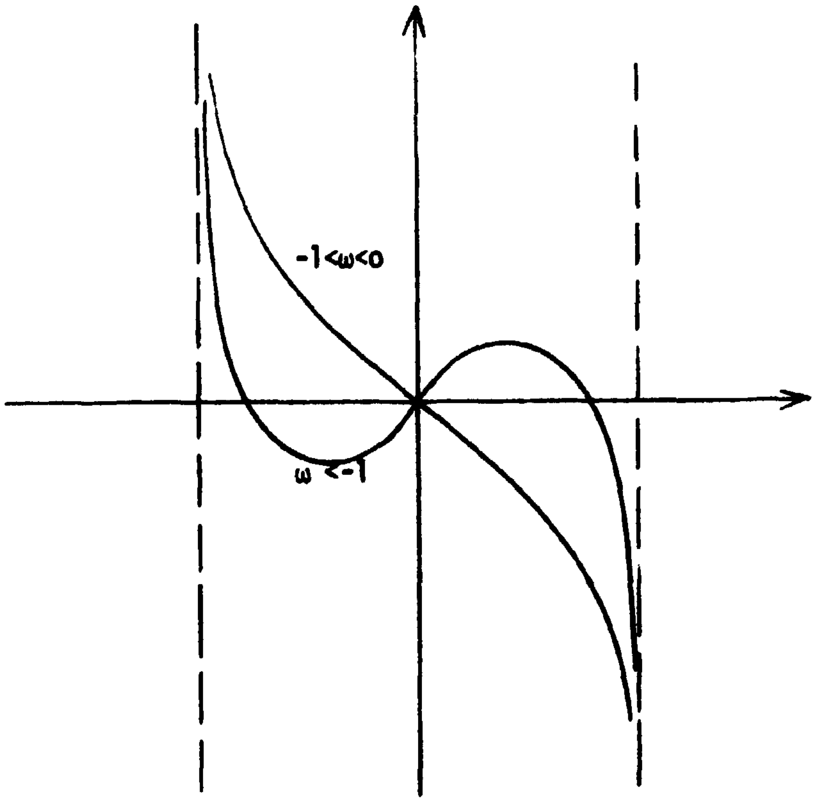


Figure 6 – Behavior of $\hat{D}(t)$ for $\omega < 0$

In either case the net change in arg $\hat{\Lambda}^+(t)$ is π, and as $\hat{\Lambda}^\pm(t)$ are complex conjugates the net change in arg $\Lambda^-(t)$ is also π. However, the change in arg t around the cut is -2π so that there is no net change in arg $\hat{\Lambda}_0(z)$ around the cut. The change around the large circle is clearly 4π as $(\hat{\Lambda}_0(z)/z^2) \to 1$ when $|z| \to \infty$, so that $\hat{\Lambda}_0(z)$ has two zeros $\pm z_0$ say, in the cut plane.

The reasoning for $\hat{\Lambda}_n(z)$ makes use of the behavior of $\hat{D}(t)$, and indicates that $\hat{\Lambda}_n(z)$, $n \geq 1$ has precisely one zero in the cut plane. We now are in a position to determine the factorizations of $\hat{\Lambda}_0(z)$ and $\hat{\Omega}_n(z)$. Now for $\hat{\Lambda}_0(z)$ the appropriate Riemann coefficient is

$$\hat{G}_0(t) = \frac{\hat{\Lambda}_0^+(t)}{\hat{\Lambda}_0^-(t)}, \; 0 < t < 1, \tag{3.31}$$

but to be consistent with our choice of taking the principal value of the argument it is convenient to consider the cases $\omega < 0$ and $\omega > 0$ separately. Now for $\omega < 0$, Arg $\Lambda_0^+(0) = \frac{\pi}{2}$ and arg $\hat{\Lambda}_0^+(1) = \pi$, while for $\omega > 0$ arg $\hat{\Lambda}_0^+(0) = -\frac{\pi}{2}$ and arg $\hat{\Lambda}_0^+(1) = 0$. Thus by the results of chapter 2 we can write.

$$X_0(z) = \frac{1}{z-1} \exp\left\{\frac{1}{\pi}\int_0^1 \arg \hat{\Lambda}_0^+(t) \frac{dt}{t-z}\right\}, \; \omega < 0, \tag{3.32a}$$

$$X_0(z) = \frac{1}{z} \exp\left\{\frac{1}{\pi}\int_0^1 \arg \hat{\Lambda}_0^-(t) \frac{dt}{t-z}\right\}, \; \omega > 0. \tag{3.32b}$$

We can combine these results however, if we make use of the identity

$$\left(\frac{z}{z-1}\right)^{1/2} = \exp\left\{-\frac{1}{2}\int_0^1 \frac{dt}{t-z}\right\},$$

deriving

$$X_0(z) = \frac{1}{\sqrt{z(z-1)}} \exp\left\{\frac{1}{\pi}\int_0^1 [\mathrm{Arg}\, \hat{\Lambda}_0^+(t) + \frac{\pi}{2}\,\mathrm{sgn}(\omega)] \frac{dt}{t-z}\right\}, \tag{3.33}$$

where it is understood that

$$\arg \Lambda_0^+(0) = \begin{cases} \pi/2, & \omega < 0, \\ -\pi/2, & \omega > 0. \end{cases}$$

The factorization for $\Lambda_o(z)$ now takes the form

$$\Lambda_o(z) = z^2 (z_o^2 - z^2) X_o(z) X_o(-z).$$

If we set z = 0, after first dividing the equation by z we deduce

$$z_o^2 = -\frac{\pi}{2\omega} \exp\{-\frac{2}{\pi} \int_0^1 [\arg \Lambda_o^+(t) + \frac{\pi}{2} \operatorname{sgn}(\omega)] \frac{dt}{t}\},$$

and so from equation (3.23)

$$\beta_o = \pm (\frac{\pi\omega}{2})^{1/2} \exp\{-\frac{1}{\pi} \int_0^1 [\arg \Lambda_o^+(t) + \frac{\pi}{2} \operatorname{sgn}(\omega)] \frac{dt}{t}\}. \tag{3.34}$$

This indicates that β_o is real if $\omega > 0$, and purely imaginary if $\omega < 0$, a result we could have deduced by other means.

The factorization for $\hat{\Omega}_n(z)$, $n \geqslant 1$, is

$$\hat{\Omega}_n(z) = (z^2 - z_n^2) X_n(z) X_n(-z),$$

where

$$X_n(z) = \exp\{\frac{1}{\pi} \int_0^1 \arg \hat{\Omega}_n^+(t) \frac{dt}{t-z}\}.$$

On setring z = 0 here, we find

$$z_n^2 = -\frac{\pi^2}{4\omega^2}(4n^2 - 1) \exp\{-\frac{2}{\pi} \int_0^1 \arg \hat{\Omega}_n^+(t) \frac{dt}{t}\},$$

and so

$$\beta_n = \pm \frac{\pi}{2} (4n^2 - 1)^{1/2} \exp\{-\frac{1}{\pi} \int_0^1 \arg \hat{\Omega}_n^+(t) \frac{dt}{t}\}, \tag{3.35}$$

which is clearly real for all ω. In figure 7 we give a graphical display of these zeros.

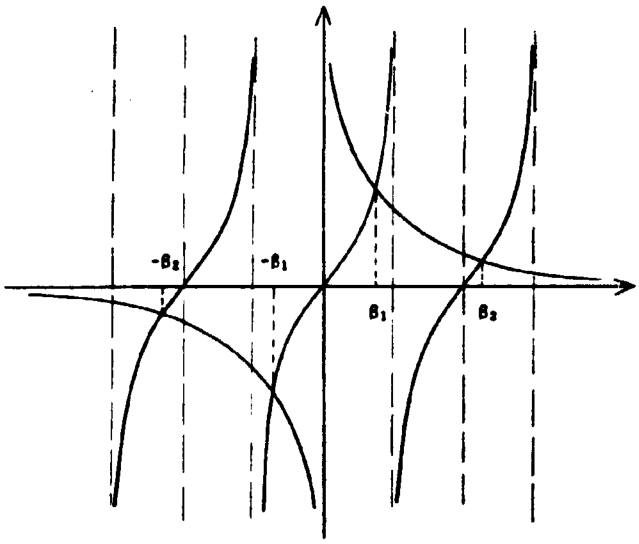


Figure 7a – Zeros of $\beta \tan \beta = \omega$, $\omega > 0$.

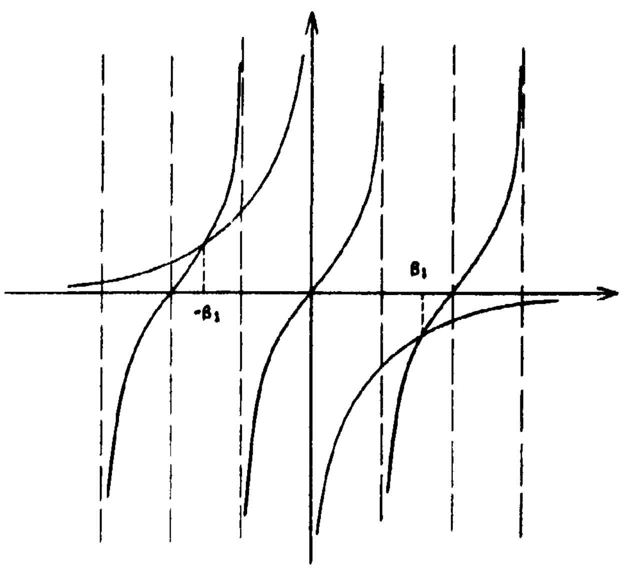


Figure 7b – Zeros of $\beta \tan \beta = \omega$, $\omega < 0$.

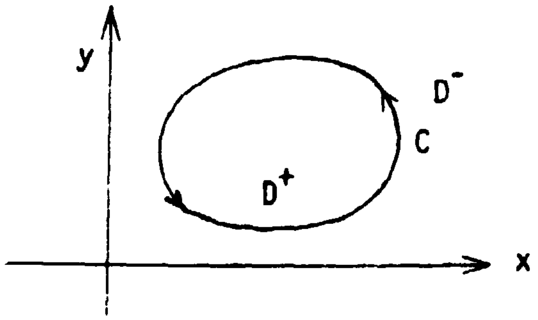
Example: Apply the argument principle to $\hat{\Lambda}_0(z)$ for complex values of ω, in the manner used for $\Lambda_0(z)$.

CHAPTER IV

The Matrix Riemann Problem

This chapter is essentially the same as in Muskhelishvili's book [3].

Notation. As usual C will denote a simple contour, D^+ the interior of C and D^- the exterior, as depicted in the figure



$\Phi(z)$ will now denote an n-vector i.e.

$$\Phi(z) = \begin{bmatrix} \Phi_1(z) \\ \Phi_2(z) \\ \cdot \\ \Phi_n(z) \end{bmatrix} . \tag{4.1}$$

When a vector or a matrix is said to be Hölder on C we shall mean that each component is Hölder on C. Likewise $\Phi(z)$ will be called sectionally analytic if each component is sectionally analytic. The principal part of $\Phi(z)$ at ∞ will be denoted by

$$\gamma(z) = \begin{bmatrix} \gamma_1(z) \\ \gamma_2(z) \\ \cdot \\ \cdot \\ \gamma_n(z) \end{bmatrix} , \tag{4.2}$$

where each γ_i is a polynomial.

The Cauchy Integral

Let $\phi(t)$ be a vector, Hölder on C and consider

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\tau)}{\tau - z} d\tau,$$

i.e.

$$\Phi_1(z) = \frac{1}{2\pi i} \int_C \frac{\phi_1(\tau)}{\tau - z} d\tau,$$

$$\Phi_2(z) = \frac{1}{2\pi i} \int_C \frac{\phi_2(\tau)}{\tau - z} d\tau.$$

etc.

It is obvious that the Plemelj formulas yield

$$\left.\begin{aligned} \Phi^+(t) &= \frac{\phi(t)}{2} + \frac{1}{2\pi i} \int_C \frac{\phi(\tau)}{\tau - t} d\tau, \\ \Phi^-(t) &= \frac{-\phi(t)}{2} + \frac{1}{2\pi i} \int_C \frac{\phi(\tau)}{\tau - t} d\tau. \end{aligned}\right] \quad (4.3)$$

By Cauchy's integral formulas it is also obvious that if $\Phi(z)$ is analytic in D^+ and continuous on C, then

$$\left.\begin{aligned} \Phi(z) &= \frac{1}{2\pi i} \int_C \frac{\Phi^+(\tau)}{\tau - z} d\tau, \; z \in D^+, \\ 0 &= \frac{1}{2\pi i} \int_C \frac{\Phi^+(\tau)}{\tau - z} d\tau \; z \in D^-. \end{aligned}\right] \quad (4.4)$$

THEOREM (4.1). The last formula of (4.4) is a necessary and sufficient condition for a continuous function to be the limiting value of a function analytic in D^+.

PROOF. The necessity is simply the above. To prove the sufficiency let

$$F(z) = \frac{1}{2\pi i} \int_C \frac{\Phi^+(\tau)d\tau}{\tau - z}, \quad z \epsilon D^+.$$

Clearly F(z) is analytic in D^+. By the Plemelj formula

$$F^+(t) - F^-(t) = \Phi^+(t).$$

But $F^-(t) = 0$. Q.E.D. An equivalent form is

$$0 = -\frac{1}{2}\Phi^+(t) + \frac{1}{2\pi i} \int_C \frac{\Phi^+(t)}{\tau - t} d\tau \tag{4.5}$$

This may be seen as it is evident that (4.5) follows from (4.4). If (4.5) is true then

$$g(z) = \frac{1}{2\pi i} \int_C \frac{\Phi^+(\tau)}{\tau - z} d\tau, z \epsilon D^-,$$

has zero limiting values on C, and hence must be zero in D^-, i.e. $g(z) = 0$. Also by Cauchy's Theorem

$$\left.\begin{aligned} \gamma(z) - \Phi(z) &= \frac{1}{2\pi i} \int_C \frac{\Phi^-(\tau)}{\tau - z} d\tau, \ z \epsilon D^-, \\ \gamma(z) &= \frac{1}{2\pi i} \int_C \frac{\Phi^-(\tau)}{\tau - z} d\tau, \ z \epsilon D^+, \end{aligned}\right] \tag{4.6}$$

for a function analytic in D^- except for a pole at ∞.

THEOREM (4.2). *The last equation of (4.6) is a necessary and sufficient condition for a continuous function to be the limiting value of a function analytic in D^-. It is equivalent to*

$$\gamma(t) = \frac{1}{2}\Phi^-(t) + \frac{1}{2\pi i} \int_C \frac{\Phi^-(\tau)}{\tau - t} d\tau. \tag{4.7}$$

PROOF. Similar to the above

The Homogenous Riemann Problem

Let G(t) be a n x n matrix which is Hölder on a given smooth contour C, and also is such that

$$|G(t) \neq 0,$$

on C

DEFINITION (4.1). The Riemann problem will be to determine a sectionally analytic function $\Phi(z)$, such that

$$\Phi^+(t) = G(t)\,\Phi^-(t) \text{ on } C, \tag{4.8}$$

i.e

$$\begin{bmatrix} \Phi_1^+ \\ \Phi_2^+ \\ \Phi_3^+ \\ \cdot \\ \\ \Phi_n^+ \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \cdot & \cdot & G_{1n} \\ G_{21} & G_{22} & \cdot & \cdot & G_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ G_{n1} & G_{n2} & & & G_{nn} \end{bmatrix} \begin{bmatrix} \Phi_1^- \\ \Phi_2^- \\ \Phi_3^- \\ \cdot \\ \cdot \\ \Phi_n^- \end{bmatrix},$$

and so

$$\Phi_1^+ = G_{11}\,\Phi_1^- + G_{12}\,\Phi_2^- + \ldots + G_{1n}\,\Phi_n^-,$$

$$\Phi_2^+ = G_{21}\,\Phi_1^- + G_{22}\,\Phi_2^- + \ldots + G_{2n}\,\Phi_n^-,$$

$$\Phi_n^+ = G_{n1}\,\Phi_1^- + G_{n2}\,\Phi_2^- + \ldots + G_{nn}\,\Phi_n^-,$$

We shall refer to this as problem I, i.e.

$$\Phi^+(t) = G(t)\;\Phi^-(t), \tag{I}$$

We first seek an equivalent statement of I, One is the following. To determine a $\Phi^-(t)$, Hölder on C such that

(1) $\Phi^-(t)$ is the boundary value of $\Phi(z)$ analytic in D^-, with a principal part $\gamma(z)$ at ∞.

(2) $\Phi^+(t)$ determined from I is the boundary value of a function $\Phi(z)$ analytic in D^+.

Now an equivalent form of (1), by Theorem (4.2), is

$$\frac{1}{2}\Phi^-(t) + \frac{1}{2\pi i}\int_C \frac{\Phi^-(\tau)}{\tau - t} d\tau = \gamma(t), \tag{4.9a}$$

and an equivalent form of (2) is, by Theorem (4.3),

$$-\frac{1}{2} G(t)\,\Phi^-(t) + \frac{1}{2\pi i}\int_C \frac{G(\tau)\,\Phi^-(\tau)}{\tau - t} d\tau = 0. \tag{4.9b}$$

Thus problem I is equivalent to determining a vector Φ^- (t) which simultaneously satisfies (4.9a) and (4.9b). If we multiply (4.9b) by $G^{-1}(t)$ we derive

$$-\frac{1}{2}\Phi^-(t) + \frac{1}{2\pi i}\int_C \frac{G^{-1}(t)G(\tau)}{\tau - t}\Phi^-(\tau)\,d\tau = 0.$$

On subtracting this from (4.9a) we have

$$\Phi^-(t) - \frac{1}{2\pi i}\int_C \frac{[G^{-1}(t)G(\tau) - I]}{\tau - t}\Phi^-(\tau)\,d\tau = \gamma(t). \tag{4.10}$$

We note two things here about (4.10)

(1) It is not necessarily equivalent to (4.9).

(2) It is quasi-regular Fredholm equation, since

$$\frac{G^{-1}(t)G(\tau) - I}{\tau - t} = G^{-1}(t)\frac{[G(\tau) - G(t)]}{\tau - t}, \tag{4.11}$$

and as G is Hölder this kernel is integrable. For this reason we will sometimes refer to (4.10) as I(F).

Now with regard to (1) we ask the following questions: (i) when is (4.10) soluble? (ii) when is a solution to (4.10) a solution to (4.9)?

We note that every continuous solution of (4.10) is Hölder, by (4.11).

The Equivalence of I(F) and Problem I

Here we assume $\Phi^-(t)$ is a solution of I(F) and we wish to show it is a solution of I. Consider the vector

$$\Psi(z) = \frac{1}{2\pi i} \int_C \frac{\Phi^-(\tau)}{\tau - z} d\tau - \gamma(z),\ z \in D^+ \tag{4.12a}$$

$$= \frac{1}{2\pi i} \int_C \frac{G(\tau)\,\Phi^-(\tau)}{\tau - z} d\tau \cdot z \in D^-. \tag{4.12b}$$

Now

$$\Psi^+(t) = \frac{1}{2}\Phi^-(t) + \frac{1}{2\pi i}\int_C \frac{\Phi^-(\tau)}{\tau - t} d\tau - \gamma(t),$$

$$\Psi^-(t) = \frac{-G(t)\Phi^-(t)}{2} + \frac{1}{2\pi i}\int_C \frac{G(\tau)\Phi^-(\tau)}{\tau - t} d\tau \cdot$$

Clearly $\Psi(z)$ vanishes at ∞, and so on writing (4.10) in the form

$$\Psi^+(t) = G^{-1}(t)\Psi^-(t), \tag{II}$$

which we term the **accompanying** problem of I, we have:

LEMMA 4.3.

If problem II has only trivial solutions which vanish at ∞, then each solution of I(F) gives a solution of I

PROOF. If $\Psi(z) = 0$ then (4.12) implies that $\Phi^-(t)$ satisfies I, as $\Psi^+(t) = \Psi^-(t) = 0$ and so the pair (4.9) is satisfied.

We now introduce the idea of a problem and a corresponding Fredholm equation. Clearly I(F) is the corresponding Fredholm equation to problem I. For problem II we rewrite

$$\Psi^-(t) = G(t)\Psi^+(t)$$

and so as we are seeking solutions which vanish at ∞ we have from (4.5) and (4.7) that

$$0 = \frac{\Psi^+(t)}{2} + \frac{1}{2\pi i}\int_C \frac{\Psi^+(\tau)}{\tau - t} d\tau,$$

$$0 = \frac{G(t)\,\Psi^+(t)}{2} + \frac{1}{2\pi i}\int_C \frac{G(\tau)\Psi^+(\tau)}{\tau - t} d\tau,$$

and so

$$\Psi^{+}(t) + \frac{1}{2\pi i}\int_{C} \frac{(G^{-1}(t)G(\tau) - I)}{\tau - t}\Psi^{+}(\tau)\,d\tau = 0 \tag{4.13}$$

Recapping then we have

$$\Phi^{+}(t) = G(t)\,\Phi^{-}(t), \tag{I}$$

$$\Phi^{-}(t) - \frac{1}{2\pi i}\int_{C} \frac{(G^{-1}(t)G(\tau) - I)}{\tau - t}\Phi^{-}(\tau)\,d\tau = \gamma(t), \tag{I(F)}$$

$$\Psi^{+}(t) = G^{-1}(t)\Psi^{-}(t), \tag{II}$$

$$\Psi^{+}(t) + \frac{1}{2\pi i}\int_{C} \frac{(G^{-1}(t)G(\tau) - I)}{\tau - t}\Psi^{+}(\tau)\,d\tau = 0. \tag{II(F)}$$

The Solubility of I

Now introduce the following problems with their corresponding Fredholm equations

$$\Phi'^{+}(t) = \tilde{G}^{-1}(t)\Phi'^{-}(t), \tag{I'}$$

$$\Phi'^{-}(t) - \frac{1}{2\pi i}\int_{C} \frac{(\tilde{G}(t)\tilde{G}^{-1}(\tau) - I)}{\tau - t}\Phi'^{-}(\tau)\,d\tau = 0, \tag{I'(F)}$$

$$\Psi'^{+}(t) = \tilde{G}(t)\Psi'^{-}(t), \tag{II'}$$

$$\Psi'^{+}(t) + \frac{1}{2\pi i}\int_{C} \frac{(\tilde{G}(t)\tilde{G}^{-1}(\tau) - I)}{\tau - t}\Psi'^{+}(\tau)\,d\tau = 0. \tag{II'(F)}$$

$\tilde{G}$ is the transpose of G. Problem I′ is the associate problem to I. We also observe that I′ is the accompanying problem to II′. Thus we have:

LEMMA (4.4). If problem I′ has only trivial solutions which vanish at ∞ then every solution of the Fredholm equation corresponding to problem II′ is a solution of II′.

PROOF. By Lemma (4.3)

THEOREM (4.5). If problem I is such that neither its accompanying nor its associate problems II and I′ have non-trivial solutions vanishing at ∞, then the Fredholm equation corresponding to problem I (4,10) is soluble for any $\gamma(t)$ and every solution of it is a solution of problem I.

PROOF. By the above lemmas there exists $\widetilde{\Psi}^+(t)$ which is the boundary value of an analytic function. $\{\widetilde{\Psi}^+(t) = 0$ is one$\}$ Now a necessary and sufficient condition for the solubility of the Fredholm equation corresponding to I is

$$\int_C \gamma(t)\widetilde{\Psi}^+(t)dt = \int_C [\gamma_1\widetilde{\Psi}_1^+ + \gamma_2\widetilde{\Psi}_2^+ + \ldots + \gamma_n\widetilde{\Psi}_n^+]dt$$
$$= 0,$$

This is a generalization of a Fredholm theorem as II'(F) is the adjoint of I(F) But, as we have stated the $\widetilde{\Psi}_i^+$ are boundary values of a function analytic in D^+, the γ_i are polynomials and so $\gamma_i(t)\widetilde{\Psi}_i^+(t)$ are boundary values of a function analytic in D^+ and so Cauchy's theorem assures us that the condition holds Q. E. D

The Solution of I when Theorem (4,5) Applies

We restrict attention to solutions which are bounded at ∞. Now the theorem guarantees that solutions of I(F) exist no matter what values we attach to γ. We thus write that

$$\overset{1}{\Phi}{}^-(t) = \begin{bmatrix} \overset{1}{\Phi}{}_1^- \\ \overset{1}{\Phi}{}_2^- \\ \cdot \\ \\ \overset{1}{\Phi}{}_n^- \end{bmatrix} \text{ is a solution when } \overset{1}{\gamma}(t) = \begin{bmatrix} 1 \\ 0 \\ \\ \cdot \\ 0 \end{bmatrix}$$

Similarly

$$\overset{2}{\Phi}{}^-(t) = \begin{bmatrix} \overset{2}{\Phi}{}_1^-(t) \\ \overset{2}{\Phi}{}_2^-(t) \\ \\ \cdot \\ \cdot \\ \overset{2}{\Phi}{}_n^-(t) \end{bmatrix} \text{ is a solution when } \overset{2}{\gamma}(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

etc. up to $\overset{n}{\Phi}{}^-(t)$ and $\overset{n}{\gamma}(t)$. Clearly then the general solution of I(F) is

$$\Phi^-(t) = \gamma_1 \overset{1}{\Phi}(t) + \quad + \gamma_n \overset{n}{\Phi}{}^-(t) + \gamma_{n+1} \overset{n+1}{\Phi}(t) + \ldots + \gamma_m \overset{m}{\Phi}{}^-(t) \tag{4.14}$$

Where $\overset{n+1}{\Phi^-}(t) \ldots \overset{m}{\Phi^-}(t)$ are all the linearly independent solutions to the homogeneous equation corresponding to I(F) i.e. $\gamma = 0$, and the $\overset{1}{\gamma_i}$ are constants. We note that if $\overset{1}{\Phi}{}_1^-(t)$ and $\overset{1}{\Phi}{}_2^-(t)$ are solutions when $\gamma = \overset{1}{\gamma}$ then $\overset{1}{\Phi}{}_1^-(t) - \overset{1}{\Phi}{}_2^-(t)$ satisfies the homogeneous equation and so is $\overset{n+1}{\Phi^-}(t)$ to $\overset{m}{\Phi}(t)$.

Also by the theorem, if we denote the solution to the Riemann problem corresponding to $\overset{1}{\Phi}{}^-(t)$ by $\overset{1}{\Phi}(z)$, then the general solution to I is

$$\Phi(z) = \gamma_1 \overset{1}{\Phi}(z) + \quad + \gamma_n \overset{n}{\Phi}(z) + \gamma_{n+1} \overset{n+1}{\Phi}(z) + \ldots + \gamma_m \overset{m}{\Phi}(z). \tag{4.15}$$

It is clear that

$$\overset{j}{\Phi}_k(\infty) = \delta_{jk}. \tag{4.16}$$

General Case.

LEMMA (4.6). Let's denote the number of linearly independet solutions of the homogeneous equation I(F) and let k be the order of any solution at ∞, then $k \leqslant s$.

PROOF. Let $\Phi(z)$ have a zero of order k at ∞ i.e., near ∞,

$$\Phi(z) = \frac{a_k}{z^k} + \frac{a_{k+1}}{z^{k+1}} + \ldots .$$

Now $z\Phi(z)$, $z^2\Phi(z) \ldots z^{k-1}\Phi(z)$ are thus solutions of I which vanish at ∞ which will yield solutions $t\Phi^-(t)$, $t^2\Phi^-(t), \ldots t^{k-1}\Phi^-(t)$ to I(F) homogeneous. Now these are clearly linearly independent, whence $k \leqslant s$.

LEMMA (4.7). An integer $r \geqslant 0$ exists such that the order of the zero of any solution to II and I'at ∞ does not exceed r

PROOF. By Lemma (4 6), s's exists for II and I' separately and so we simply take the larger.

To determine the solution of I of degree at ∞ not exceeding r.

Consider the problem

$$\overset{*}{\Phi^+}(t) = (t-a)^r G(t) \overset{*}{\Phi^-}(t), \tag*{$\overset{*}{\text{I}}$}$$

where a is in D^+. Thus $G(t) \to (t-a)^r G(t)$ and we suppose we are seeking solutions bounded at ∞. Now the accompanying problem is

$$\overset{*}{\Psi}{}^{+}(t) = \frac{G^{-1}(t)}{(t-a)^r}\overset{*}{\Psi}{}^{-}(t), \qquad \overset{*}{\text{II}}$$

and the associate problem is

$$\overset{*}{\tilde{\Phi}}{}^{+}(t) = \frac{\tilde{G}^{-1}(t)}{(t-a)^r}\overset{*}{\tilde{\Phi}}{}^{-}(t). \qquad \overset{*}{\text{I}'}$$

Now if either $\overset{*}{\text{II}}$ or $\overset{*}{\text{I}'}$ has a solution vanishing at ∞ then

$$\Psi(z) = \overset{*}{\Psi}(z), \qquad z \in D^+$$
$$= \frac{\overset{*}{\Psi}(z)}{(z-a)^r}, \quad z \in D^-$$

and

$$\Phi(z) = \overset{*}{\Phi}(z), \qquad z \in D^+$$
$$= \frac{\overset{*}{\Phi}(z)}{(z-a)^r}, \qquad z \in D^-$$

are clearly solutions of II and I′ respectively, which vanish at ∞ with an order larger than r, which contradicts the original statement about r. Consequently, by Theorem (4.5) $\overset{*}{\text{I}}$ possesses solutions which are bounded at ∞. Then

$$\Phi(z) = \overset{*}{\Phi}(z), \qquad z \in D^+,$$
$$= (z-a)^r\,\overset{*}{\Phi}(z), \qquad z \in D^-,$$

is clearly then a solution of degree r or less of I.

We thus have the following theorem.

THEOREM (4.7).

Every solution of I of degree at most r at ∞, where r is sufficiently large, is given by

$$\Phi(z) = \gamma_1 \overset{1}{\Phi}(z) + \ldots + \gamma_n \overset{n}{\Phi}(z) + \gamma_{n+1} \overset{n+1}{\Phi}(z) + \ldots + \gamma_m \overset{m}{\Phi}(z), \qquad (4.17)$$

where $\overset{1}{\Phi}(z) \ldots \overset{n}{\Phi}(z)$ satisfy

$$\lim_{z\to\infty} z^{-r} \overset{\beta}{\Phi}_{\alpha}(z) = \delta_{\alpha\beta} \qquad (4.18)$$

and $\overset{n+1}{\Phi}(z) \ldots \overset{m}{\Phi}(z)$ are solutions of degree less than r at ∞. Clearly m depends on r.

The Homogeneous Riemann Problem and its General Solution

We will now assume that an integer r satisfying Lemma 4.7 has been assigned, and also now it has been established in Theorem (4.7) that all solutions of degree of at most r at ∞, are of the form (4.17).

LEMMA (4.8)

The solutions $\overset{1}{\Phi}(z), \overset{2}{\Phi}(z) \ldots \overset{n}{\Phi}(z)$ are not connected by any relations of the form

$$Q_1(z)\overset{1}{\Phi}(z) + Q_2(z)\overset{2}{\Phi}(z) + \ldots + Q_n(z)\overset{n}{\Phi}(z) = 0, \tag{4.19}$$

where the $Q_1(z), \ldots Q_n(z)$ are polynomials, not all identically zero.

PROOF.

Now (4.19) is equivalent to the n equations

$$Q_1(z)\overset{1}{\Phi}_\alpha(z) + Q_2(z)\overset{2}{\Phi}_\alpha(z) + \ldots + Q_n(z)\overset{n}{\Phi}_\alpha(z) = 0,$$

$$\alpha = 1, 2, \ldots n.$$

Now if polynomials $Q_1(z) \ldots Q_n(z)$ could be found so that (4.19) were true then necessarily the determinant

$$|\overset{\beta}{\Phi}_\alpha(z)|$$

would have to be identically zero which is clearly not so from (4.18)

The Construction of a Fundamental or Canonical Solution

Among the solutions in (4,17) there will be some with lowest degree at ∞.

[This is so because the order of the zero at ∞ is finite, see lemma (4.6)]. Denote this degree by $-\kappa_1$ and a solution having this degree by $\overset{1}{\chi}(z)$. Note that if $\overset{1}{\chi}(z)$ vanishes at ∞ $\kappa_1 > 0$, and if $\chi(z)$ has a pole at ∞ $\kappa_1 < 0$.

Now denote by $-\kappa_2$ the lowest degree of those solutions which are not related to $\overset{1}{\chi}(z)$ by any relation of the form

$$P_1(z)\overset{1}{\chi}(z)$$

where $P_1(z)$ is a polynomial. Let $\overset{2}{\chi}(z)$ be one of these solutions Clearly $\kappa_2 \geqslant \kappa_1$ i.e. $\kappa_2 \leqslant \kappa_1$

Denote by $-\kappa_3$ the lowest degree of those solutions which are not related to $\overset{1}{\chi}(z)$ and $\overset{2}{\chi}(z)$ by any relation of the form

$$P_1(z)\overset{1}{\chi}(z) + P_2(z)\overset{2}{\chi}(z)$$

where $P_1(z)$ and $P_2(z)$ are polynomials Let $\overset{3}{\chi}(z)$ be one of these solutions. Clearly $-\kappa_3 \geqslant -\kappa_2 \geqslant -\kappa_1$ i.e

$$\kappa_3 \leqslant \kappa_2 < \kappa_1$$

To demonstrate that this can be done until some solution $\overset{n}{\chi}(z)$ is obtained, suppose that k solutions $\overset{1}{\chi}(z), \dots \overset{k}{\chi}(z)$ have been constructed in this way and assume that all other solutions in (4.17) are expressible in the form

$$P_1(z)\overset{1}{\chi}(z) + P_2(z)\overset{2}{\chi}(z) + \dots + P_k(z)\overset{k}{\chi}(z)$$

Consequently the solutions $\overset{1}{\Phi}(z) \dots \overset{n}{\Phi}(z)$ could be written in this form, i.e. by successive elimination it is clear that this would imply a relation of the form (4.19) which is impossible Thus n solutions $\overset{1}{\chi}(z), \dots \overset{n}{\chi}(z)$ can be constructed so that they are not related by polynomials with degrees $-\kappa_1, -\kappa_2 \dots -\kappa_n$ such that

$$\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \kappa_4 \dots \geqslant \kappa_n$$

We will prove later that the process cannot be continued further

LEMMA (4.9)

Any solution of degree less than $-\kappa_k$ can be represented in the form

$$\chi(z) = P_1(z)\overset{1}{\chi}(z) + \dots + P_{k-1}(z)\overset{k-1}{\chi}(z).$$

PROOF. Assume the contrary. Thus a solution of degree less than κ_k would exist not of the above form. This contradicts our method of selection however, as $\overset{k}{\chi}(z)$ is supposed to be the solution of lowest degree not representable in the above form.

THEOREM (4.10). The expression

$$\chi(z) = a_1\overset{1}{\chi}(z) + a_2\overset{2}{\chi}(z) + \dots + a_n\overset{n}{\chi}(z), \tag{4.20}$$

where the a_i are constants, not all zero, is non-vanishing in the finite plane.

PROOF. We prove the result by contradiction and so assume that:

Case (i) $\chi(z)$ vanishes at $z = c$ not on C. As $\chi(z)$ is analytic at c it has an integer order zero so clearly we can write

$$\chi(z) = (z - c)\Phi(z),$$

where $\Phi(z)$ is analytic at $z = c$. Now clearly $\Phi(z)$ is a solution of I as

$$\chi^+(t) = (t - c)\Phi^+(t),$$

$$\chi^-(t) = (t - c)\Phi^-(t).$$

However $$\chi^+(t) = G(t)\chi^-(t),$$

$$\Rightarrow \Phi^+(t) = G(t)\Phi^-(t).$$

Let a_k be the last coefficient which is non-zero. Thus the degree of $\Phi(z)$ is less than $\overset{k}{\chi}(z)$ and so by Lemma (4.9) can be expressed as

$$\Phi(z) = P_1(z)\overset{1}{\chi}(z) + \ldots + P_{k-1}(z)\overset{k-1}{\chi}(z),$$

whilst from (4.20)

$$(z - c)\Phi(z) = a_1\overset{1}{\chi}(z) + a_2\overset{2}{\chi}(z) + \ldots + a_k\overset{k}{\chi}(z).$$

These two equations however imply a relation

$$\hat{P}_1(z)\overset{1}{\chi}(z) + \hat{P}_2(z)\overset{1}{\chi}(z) + \ldots \hat{P}_k(z)\overset{k}{\chi}(z) = 0,$$

which contradicts the properties of the $\overset{i}{\chi}(z)$.

Case (ii). Assume $\chi(z)$ vanishes on C, i.e. there is a point c on C such that $\chi^+(c) = \chi^-(c) = 0$. Note that if one of these vanishes the other does as $G(t)$ is non-singular. Consider now

$$\Phi(z) = \frac{\chi(z)}{z - c},$$

Clearly $\Phi(z)$ is a sectionally analytic function with continuous values on C except possibly at $z = c$. Now as $\chi^{\pm}(t)$ are Hölder on C

$$\Phi^+(t) = \frac{\overset{\circ}{\Phi}{}^+(t)}{|t-c|^\alpha}$$

$$0 \leqslant \alpha < 1$$

$$\Phi^-(t) = \frac{\overset{\circ}{\Phi}{}^-(t)}{|t-c|^\alpha}$$

where $\overset{\circ}{\Phi}{}^{\pm}(t)$ are continuous, and so equations (4.4), (4.6) will hold and so will equations (4.5) and (4.7), except possibly at $t = c$. Consequently $\Phi^-(t)$ will satisfy the I(F) for some $\gamma(t)$ except possibly at $t = c$. That is

$$\chi^-(t) - \frac{1}{2\pi i}\int_C \frac{G^{-1}(t)G(\tau) - I}{\tau - t}\chi^-(\tau)d\tau = \gamma(t),$$

and

$$\Phi^-(t) - \frac{1}{2\pi i}\int_C \frac{G^{-1}(t)G(\tau) - I}{\tau - t}\Phi^-(\tau)d\tau = \gamma_1(t),\ t \neq c.$$

However both these equations have the same kernel, viz.,

$$\frac{G^{-1}(t)G(\tau) - I}{\tau - t} = \frac{K(t,\tau)}{|\tau - t|^\mu},\quad 0 \leqslant \mu < 1,$$

where $K(t,\tau)$ is continuous, and thus they have the same resolvent $R(t,\tau)$ which is also of the form

$$R(t,\tau) = \frac{J(t,\tau)}{|t-c|^\mu},\quad 0 \leqslant \mu < 1,$$

and so

$$\Phi^-(t) = \gamma_1(t) + \frac{1}{2\pi i}\int_C R(t,\tau)\gamma_1(\tau)d\tau,$$

except possibly at $t = c$. But the R.H.S. of this equation is bounded for all $t \in C$, consequently $\Phi^-(c)$ is bounded. This implies that $\alpha = 0$ and hence $\Phi^-(t)$ is continuous at c and hence Hölder at c and hence on C. (This requires defining $\Phi^-(c) = \lim_{t \to c} \Phi^-(t)$). Thus $\Phi(z)$ is a solution of I and so the argument of case (i) now applies.

We now have the following properties for the $\overset{1}{\chi}(z), \overset{2}{\chi}(z), \ldots \overset{n}{\chi}(z)$:

Property 1.

$$\Delta(z) = \det | \overset{\beta}{\chi}_\alpha |$$

does not vanish anywhere in the finite part of the plane. If it did vanish at $z = c$ then the system

$$a_1\overset{1}{\chi}(c) + a_2\overset{2}{\chi}(c) \ldots + a_n\overset{n}{\chi}(c) = 0$$

would have a non-trivial solution for the a_i, which contradicts (4.10).

Property 2.

Set

$$\overset{\beta}{\chi}{}^{o}(z) = z^{\kappa_\beta}\overset{\beta}{\chi}(z), \quad (\beta = 1, 2, \ldots n),$$

then

$$\Delta^o(z) = \det | \overset{\beta}{\chi}{}^{o}_{\alpha}(z) |$$

is non-zero is non-zero and finite at ∞. It is clear that $\Delta^\circ(\infty)$ is finite. To show that it is non-zero suppose it is. This implies that non-trivial a_i can be found so that

$$a_1 z^{\kappa_1} \overset{1}{\chi}(z) + a_2 z^{\kappa_2} \overset{2}{\chi}(z) + \ldots + a_n z^{\kappa_n} \overset{n}{\chi}(z)$$
$$= 0\left(\frac{1}{z}\right),$$

as $z \to \infty$. Let a_k be the last non-zero a. Then

$$\chi(z) = a_1 z^{\kappa_1 - \kappa_k} \overset{1}{\chi}(z) + \ldots + a_k \overset{k}{\chi}(z)$$
$$= O(z^{-\kappa_k - 1})$$

Consequently $\chi(z)$ can be represented (Lemma (4.9)) by

$$\chi(z) = P_1 \overset{1}{\chi}(z) + \ldots + P_{k-1} \overset{k-1}{\chi}(z).$$

By equating these two expressions for $\chi(z)$ we have

$$\overset{k}{\chi}(z) = Q_1(z)\overset{1}{\chi}(z) + Q_2(z)\overset{2}{\chi}(z) + \ldots + Q_k(z)\overset{k}{\chi}(z),$$

which is not possible. It follows from property 2 that any expression of the form

$$P_1(z)\overset{1}{\chi}(z) + P_2(z)\overset{2}{\chi}(z) + \ldots + P_n(z)\overset{n}{\chi}(z)$$

has a degree at infinity equal to the degree of those terms which have the highest degree. i.e. max.$(m_1 - \kappa_1, m_2 - \kappa_2, \ldots, m_n - \kappa_n)$, thus the terms of higher degree cannot cancel. Substituting for $\overset{i}{\chi}(z)$ gives

$$\chi(z) = z^{-\kappa_1} P_1(z)\overset{1}{\chi}{}^{o}(z) + \ldots + z^{-\kappa_n} P_n(z)\overset{n}{\chi}{}^{o}(z),$$

and so near ∞

$$\chi(z) \sim z^{m_1 \kappa_1} \begin{bmatrix} \overset{1}{\chi}{}^{o}(\infty) \\ \cdot \\ \cdot \\ \cdot \\ \overset{1}{\chi}{}^{o}_{n}(\infty) \end{bmatrix} + \ldots + z^{m_n - \kappa_n} \begin{bmatrix} \overset{n}{\chi}{}^{o}_{1}(\infty) \\ \cdot \\ \cdot \\ \cdot \\ \overset{n}{\chi}{}^{o}_{n}(\infty) \end{bmatrix}.$$

Suppose that the highest order terms are $z^{m_1 - \kappa_1}$ and $z^{m_2 - \kappa_2}$ and that they cancel. This implies however that column 1 and column 2 are linearly dependent which is not so as det $|\overset{o}{\chi}(\infty)| \neq 0$.

Any n solutions of problem I having properties 1 and 2 will be called a **fundamental or canonical** system of solutions.

The matrix

$$X(z) = \begin{bmatrix} \overset{1}{X}_1 & \overset{2}{X}_1 & \cdots & \overset{n}{X}_1 \\ \overset{1}{X}_2 & \overset{2}{X}_2 & \cdots & \overset{n}{X}_2 \\ \cdot & & & \vdots \\ \cdot & & & \cdot \\ \cdot & & & \cdot\cdot \\ \overset{1}{X}_n & \overset{2}{X}_n & \cdots & \overset{n}{X}_n \end{bmatrix}$$

will be called the **fundamental or canonical** matrix for I.

Also as

$$X^+(t) = G(t)X^-(t),$$

it follows that

$$G(t) = X^+(t)[X^-(t)]^{-1}, \quad t \in C. \tag{4.21}$$

THEOREM (4.11) All solutions of I can be expressed as

$$\Phi(z) = P_1(z)\overset{1}{X}(z) + P_2(z)\overset{2}{X}(z) + \ldots P_n(z)\overset{n}{X}(z),$$

i.e.

$$\Phi(z) = X(z)P(z),$$

where

$$P(z) = \begin{bmatrix} P_1(z) \\ P_2(z) \\ \\ \\ P_n(z) \end{bmatrix}$$

is a vector of polynomials

PROOF. If $\Phi(z)$ is a solution, then

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in C.$$

From (4.2 i) this is

$$[X^+(t)]^{-1}\Phi^+(t) = [X^-(t)]^{-1}\Phi^-(t),$$

i.e. $[X(z)]^{-1}\Phi(z)$ is analytic in the entire plane and of finite degree at ∞ and so

$$[X(z)]^{-1}\Phi(z) = P(z), \text{ etc.}$$

Conversely it is also clear that

$$\Phi(z) = X(z)P(z)$$

is a solution of I.

COROLLARY A solution matrix X(z) possessing only property 1 is sufficient for this theorem, as the only fact used is that X(z) is non-singular in the finite part of the plane.

The integers $\kappa_1, \kappa_2, \kappa_3 \ldots \kappa_n$ are called the **component or partial indices** of I and their sum

$$\kappa = \kappa_1 + \kappa_2 + \ldots + \kappa_n$$

the **index** or **total index**.

THEOREM (4.12). The component indices are the same for all fundamental systems.
PROOF. Let $\overset{i}{\chi}(z)$ and $\zeta(z)$ denote two different fundamental systems. Then by the above theorem (4.11) we can write

$$\overset{i}{\chi}(z) = P_{i1}\overset{1}{\zeta}(z) + P_{i2}\overset{2}{\zeta}(z) + \ldots + P_{in}\overset{n}{\zeta}(z), \qquad \text{(a)}$$

and

$$\overset{i}{\zeta}(z) = Q_{i1}\overset{1}{\chi}(z) + Q_{i2}\overset{2}{\chi}(z) + \ldots + Q_{in}\overset{n}{\chi}(z), \qquad \text{(b)}$$

where the P_{ij} and Q_{ij} are polynomials. Now assume that the component indices for the $\overset{i}{\chi}(z)$ have been ordered so that $\kappa_1 \geqslant \kappa_2 \geqslant \ldots \geqslant \kappa_n$ and similarly for the $\overset{i}{\zeta}(z)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$.

Suppose that $\kappa_1 = \kappa_2 = \ldots = \kappa_k > \kappa_{k+1}$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_\ell > \lambda_{\ell+1}$

Now degree of the R.H.S. of (a) $\geqslant -\lambda_1$

$$\text{i.e. } -\kappa_1 \geqslant -\lambda_1,$$

and the degree of the R.H.S. of (b) $\geqslant -\lambda_1$,

$$-\lambda_1 \geqslant -\kappa_1,$$

i.e

$$\lambda_1 = \kappa_1.$$

Now we need to show $k = \ell$ from (b)

$$\overset{i}{\zeta}(z) = a_1\overset{1}{\chi}(z) + a_2\overset{2}{\chi}(z) + \ldots + a_k\overset{k}{\chi}(z),$$

for $i = 1,2,\ldots \ell$, and the a_k are constants. Suppose $\ell > k$ we could successively eliminate the $\overset{i}{\chi}(z)$ $i = 1,\ldots k$ to derive

$$b_1\overset{1}{\zeta}(z) + b_2\overset{2}{\zeta}(z) + \ldots + b_\ell\overset{\ell}{\zeta}(z) = 0,$$

which is clearly impossible. Similarly $k \ngtr \ell$ and so $k = \ell$. *The argument proceeds in this manner.*

THEOREM (4.13) The total index can be determined from G(t) directly.

PROOF. *Consider the scalar problem*

$$\Phi^+(t) = [\det G(t)]\,\Phi^-(t), \qquad t \in C.$$

Clearly

$$\Phi(z) = \Delta(z)$$

is a solution to this and

$$[\log \Delta^+(t)]_C = [\log \det G(t)]_C + [\log \Delta^-(t)]_C.$$

Now $\Delta(z)$ is analytic and non-zero in $D^+ + C$ and so

$$[\log \Delta^+(t)]_C = 0.$$

Furthermore $\Delta(z)$ is analytic and non-zero in D^- except possibly at ∞ where it has the form

$$\Delta(z) = \frac{\Delta^o(z)}{z^\kappa}$$

and so

$$[\log \Delta^-(t)]_C = -2\pi i\kappa.$$

Thus

$$\kappa = \frac{1}{2\pi i}[\log \det G(t)]_C. \qquad (4\ 22)$$

THEOREM (4.14). If $X(z)$ is a fundamental matrix for I. i.e,

$$\Phi^+(t) = G(t)\Phi^-(t),$$

then $[\tilde{X}(z)]^{-1}$ is a canonical matrix for I′, the associate problem, i.e.

$$\Psi^+(t) = [\tilde{G}(t)]^{-1}\Psi^-(t).$$

PROOF. From

$$X^+(t) = G(t)X^-(t)$$

we easily derive

$$[\tilde{X}^+(t)]^{-1} = [\tilde{G}(t)]^{-1}[\tilde{X}^-(t)]^{-1}$$

and so it is a solution. To establish it as a canonical solution we need:

Property 1. The determinant is non-vanishing in the finite plane. This is immediate, however, from

$$\det[X(z)]^{-1} = \frac{1}{\Delta(z)} \neq 0$$

Property 2. Write

$$[X(z)]^{-1} = [\overset{\beta}{\zeta}_\alpha(z)]$$

and so

$$\overset{\beta}{\zeta}_\alpha(z) = \frac{\overset{\beta}{\Delta}_\alpha(z)}{\Delta(z)}$$

where $\overset{\beta}{\Delta}_\alpha(z)$ is the co-factor of the element $\overset{\beta}{\chi}_\alpha$ in the determinant $\Delta(z)$. Consequently

$$\overset{\beta}{\zeta}_\alpha(z) = \frac{\overset{\beta}{\Delta}{}^{o}_\alpha(z)}{\Delta^{o}(z)}\, z^{\kappa_\beta}$$

and so the degree of $\overset{\beta}{\zeta}(z)$ at infinity is exactly κ_β. Also

$$\det|z^{-\kappa_\beta}\, \overset{\beta}{\zeta}_\alpha(z)| = \frac{1}{\Delta^{o}(z)}$$

which is non-zero at ∞. Q.E.D.

Corollary. If $\kappa_1, \kappa_2 \ldots \kappa_n$ are the component indices of G(t) then $-\kappa_1, -\kappa_2, \ldots, -\kappa_n$ are the component indices of $[\tilde{G}(t)]^{-1}$.

Examples. When the G(t) is a rational matrix, we can write

$$G_{ij}(t) = \frac{P_{ij}(t)}{q_{ij}(t)}$$

where $q_{ij}(t)$ does not vanish on C.

Write

$$G(t) = \frac{P(t)}{r(t)}$$

where P(t) is a matrix of polynomials and r(t) is a scalar polynomial. Further factor r(t) as

$$r(t) = r_+(t)r_-(t)$$

where $r_+(t)$ is a polynomial with no zeros in D^+

$r_-(t)$ is a polynomial with no zeros in D^-.

Thus the problem

$$\Phi^+(t) = G(t)\Phi^-(t) \qquad \text{(i)}$$

becomes

$$\Psi^+(t) = P(t)\Psi^-(t) \qquad \text{(ii)}$$

where

$$\left.\begin{aligned} \Psi(z) &= r_+(z)\Phi(z), \quad z \in D^+ \\ &= \frac{1}{r_-(z)}\Phi(z), \ z \in D^- \end{aligned}\right] \qquad \text{(iii)}$$

Now we know that det $G(t) \neq 0$ on C and so det $P(t) \neq 0$ on C For the moment assume that det $P \neq 0$ in D^+, then

$$\overset{k}{\Psi}(z) = P(z)\overset{k}{\gamma}, \ z \in D^+,$$

$$\overset{k}{\Psi}(z) = \overset{k}{\gamma}, \quad z \in D^-,$$

where

$$\overset{k}{\gamma} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow k, \\ \\ \\ \\ \\ \\ \end{matrix}$$

is a fundamental solution to (ii). Note as det $P \neq 0$ in D^+ (ii) has a zero total index

Consequently by (iii)

$$\overset{k}{\Phi}(z) = \frac{1}{r_+(z)} P(z)\overset{k}{\gamma}, \quad z \in D^+$$
$$= r_-(z)\overset{k}{\gamma}, \quad z \in D^-$$

represents a fundamental solution of (i).

Ex. (1)

$$G(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here $r(t) = r_+(t) = r_-(t) = 1$, i.e. $G(t) = P(t)$,

$$X(z) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, z \in D^+,$$

and

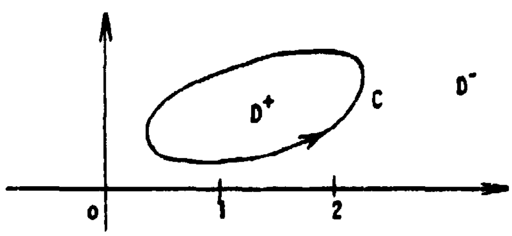
$$X(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, z \in D^-.$$

Note $\kappa = 0$.

Ex. (2)

$$G(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & t-1 & 0 \\ 0 & 0 & t-2 \end{bmatrix}$$

Again $r_+(t) = r_-(t) = 1$ and $P(t) = G(t)$. However C cannot contain 0,1,2.



In this case

$$X(z) = \begin{bmatrix} z & 0 & 0 \\ 0 & z-1 & 0 \\ 0 & 0 & z-2 \end{bmatrix}, \; z \in D^+,$$

and

$$X(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; z \in D^-.$$

Ex 3.

$$G(t) = \begin{bmatrix} \frac{t-1}{t+2} & t & \frac{t+1}{t+2} \\ 0 & t^2 & 0 \\ 0 & \frac{t-1}{t+1} & 1 \end{bmatrix}$$

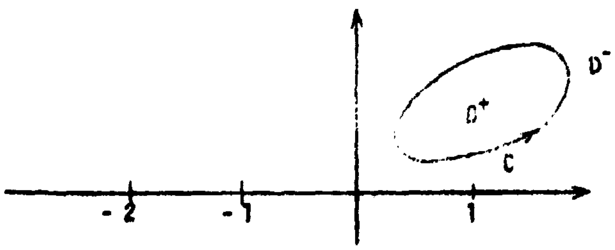
$$= \frac{1}{(t+1)(t+2)} \begin{bmatrix} (t-1)(t+1) & t(t+1)(t+2) & (t+1)^2 \\ 0 & t^2(t+1)(t+2) & 0 \\ 0 & (t-1)(t+2) & (t+1)(t+2) \end{bmatrix}$$

and so

$$r(t) = (t+1)(t+2)$$

with

$$\det P = t^2 (t-1)(t+1)^3 (t+2)^2.$$



For a contour not containing -2, -1, 0 and 1 we have

$$r_+ = (t+1)(t+2), \; r_- = 1$$

Thus

$$X(z) = \frac{1}{(z+1)(z+2)} \begin{bmatrix} (z-1)(z+1) & z(z+1)(z+2) & (z+1)^2 \\ 0 & z^2(z+1)(z+2) & 0 \\ 0 & (z-1)(z+2) & (z+1)(z+2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{z-1}{z+2} & z & \frac{z+1}{z+2} \\ 0 & z^2 & 0 \\ 0 & \frac{z-1}{z+1} & 1 \end{bmatrix}, z \in D^+$$

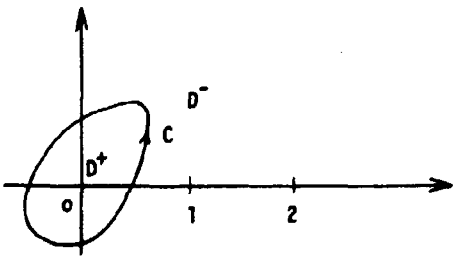
and

$$X(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, z \in D^-$$

Ex. 4

$$G(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & t-1 & 0 \\ 0 & 0 & t-2 \end{bmatrix}$$

where C encloses origin bu not 1 or 2.



We can easily show that the total index is 1.

Note

$$\Phi^+(t) = G(t)\,\Phi^-(t)$$

gives the 3 scalar problems

$$\begin{aligned} \overset{1}{\Phi}{}^+(t) &= t\overset{1}{\Phi}{}^-(t), \\ \overset{2}{\Phi}{}^+(t) &= (t-1)\overset{2}{\Phi}{}^-(t), \\ \overset{3}{\Phi}{}^+(t) &= (t-2)\overset{3}{\Phi}{}^-(t). \end{aligned}$$

Thus we can write

$$\overset{1}{\Phi}(z) = 1, \qquad z \in D^+,$$

$$= \frac{1}{z}, \qquad z \in D^-,$$

$$\overset{2}{\Phi}(z) = z - 1, \qquad z \in D^+,$$

$$= 1, \qquad z \in D,$$

$$\overset{3}{\Phi}(z) = z - 2, \qquad z \in D^+,$$

$$= 1, \qquad z \in D^-,$$

giving us

$$X(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z-1 & 0 \\ 0 & 0 & z-2 \end{bmatrix}, z \in D^+$$

$$= \begin{bmatrix} \frac{1}{z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, z \in D^-$$

with

$$\kappa_1 = 1, \ \kappa_2 = 0, \ \kappa_3 = 0,$$

Note if C includes 0 and 1 not 2

$$X(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z-2 \end{bmatrix}, z \in D^+, \ X(z) = \begin{bmatrix} \frac{1}{z} & 0 & 0 \\ 0 & \frac{1}{z-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, z \in D^-.$$

CHAPTER V

The Matrix Riemann Problem in Transport Theory

The method of singular-eigenfunction expansions, introduced by Case[6], has in recent years been utilized[7-12], in the degenerate-kernel approximation of the energy-dependent transport equation, or the multi-group transport equations[7] A crucial point in its use is the proof of the basic completeness theorem, i e the demonstration of the expansion properties of the elementary functions Generally this is based on the reduction of a system of singular-integral equations to a matrix Riemann problem of the form,

$$\underset{\sim}{\Phi}^{+}(t) = \underset{\sim}{G}(t)\,\underset{\sim}{\Phi}^{-}(t) + \underset{\sim}{\kappa}(t),\ t \in L, \tag{5.1}$$

were L is an interval on the real line, $\underset{\sim}{G}(t)$ and $\underset{\sim}{\kappa}(t)$ are given n x n matrices, and $\underset{\sim}{\Phi}^{\pm}(t)$ are limiting values of the n x n sectionally analytic matrix $\underset{\sim}{\Phi}(z)$ to be determined, i.e.

$$\underset{\sim}{\Phi}^{\pm}(t) = \lim_{y \to 0^{+}} \underset{\sim}{\Phi}(t \pm iy)$$

The interesting cases are the half-ranges[7,9,11,12], or the two half-space applications[10], for which we require a so-called solution of

$$\underset{\sim}{\Phi}^{+}(t) = \underset{\sim}{G}(t)\,\underset{\sim}{\Phi}^{-}(t),\ t \in L, \tag{5.2a}$$

where the matrix $\underset{\sim}{G}(t)$ is of the form

$$\underset{\sim}{G}(t) = \underset{\sim}{\Lambda}^{+}(t)\,[\,\underset{\sim}{\Lambda}^{-}(t)\,]^{-1}, \tag{5.2b}$$

with

$$\underset{\sim}{\Lambda}(z) = \underset{\sim}{I} + z\int_{0}^{1} \underset{\sim}{\Psi}(t)\,\frac{dt}{t-z} \tag{5.3}$$

Here $\underset{\sim}{\Psi}(t)$ is a real function of t, and, without loss of generality, we take L to be the interval (0,1) on the real line If $\underset{\sim}{G}(t)$ is continuous and non-singular on L, with $\underset{\sim}{G}(0) = \underset{\sim}{G}(1) = \underset{\sim}{I}$, then as explained in [12] the theory of Mandžavidze and Hvedelidze[13] guarantees the existence of a canonical solution $\underset{\sim}{\Phi}(z)$ to the boundary-value problem. We use the term canonical to mean that det $\underset{\sim}{\Phi}(z) \neq 0$ in the finite plane In addition to this property a canonical solution may also be of ordered normal form at infinity, i e it satisfies the equation

$$\lim_{|z| \to \infty} \underset{\sim}{\Phi}(z)\underset{\sim}{Z}^{\kappa} = \underset{\sim}{\Phi}^{(0)}, \tag{5.4}$$

$$\underset{\sim}{Z}^{\kappa} = \text{diag}\ \{z^{\kappa_1}, z^{\kappa_2}, \ldots, z^{\kappa_n}\},$$

where $\underset{\sim}{\Phi}^{(0)}$ is a non-singular constant matrix and where the integers $\{\kappa_i\}_{i=1}^{n}$ are the partical indices[13], of the problem. We will suppose that the partial indices are ordered so that $\kappa_1 \geqslant \kappa_2 \geqslant \ldots \geqslant \kappa_n$. It is readily shown, see for example[7,12], that proving half-range completeness is equivalent to showing that the κ_i are non-negative [Note that $\kappa = \sum_{i=1}^{n} \kappa_i$ is calculable from the result $\kappa = \frac{1}{\pi}\,[\arg \det \underset{\sim}{G}(t)]_0^1$, where $[\ ]_0^1$ denotes the change as t proceeds from 0 to 1] One purpose of the present note is to show, for the general n x n Riemann problem (5 2) that a canonical solution of ordered normal form at infinity exists which satisfies

$$\underset{\sim}{\Phi}(z) = \bar{\underset{\sim}{\Phi}}(z) \equiv \overline{\underset{\sim}{\Phi}(\bar{z})} \tag{5.5}$$

We begin by citing some straight forward lemmas:

Lemma 1. If $\underset{\sim}{\Phi}(z)$ is a solution of the Riemann problem (5 2) then so is $\bar{\underset{\sim}{\Phi}}(z)$

Lemma 2. If $\underset{\sim}{\Phi}(z)$ is a canonical solution of the Riemann problem (5 2) then so is $\underset{\sim}{\Phi}(z)\underset{\sim}{P}(z)$, where $\underset{\sim}{P}(z)$ is an elementary matrix, i e a matrix of polynomials with det $P(z) = \text{const}$ $(\neq 0)$.

Lemma 3 If $\Phi(z)$ is a canonical solution of ordered normal form at infinity of the Riemann problem (5 2) then so is $\underset{\sim}{\Phi}(z)\ P(z)$ where $P(z)$ is an elementary matrix whose elements satisfy the following conditions:

(a) if $i < j$, degree of $P_{ij}(z) \leqslant \kappa_i - \kappa_j$,

(b) if $i = j$, $P_{ij}(z) = \text{const}$,

(c) if $i > j$, $P_{ij}(z) = 0$, $\kappa_i > \kappa_j$,

$= \text{const.}$, $\kappa_i = \kappa_j$.

For convenience we shall call such elementary matrices **allowable.** We now proceed to our main result:

Theorem (5.1). For the Riemann problem (5.2) there exists a canonical solution of ordered normal form at infinity, such that

$$\bar{\underset{\sim}{\Phi}}(z) = \underset{\sim}{\Phi}(z)$$

PROOF. The proof is constructive. Now for large $|z|$ we have

$$\underset{\sim}{\Phi}(z) \sim \{\ \underset{\sim}{\Phi}^{(0)} + \frac{1}{z}\underset{\sim}{\Phi}^{(1)} + \frac{1}{z^2}\ \underset{\sim}{\Phi}^{(2)} + \ldots\ \}\ \underset{\sim}{Z}^{-\kappa}, \tag{5.6}$$

where $\underset{\sim}{Z}^{-\kappa} = \text{diag}\ \{z^{-\kappa_1}, z^{-\kappa_2}, \ldots, z^{-\kappa_n}\}$, and the $\underset{\sim}{\Phi}^{(i)}$, $i \geqslant 0$ are constants matrices with det $\Phi^{(0)} \neq 0$. Now as $\underset{\sim}{\Phi}^{(0)}$ is non-singular there is at least one non-zero element in its first column. We take the first such element. $\Phi^{(0)}_{\ell_1}$ say and without loss of generality assume it is unity, (if $\underset{\sim}{\Phi}^{(0)}_{\ell_1} \neq 1$ we may divide the first column of $\Phi(z)$ by $\Phi^{(0)}_{\ell_1}$). Now by multiplying $\underset{\sim}{\Phi}(z)$ by a suitable allowable elementary matrix the remaining columns of $\Phi(z)$ may be modified so that

$$\Phi^{(i)}_{\ell j} = 0, 0 \leqslant i \leqslant \kappa_1 - \kappa_j, 2 \leqslant j \leqslant n, \tag{5.7}$$

(To avoid the introduction of further notation we will still call the modified solution $\underset{\sim}{\Phi}(z)$.) We now select the first non-zero element $\Phi^{(0)}_{m2}$ ($m \neq \ell$) in the second column of $\underset{\sim}{\Phi}^{(0)}$ and assume it is unity. The succeeding columns are now modified, again by multiplying by a suitable allowable elementary matrix so that,

$$\Phi^{(i)}_{mj} = 0,\ 0 \leqslant i \leqslant \kappa_2 - \kappa_j,\ 3 \leqslant j \leqslant n. \tag{5.8}$$

This processes is repeated for the remaining columns. If some of the κ_i are equal, say $\kappa_p = \kappa_{p+1} = \ldots = \kappa_q$, then in addition to the above steps we modify $\underset{\sim}{\Phi}(z)$ so that if the first unit element in the n-th column of $\underset{\sim}{\Phi}^{(0)}$, $p \leqslant r \leqslant q$, is $\Phi^{(0)}_{sr}$ then $\Phi^{(0)}_{sm} = 0$, $p \leqslant m \leqslant r$. Thus $\underset{\sim}{\Phi}(z)$ is now such that $\underset{\sim}{\Phi}^{(0)}$ is the sum of a permutation matrix $\underset{\sim}{P}$ and a singular matrix S such that if $P_{ij} = 1$ then $S_{pj} = 0, 1 \leqslant p \leqslant i$ and $S_{iq} = 0$, $j \leqslant q \leqslant n$. Some of the remaining $\underset{\sim}{\Phi}^{(i)}$, $i \geqslant 1$ will also have, in general, certain zero entries. Our claim is that the $\underset{\sim}{\Phi}(z)$ so constructed satisfies the equation $\bar{\underset{\sim}{\Phi}}(z) = \underset{\sim}{\Phi}(z)$.

Consider the following matrix,

$$\underset{\sim}{\Psi}(z) = \underset{\sim}{\Phi}(z) - \bar{\underset{\sim}{\Phi}}(z). \tag{5.9}$$

Clearly $\underset{\sim}{\Psi}(z)$ is a sectionally analytic solution of the Riemann problem (2) and so may be written[13], as

$$\underset{\sim}{\Psi}(z) = \underset{\sim}{\Phi}(z)\,\underset{\sim}{P}(z) \tag{5.10}$$

where $\underset{\sim}{P}(z)$ is a matrix of polynomials. It is now a straightforward matter to argue that $\underset{\sim}{P}(z) = \underset{\sim}{0}$. Let $\underset{\sim}{\Psi}_r(z)$ and $\underset{\sim}{\Phi}_r(z)$ denote the r-th columns of $\underset{\sim}{\Psi}(z)$ and $\underset{\sim}{\Phi}(z)$ respectively, so that from (10)

$$\underset{\sim}{\Psi}_r(z) = \sum_{j=1}^{n} \underset{\sim}{\Phi}_j P_{jr} \tag{5.11}$$

We now examine $\underset{\sim}{\Psi}_r(z)$, noting that only two cases may arise.

<u>(i) $\kappa_1 > \kappa_2$</u>. From (5.11) it is clear that $P_{j1} = 0, 2 \leqslant j \leqslant n$ and so

$$\underset{\sim}{\Psi}_1(z) = \underset{\sim}{\Phi}_1 P_{11},$$

where P_{11} is a constant. On recalling that $\Phi^{(0)}_{\ell_1} = 1$ and $\Psi^{(0)}_{\ell_1} = 0$ it follows that $P_{11} = 0$, and so $\underset{\sim}{\Psi}_1(z) = \underset{\sim}{0}$.

<u>(ii) $\kappa_1 = \kappa_2 = \ldots = \kappa_q$</u>. Here, we have that $P_{jr} = 0$, $q+1 \leqslant j \leqslant n$, $1 \leqslant r \leqslant q$, and so

$$\underset{\sim}{\Psi}_r(z) = \sum_{j=1}^{q} \underset{\sim}{\Psi}_j(z) P_{jr},\ 1 \leqslant r \leqslant q,$$

where again the P_{jr} are constants. However, by reasoning similar to that used in (i), it is evident that $P_{jr} = 0, 1 \leqslant j \leqslant q$, $1 \leqslant r \leqslant q$, so that $\underset{\sim}{\Psi}_r(z) \equiv \underset{\sim}{0}, 1 \leqslant r \leqslant q$.

By proceeding in this manner for the remaining columns of $\underset{\sim}{\Psi}(z)$ we deduce that indeed $\underset{\sim}{P}(z) = 0$.

We are now in a position to show that the partial indices are non-negative:

THEOREM (5.2). The partial indices for the Riemann problem (5.2) (5.3), where in addition $\underset{\sim}{\Psi}(t)$ is symmetric, are non-negative.

PROOF. We first observe that as $\underset{\sim}{\Psi}(t)$ is symmetric so is $\underset{\sim}{\Lambda}(z)$. In addition it is also clear that $\underset{\sim}{\Lambda}(z) = \underset{\sim}{\Lambda}(-z)$. By making use of these properties it can be shown that if $\underset{\sim}{\Phi}(z)$ is a canonical solution of the Riemann problem (5.2) then the matrix $\underset{\sim}{\Lambda}(z)\, \underset{\sim}{\tilde{\Phi}}^{-1}(-z)$ is also a solution, where $\underset{\sim}{\tilde{\Phi}}$ denotes the transpose of $\underset{\sim}{\Phi}$. Consequently we may write

$$\underset{\sim}{\Lambda}(z) = \underset{\sim}{\Phi}(z)\underset{\sim}{P}(z)\, \underset{\sim}{\tilde{\Phi}}(-z) \tag{5.12}$$

where $\underset{\sim}{P}(z)$ is a matrix of polynomials. We now choose $\underset{\sim}{\Phi}(z)$ to be of ordered normal form at infinity possessing the Schwarz reflection property.

Suppose now that $\kappa_n < 0$. By examining the Laurent series of both sides of equation (5.12) near the point at infinity, it is clear that this assumption leads to a contradiction unless $P_{nn}(z) \equiv 0$. However from equations (5.3) and (5.12),

$$\underset{\sim}{P}(0) = \underset{\sim}{\Phi}^{-1}(0)\, \underset{\sim}{\tilde{\Phi}}^{-1}(0),$$

so that as $\underset{\sim}{\Phi}(0)$ is real and non-singular $P_{nn}(0) \neq 0$. Thus $\kappa_n \not< 0$.

RESUMO

Dentre os vários processos de resolução da equação de transporte linear, o método de Case de expansão singular e considerado o mais elegante das soluções analíticas. Apesar desta técnica ter sido aplicada por muitos pesquisadores em vários problemas e a sua solução possibilitar um tratamento numérico com alto grau de acuidade, ela requer conhecimentos de matemática não convencionais

Neste relatório, são apresentados os conceitos fundamentais e os teoremas matemáticos requeridos.

É feita um rápido recordação da teoria de funções de variáveis complexas seguidas da definição da integral do valor-principal de Cauchy e os problemas de contorno de Riemann para uma função, são apresentados de maneira sucinta Como aplicação da teoria aqui desenvolvida, são encontradas as soluções analíticas de uma classe de equações transcedentais. São discutidos, também, os sistemas de equações integrais singulares e os problemas de matriz de Riemann requeridos no modelo multi-grupo da teoria de transporte. Finalmente, como exemplo, um problema típico de aplicação da teoria de um grupo é resolvido em detalhes

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