THE MILNE PROBLEM FOR TWO ADJACENT HALF-SPACES IN THE THEORY OF NEUTRON DIFFUSION WITH ANISOTROPIC SCATTERING

C. E. Siewert and Yuji Ishiguro

PUBLICAÇÃO IEA 409 CEN 57	MARÇO/1976

PUBL. IEA 400
CEN 57

MARÇO/1976

THE MILNE PROBLEM FOR TWO ADJACENT HALF-SPACES IN THE THEORY OF NEUTRON DIFFUSION WITH ANISOTROPIC SCATTERING

C. E. Siewert and Yuji Ishiguro

COORDENADORIA DE ENGENHARIA NUCLEAR
(CEN)

INSTITUTO DE ENERGIA ATÔMICA
SÃO PAULO – BRASIL

APROVADO PARA PUBLICAÇÃO EM OUTUBRO/1975

INSTITUTO DE ENERGIA ATÔMICA
Caixa Postal 11.049 (Pinheiros)
Cidade Universitária "Armando de Salles Oliveira"
SÃO PAULO — BRASIL

NOTA: Este trabalho foi conferido pelo autor depois de composto e sua redação está conforme o original, sem qualquer correção ou mudança.

THE MILNE PROBLEM FOR TWO ADJACENT HALF-SPACES IN THE THEORY OF NEUTRON DIFFUSION WITH ANISOTROPIC SCATTERING

C. E. Siewert and Yuji Ishiguro

ABSTRACT

The elementary solutions of the one-speed neutron transport equation with linearly anisotropic scattering are used in conjunction with Chandrasekhar's invariance principles to solve in a concise manner the Milne problem for two adjacent half-spaces.

I – INTRODUCTION

An early paper by Kuszell[7] was the first to use the newly developed, by Case[2], theory of elementary solutions of the neutron transport equation to study neutron diffusion for problems defined by the presence of two dissimilar media. The early work of Kuszell[7] was limited to the case of isotropic scattering and was left in a rather cumbersome final form. Later work by Mendelson and Summerfield[11] also contributed to the general area of multi-region problems; however, it is to the basic work of McCormick[8] and McCormick and Doyas[9] that we must look for the most significant contributions to two media problems with the effects of anisotropic scattering included.

It is, of course, well known that the treatise by Chandrasekhar[5] established a vast amount of knowledge about the equation of transfer, for radiative heat-transfer applications. Chandrasekhar[5] developed extensively the invariance principles associated with the equation of transfer for anisotropic scattering, as well as for isotropic scattering. Today in the field of neutron transport theory many researchers consider the fundamental paper published by Case[2] in 1960 to be the cornerstone of the theory of "exact" solutions. In 1969 Pahor and Zweifel[13] in an elegantly written paper demonstrated how the work of Chandrasekhar[5] and Case[2] could be coupled and utilized at the same time to obtain in a profitable and concise manner certain results for a variety of single medium (semi-infinite or finite slabs) problems.

In an early important, but not so well known, paper Chandrasekhar[4] put forward the idea of using the S function, a consequence of his principles of invariance, to study adjacent half-space problems. This idea has been used recently by Ishiguro and Maiorino[6] to investigate several problems based on the two-group model; the idea has also been used by Siewert and Burkart[14] for the critical reactor problem for a reflected slab. Though the paper by Siewert and Burkart[14] is thought to illustrate the advantages of using the S function for problems involving two dissimilar media, we shall see here more dramatically the considerable improvement, over earlier efforts, the use of the S function provides.

II – FORMULATION OF THE MILNE PROBLEM

Let us consider the one-speed neutron transport equations for region 1, $x > 0$, and region 2, $x < 0$, written in the familiar manner

$$\mu \frac{\partial}{\partial x} \Psi_\alpha (x,\mu) + \Psi_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} \Psi_\alpha(x,\mu')(1 + b_\alpha \mu\mu') d\mu' \tag{1}$$

Here $\Psi_\alpha(x,\mu)$ denotes the neutron angular density in region α, as a function of position (in optical units) x and the direction cosine of the propagating neutrons, μ. In addition, c_α denotes the mean number of secondaries per collision in region α, and b_α is the coefficient of anisotropic scattering.

For the considered Milne problem, we seek a diverging (as $x \to \infty$) solution of Eq. (1) such that

$$\text{(i)} \quad \lim_{x \to +\infty} \Psi_1(x,\mu)\exp(-x/\nu_0) < \infty, \tag{2a}$$

$$\text{(ii)} \quad \lim_{x \to -\infty} \Psi_2(x,\mu) = 0 \tag{2b}$$

and

$$\text{(iii)} \quad \Psi_1(0,\mu) = \Psi_2(0,\mu), \quad \mu \in (-1,1) \tag{2c}$$

Here ν_0 denotes the discrete eigenvalue in region 1. We assume in this work that $c_1 < 1$ and $c_2 < 1$; but clearly with some slight modifications in Eqs. (2), the problem (and solution) will be meaningful in the limits $c_1 \to 1$ and/or $c_2 \to 1$.

Relying on the basic work of McCormick and Kuščer[10], we can immediately write solutions to Eq. (1) that satisfy the boundary conditions listed as Eqs. (2a) and (2b):

$$\Psi_1(x,\mu) = A(\nu_0)\phi_1(\nu_0,\mu)\exp(-x/\nu_0) + \phi_1(-\nu_0,\mu)\exp(x/\nu_0) + \int_0^1 A(\nu)\phi_1(\nu,\mu)\exp(-x/\nu)d\nu \tag{3}$$

and

$$\Psi_2(x,\mu) = B(-\eta_0)\phi_2(-\eta_0,\mu)\exp(x/\eta_0) + \int_0^1 B(-\eta)\phi_2(-\eta,\mu)\exp(x/\eta)d\eta \tag{4}$$

Here we are using the elementary solutions written for region 1 as

$$\phi_1(\nu_0,\mu) = \frac{c_1\nu_0}{2} R_1(\nu_0,\mu)\frac{1}{\nu_0 - \mu}, \tag{5}$$

where

$$R_1(x,y) = 1 + \ell_1 xy, \text{ with } \ell_1 = b_1(1 - c_1), \tag{6}$$

$\nu_0 \notin [-1,1]$ is the positive zero of

$$\Lambda_1(z) = 1 - c_1 z R_1(z,z)\tanh^{-1}\frac{1}{z} + c_1\ell_1 z^2, \tag{7}$$

and

$$\phi_1(\nu,\mu) = \frac{c_1\nu}{2} R_1(\nu,\mu)\frac{P}{\nu - \mu} + \lambda_1(\nu)\delta(\nu - \mu), \tag{8}$$

with

$$\lambda_1(\nu) = 1 - c_1\nu R_1(\nu,\nu)\tanh^{-1}\nu + c_1\ell_1\nu^2 \tag{9}$$

The symbol P is used to denote the principal-value functional, and $\delta(x)$ is Dirac's functional

In a similar manner, we have for region 2 the following:

$$\phi_2(\eta_o,\mu) = \frac{c_2\eta_o}{2} R_2(\eta_o,\mu) \frac{1}{\eta_o - \mu}, \tag{10}$$

where

$$R_2(x,y) = 1 + \ell_2 xy, \text{ with } \ell_2 = b_2(1 - c_2), \tag{11}$$

and

$$\Lambda_2(\eta_o) = 1 - c_2\eta_o R_2(\eta_o,\eta_o)\tanh^{-1}\frac{1}{\eta_o} + c_2\ell_2\eta_o^2 = 0; \tag{12}$$

also

$$\phi_2(\eta,\mu) = \frac{c_2\eta}{2} R_2(\eta,\mu) \frac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta - \mu), \tag{13}$$

where

$$\lambda_2(\eta) = 1 - c_2\eta R_2(\eta,\eta)\tanh^{-1}\eta + c_2\ell_2\eta^2 \tag{14}$$

III – BASIC ANALYSIS

Since the solutions given by Eqs. (3) and (4) inherently satisfy Eqs. (2a) and (2b), we need simply to constrain the solutions to obey Eq. (2c), which we choose to write as

$$\Psi_1(0,\mu) = \Psi_2(0,\mu),\ \mu\epsilon(0,1), \tag{15a}$$

and

$$\Psi_1(0,-\mu) = \Psi_2(0,-\mu),\ \mu\epsilon(0,1) \tag{15b}$$

At this point, we can use Chandrasekhar's S function[5] to write

$$\Psi_2(0,\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\Psi_2(0,-\mu')d\mu',\ \mu\in(0,1), \tag{16}$$

where

$$S_2(\mu',\mu) = \frac{c_2\mu\mu'}{\mu+\mu'}[1 - c_2(\mu+\mu') - \ell_2\mu\mu']H_2(\mu')H_2(\mu) \tag{17}$$

Here $H_2(\mu)$ is the H function for region 2; since we shall also need $H_1(\mu)$, we list

$$H_\alpha(\mu) = \frac{1+\mu}{(\xi_\alpha + \mu)\sqrt{1 - c_\alpha - \frac{1}{3} c_\alpha \ell_\alpha}} \exp\left[-\frac{1}{\pi}\int_0^1 \tan^{-1}\left(\frac{\pi c_\alpha x R_\alpha(x,x)}{2\lambda_\alpha(x)}\right)\frac{dx}{x+\mu}\right], \tag{18}$$

where $\xi_1 = \nu_0$ and $\xi_2 = \eta_0$. Also required to define $S_2(\mu', \mu)$ are the constants

$$\dot{c}_2 = \frac{c_2 \ell_2 \alpha_{2,1}}{2 - c_2 \alpha_{2,0}}, \quad \alpha_{\beta,\gamma} = \int_0^1 H_\beta(\mu)\,\mu^\gamma\,d\mu \tag{19}$$

If we now enter Eqs. (15) into Eq. (16), we obtain

$$\Psi_1(0,\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\Psi_1(0,-\mu')d\mu', \; \mu\in(0,1) \tag{20}$$

We consider that Eq. (20) is the basic equation now to be satisfied, since if the expansion coefficients $A(\nu_0)$ and $A(\nu)$ can be chosen such that eq. (20) is obeyed, $B(-\eta_0)$ and $B(-\eta)$, if desired, can readily be obtained from Eqs. (15) by utilizing the available [3] half-range completeness and orthogonality theorems for region 2. On substituting Eq. (3) into Eq. (20), we find that the integral over μ' can be evaluated by invoking several of the H-function identities given by Chandrasekhar[5]. Because the algebra involved in evaluating the integral over μ' is tedious for the case of anisotropic scattering, we shall list here only the result obtained; however, the interested reader will find in Appendix A of this report, a detailed development for the case of isotropic scattering. Thus

$$\frac{A(\nu_0)}{H_2(\nu_0)}\left[\phi_1(\nu_0,\mu) - W(\nu_0)\right] + \int_0^1 \frac{A(\nu)}{H_2(\nu)}\left[\phi_1(\nu,\mu) - W(\nu)\right]d\nu$$

$$= \frac{1}{H_2(-\nu_0)}\left[\phi_1(-\nu_0,\mu) - W(-\nu_0)\right], \; \mu\in(0,1), \tag{21}$$

where

$$W(\xi) = \frac{1}{2}\,\frac{c_1 \xi}{R_2(\xi,\xi)}\left[\xi(\ell_1 - \ell_2)(q_2 H_2(\xi) - 1) - \dot{c}_2 R_1(\xi,\xi)\right], \tag{22}$$

with

$$q_2 = \frac{1 - c_2}{1 - \frac{1}{2} c_2 \alpha_{2,0}}, \tag{23}$$

Equation (21) clearly is a singular integral equation that can be regularized by using the well known[3] half-range orthogonality relations for one medium (in this case, region 1). Thus, if we multiply Eq. (21) by $\mu H_1(\mu)\tilde{\phi}_1(\nu_0,\mu)$, where

$$\tilde{\phi}_1(\xi,\mu) = \phi_1(\xi,\mu) + \frac{c_1 \xi}{2}\,\dot{c}_1, \tag{24}$$

and integrate over μ from zero to one, we can use the orthogonality relations summarized in Appendix B to obtain

$$\frac{A(\nu_0)}{H_2(\nu_0)}[N_1(\nu_0)H_1(\nu_0) - \nu_0\dot{q}_1 W(\nu_0)] - \nu_0\dot{q}_1\overline{A}$$

$$= -\frac{1}{H_2(-\nu_0)}[J(-\nu_0,\nu_0) - \nu_0\dot{q}_1 W(-\nu_0)], \tag{25}$$

where

$$\overline{A} = \int_0^1 \frac{A(\nu)}{H_2(\nu)} W(\nu)d\nu, \tag{26}$$

$$N_1(\nu_0) = \frac{c_1\nu_0^2}{2} R_1(\nu_0,\nu_0)[\frac{c_1 R_1(\nu_0,\nu_0)}{\nu_0(\nu_0^2-1)} - \frac{(1-c_1)R_1(3\nu_0,\nu_0)}{\nu_0 R_1(\nu_0,\nu_0)}], \tag{27}$$

and

$$J(-\nu_0,\nu_0) = -\frac{c_1\nu_0}{4}\frac{1}{H_1(\nu_0)}[1-\ell_1\nu_0^2+2\nu_0\hat{c}_1], \tag{28}$$

with

$$\dot{q}_1 = \frac{1-c_1}{1-\frac{1}{2}c_1\alpha_{1,0}} \text{ and } \hat{c}_1 = \frac{c_1\ell_1\alpha_{1,1}}{2-c_1\alpha_{1,0}} \tag{29}$$

In a similar manner, we can multiply Eq. (21) by $\mu H_1(\mu)\widetilde{\phi}_1(\nu',\mu)$, $\nu'\epsilon(0,1)$, and integrate over μ to find (again after utilizing Appendix B)

$$\frac{A(\nu_0)}{H_2(\nu_0)}\nu'\dot{q}_1 W(\nu_0) + \frac{A(\nu')}{H_2(\nu')}N_1(\nu')H_1(\nu') - \nu'\dot{q}_1\overline{A} = -\frac{1}{H_2(-\nu_0)}$$

$$[J(-\nu_0,\nu') - \nu'\dot{q}_1 W(-\nu_0)], \nu'\in(0,1) \tag{30}$$

Here

$$N_1(\nu) = \nu[\{\lambda_1(\nu)\}^2 + \{\frac{c_1\nu\pi}{2} R_1(\nu,\nu)\}^2] \tag{31}$$

and

$$J(-\nu_0,\nu) = -\frac{c_1\nu_0\nu}{2(\nu_0+\nu)H_1(\nu_0)}\{1-\ell_1\nu_0\nu+\dot{c}_1(\nu_0+\nu)\} \tag{32}$$

Eliminating $\overline{A}$ between Eqs. (25) and (30), we can write

$$\frac{A(\nu)}{H_2(\nu)}[\frac{N_1(\nu)H_1(\nu)}{\nu}] = \frac{A(\nu_0)}{H_2(\nu_0)}[\frac{N_1(\nu_0)H_1(\nu_0)}{\nu_0}] - \frac{c_1(\nu_0-\nu)}{4(\nu_0+\nu)}\frac{R_1(\nu_0,\nu_0)}{H_2(-\nu_0)H_1(\nu_0)} \tag{33}$$

If now we rearrange Eq. (33) and multiply by $W(\nu)$, we can integrate to find

$$\bar{A} = -\frac{c_1 K_1 R_1(\nu_0, \nu_0)}{4H_1(\nu_0)H_2(-\nu_0)} + \frac{A(\nu_0)H_1(\nu_0)N_1(\nu_0)K_2}{\nu_0 H_2(\nu_0)}, \tag{34}$$

where the two constants K_1 and K_2 are given by

$$K_1 = \int_0^1 \frac{\nu W(\nu)(\nu_0 - \nu)}{N_1(\nu)H_1(\nu)(\nu_0 + \nu)} d\nu. \tag{35}$$

and

$$K_2 = \int_0^1 \frac{\nu W(\nu)}{N_1(\nu)H_1(\nu)} d\nu. \tag{36}$$

Eq. (34) can be substituted into Eq. (26) to yield $A(\nu_0)$; that result, along with Eq. (34), can subsequently be used in Eq. (30) to give $A(\nu)$:

$$A(\nu_0) = -\frac{H_2(\nu_0)}{H_2(-\nu_0)} \frac{[c_1\nu_0\{1 - \ell_1\nu_0^2 + 2\nu_0\hat{c}_1 + \hat{q}_1 K_1 R_1(\nu_0,\nu_0)\} - 4\nu_0 H_1(\nu_0)\hat{q}_1 W(-\nu_0)]}{[4N_1(\nu_0)H_1(\nu_0)H_1(\nu_0)(1 - \hat{q}_1 K_2) - 4\nu_0 H_1(\nu_0)\hat{q}_1 W(\nu_0)]}, \tag{37}$$

and

$$A(\nu) = \frac{\nu H_2(\nu)}{N_1(\nu)H_1(\nu)}\left[\frac{A(\nu_0)H_1(\nu_0)N_1(\nu_0)}{\nu_0 H_2(\nu_0)} - \frac{c_1(\nu_0 - \nu)}{4(\nu_0 + \nu)} \frac{R_1(\nu_0,\nu_0)}{H_2(-\nu_0)H_1(\nu_0)}\right] \tag{38}$$

Equations (37) and (38) are explicit expressions for the expansion coefficients $A(\nu_0)$ and $A(\nu)$; as previously mentioned the coefficients $B(-\eta_0)$ and $B(-\eta)$ can now be easily established, should they be required. Though our final results for $A(\nu_0)$ and $A(\nu)$ were obtained very differently in appearance from those of McCormick[8] and McCormick and Doyas[9], they are similar in that there appear extra terms, in our case, W terms, in regard to either the isotropic scattering case, or the single medium result, $c_2 \to 0$.

We note that the neutron density in region 1 is given by

$$\rho_1(x) = \int_{-1}^{1} \Psi_1(x,\mu)d\mu = A(\nu_0)e^{-x/\nu_0} + e^{x/\nu_0} + \int_0^1 A(\nu)e^{-x/\nu}d\nu. \tag{39}$$

Also, an asymptotic solution of the Milne problem can be written as

$$\Psi_{1_a}(x,\mu) = A(\nu_0)\phi_1(\nu_0,\mu)e^{-x/\nu_0} + \phi_1(-\nu_0,\mu)e^{x/\nu_0} \tag{40}$$

or

$$\rho_{1_a}(x) = \int_{-1}^{1} \Psi_{1_a}(x,\mu)d\mu = A(\nu_0)e^{-x/\nu_0} + e^{x/\nu_0}. \tag{41}$$

Thus if we write

$$A(\nu_0) = -e^{-2z_0/\nu_0} \tag{42}$$

or

$$z_o = -\frac{\nu_o}{2}\,\ell n\,[-A(\nu_o)], \qquad (43)$$

then Eq. (41) can be written as

$$\rho_{l_a}(x) = e^{x/\nu_o} - e^{-(x+2z_o)/\nu_o}. \qquad (44)$$

It is therefore clear that z_o is the extrapolated endpoint for the considered Milne problem.

IV – NUMERICAL RESULTS

In this section, we would like to list the results obtained by numerically evaluating the expressions given for the solution of the Milne problem. The required H functions were established by solving iteratively the non-linear integral equation; all integrals that could not be evaluated analytically were represented by a Gaussian quadrature scheme.

In Table I, we list some typical values of the Milne extrapolated endpoint, as defined by Eqs. (37) and (43), and in Figures I-IV, we plot the density $\rho(x)$ and the asymptotic density $\rho_{l_a}(x)$, as given by Eqs. (39) and (41). Our values are in general agreement with McCormick and Doyas[9] and Burkart[1].

Table I

Milne Results

c_1	b_1	c_2	b_2	ν_o	z_o
0.9	0.8	0.4	0.8	2.18132124	1.2206286
0.9	0.8	0.4	0.4	2.18132124	1.2636065
0.9	0.8	0.2	0.8	2.18132124	1.1134772
0.9	0.8	0.2	0.4	2.18132124	1.1312096
0.9	0.4	0.4	0.8	2.02796762	1.0422944
0.9	0.4	0.4	0.4	2.02796762	1.0777815
0.9	0.4	0.2	0.8	2.02796762	0.9536907
0.9	0.4	0.2	0.4	2.02796762	0.9684731
0.8	1.0	0.4	1.0	1.63709405	1.4495838
0.8	1.0	0.4	0.0	1.63709405	1.6297970
0.8	1.0	0.2	0.0	1.63709405	1.3819456
0.8	0.5	0.2	0.0	1.50882202	1.1387550
0.8	0.0	0.4	1.0	1.40763431	1.0128597
0.8	0.0	0.4	0.0	1.40763431	1.1159224
0.8	0.0	0.2	0.0	1.40763431	0.9748628
0.7	0.0	0.2	0.0	1.20680425	1.1248428
0.6	0.8	0.4	0.8	1.18701913	1.8148953
0.6	0.8	0.4	0.4	1.18701913	2.0016407
0.6	0.8	0.2	0.8	1.18701913	1.5447531
0.6	0.8	0.2	0.4	1.18701913	1.5949249
0.6	0.4	0.4	0.8	1.14107736	1.5693281
0.6	0.4	0.4	0.4	1.14107736	1.6907950
0.6	0.4	0.2	0.8	1.14107736	1.3780432
0.6	0.4	0.2	0.4	1.14107736	1.4162732
0.6	0.0	0.2	0.0	1.10213202	1.3299205
0.5	0.8	0.2	0.0	1.10019110	1.9736146
0.5	0.0	0.2	0.0	1.04438203	1.6275351

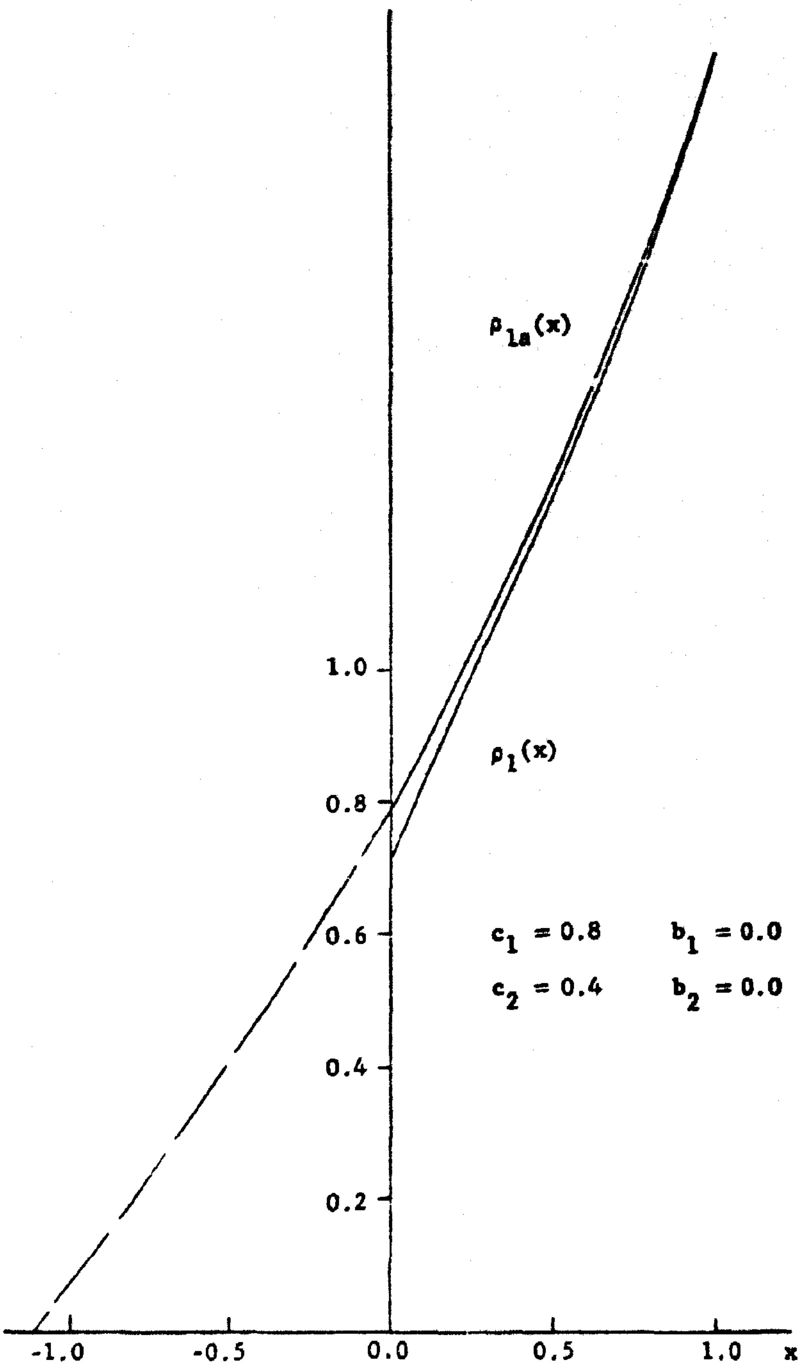

Figure 1 – The Neutron Density

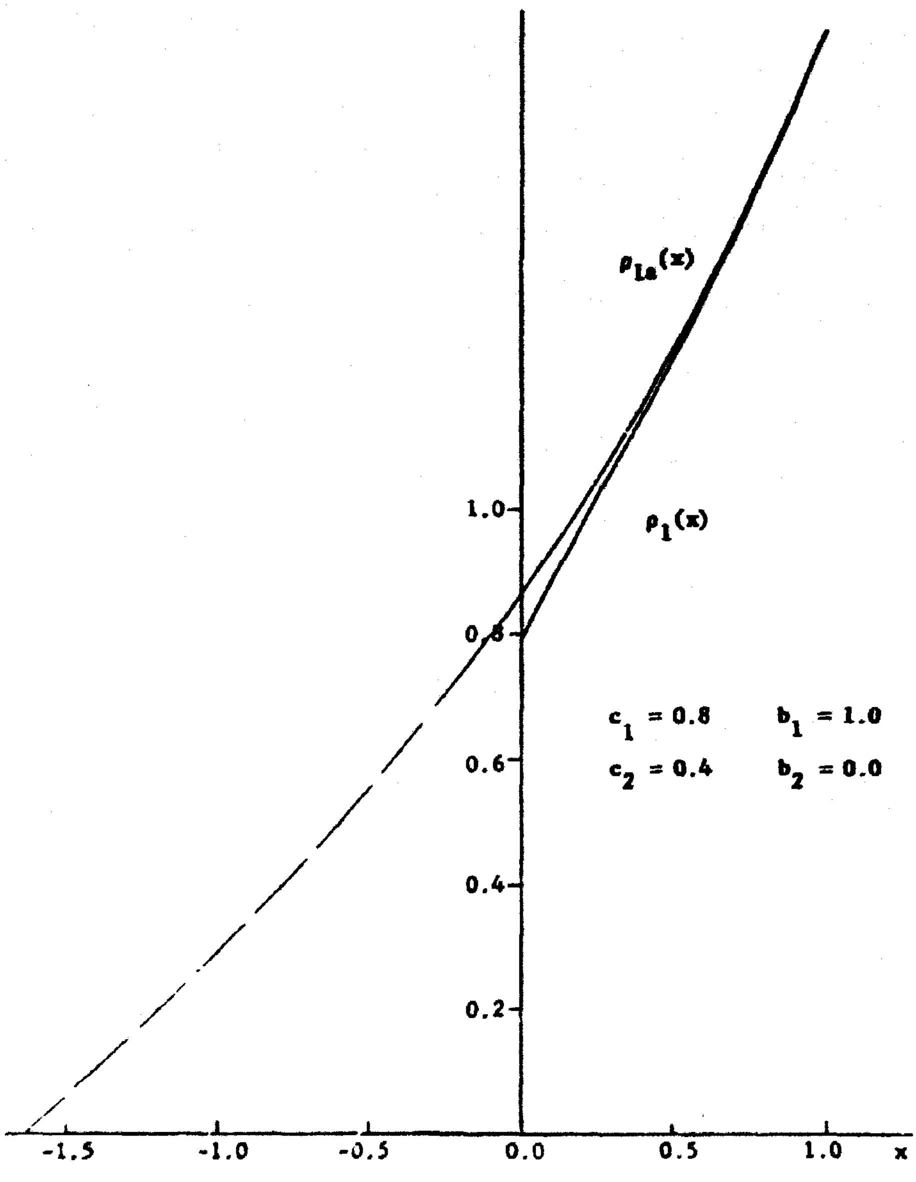

Figure 2 — The Neutron Density

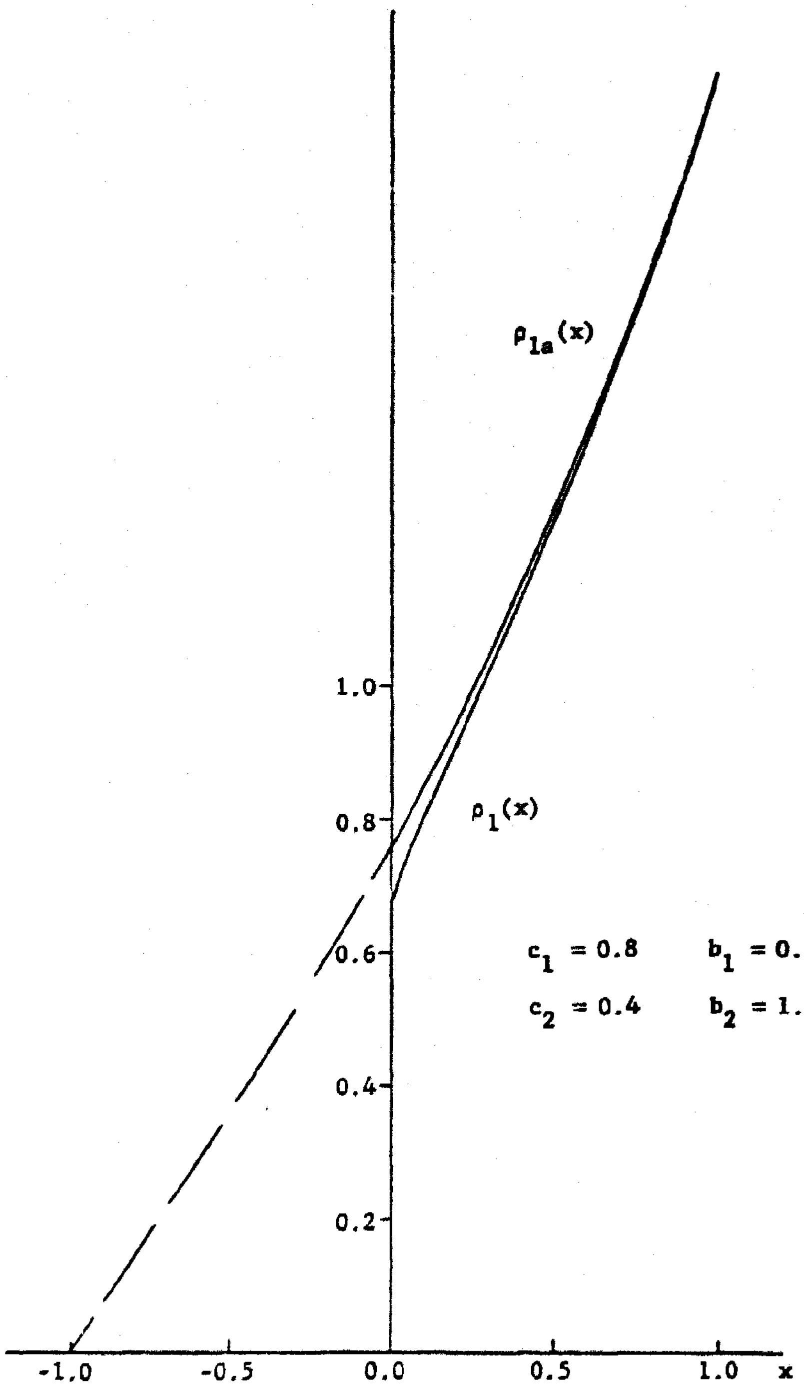

Figure 3 — The Neutron Density

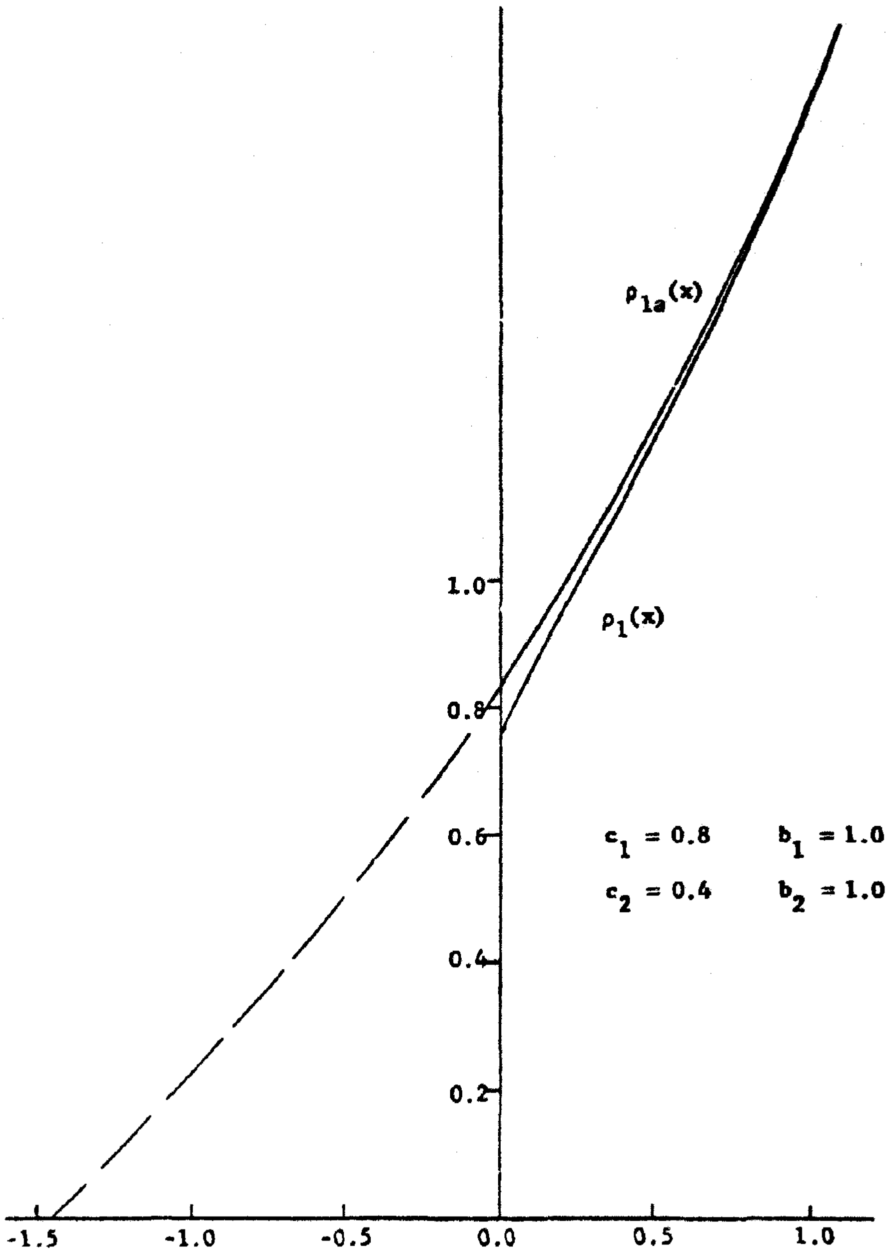

Figure 4 – The Neutron Density

APPENDIX A:

DEVELOPMENT OF EQ. (21) FOR THE CASE OF ISOTROPIC SCATTERING

For $b_1 = 0$, we see that the S_1 function is simply

$$S_1(\mu',\mu) = \frac{c_1\mu\mu'}{\mu'+\mu} H_1(\mu')H_1(\mu) \tag{A-1}$$

for $b_1 = 0$, $\Psi_1(x,\mu)$ reduces to

$$\Psi_1(x,\mu) = A(\nu_0)\phi_1(\nu_0,\mu)e^{-x/\nu_0} + \phi_1(-\nu_0,\mu)e^{x/\nu_0} + \int_0^1 A(\nu)\phi_1(\nu,\mu)e^{-x/\nu}d\nu, \tag{A-2}$$

where

$$\phi_1(\nu_0,\mu) = \frac{c_1\nu_0}{2}\,\frac{1}{\nu_0-\mu} \tag{A-3}$$

and

$$\phi_1(\nu,\mu) = \frac{c_1\nu}{2}\,\frac{P}{\nu-\mu} + [1 - c_1\nu\tanh^{-1}\nu]\,\delta(\nu-\mu) \tag{A-4}$$

On entering Eq. (A-2) into

$$\Psi_1(0,\mu) = \frac{1}{2\mu}\int_0^1 S_1(\mu',\mu)\Psi_1(0,-\mu')d\mu',\ \mu\in(0,1), \tag{A-5}$$

we find

$$A(\nu_0)\phi_1(\nu_0,\mu) + \phi_1(-\nu_0,\mu) + \int_0^1 A(\nu)\phi_1(\nu,\mu)\,d\nu =$$

$$H_1(\mu)[A(\nu_0)L(\nu_0,\mu) + L(-\nu_0,\mu) + \int_0^1 A(\nu)L(\nu,\mu)d\nu], \tag{A-6}$$

where

$$L(\xi,\mu) = \frac{c_1c_1}{4}\int_0^1 \frac{\mu'}{\mu'+\mu}H_1(\mu')\frac{d\mu'}{\mu'+\xi},\ \xi = \pm\nu_0 \text{ or } \nu\in(0,1) \tag{A-7}$$

Clearly, after some partial-fraction analysis, we can write Eq. (A-7) as

$$L(\xi,\mu) = \frac{c_1c_1\xi}{4}\,\frac{1}{\xi-\mu}\int_0^1 H_1(\mu')\left[\frac{\xi}{\xi+\mu'} - \frac{\mu}{\mu+\mu'}\right]d\mu', \tag{A-8}$$

and thus after using the non-linear H equation[illegible],

$$\frac{1}{H_1(\xi)} = 1 - \frac{c_1\xi}{2}\int_0^1 H_1(x)\,\frac{dx}{x+\xi}, \tag{A-9}$$

we find

$$L(\xi,\mu) = \phi_1(\xi,\mu)\left[\frac{1}{H_2(\mu)} - \frac{1}{H_2(\xi)}\right] \tag{A-10}$$

We can now enter Eq. (A - 10) into Eq (A - 6) to obtain

$$A(\nu_0)\phi_1(\nu_0,\mu) + \phi_1(-\nu_0,\mu) + \int_0^1 A(\nu)\phi_1(\nu,\mu)d\nu = H_2(\mu)\{A(\nu_0)\phi_1(\nu_0,\mu)\left[\frac{1}{H_2(\mu)} - \frac{1}{H_2(\nu_0)}\right]$$

$$+ \phi_1(-\nu_0,\mu)\left[\frac{1}{H_2(\mu)} - \frac{1}{H_2(-\nu_0)}\right] + \int_0^1 A(\nu)\phi_1(\nu,\mu)\left[\frac{1}{H_2(\mu)} - \frac{1}{H_2(\nu)}\right]d\nu\}. \tag{A-11}$$

Finally, upon cancelling like terms and common factors, we arrive at

$$\frac{A(\nu_0)}{H_2(\nu_0)}\phi_1(\nu_0,\mu) + \int_0^1 \frac{A(\nu)}{H_2(\nu)}\phi_1(\nu,\mu)d\nu = \frac{(-1)}{H_2(-\nu_0)}\phi_1(-\nu_0,\mu), \mu\in(0,1), \tag{A-12}$$

the version of Eq. (21) corresponding to isotropic scattering.

APPENDIX B:

ORTHOGONALITY RELATIONS, RELATED INTEGRALS AND H-FUNCTION IDENTITIES

In this appendix we wish to summarize some of the important orthogonality relations that are useful for half range analysis, with linearly anisotropic scattering. We will also list here additional identities and expressions of interest in regard to Chandrasekhar's H function for linearly anisotropic scatering.

The function

$$H(\mu) = \frac{1+\mu}{(\nu_0+\mu)\sqrt{1-c-\frac{1}{3}c\ell}}\exp\{-\frac{1}{\pi}\int_0^1 \tan^{-1}\left[\frac{\pi\, cxR(x,x)}{2\lambda(x)}\right]\frac{dx}{x+\mu}\}, \tag{B-1}$$

where

$$\lambda(\nu) = 1 - c\nu R(\nu,\nu)\tanh^{-1}\nu + c\ell\nu^2, \tag{B-2}$$

$\ell = b(1-c)$, $R(x,x) = 1 + \ell x^2$, and ν_0 is the "positive" zero of

$$\Lambda(z) = 1 - czR(z,z)\tanh^{-1}\frac{1}{z} + c\ell z^2, \tag{B-3}$$

is the unique[12] solution of the singular integral equation

$$H(\nu)\lambda(\nu) + \frac{c\nu}{2}\int_0^1 H(\mu)R(\mu,\mu)\frac{d\mu}{\nu-\mu} = 1, \nu\in(0,1), \tag{B-4a}$$

and the linear constraint

$$\frac{c\nu_0}{2} \int_0^1 H(\mu)R(\mu,\mu)\,\frac{d\mu}{\nu_0 - \mu} = 1 \qquad (B\cdot4b)$$

Alternatively, $H(\mu)$ is the unique solution of the non linear equation

$$\frac{1}{H(\mu)} = 1 - \frac{c}{2}\mu \int_0^1 H(x)R(x,x)\,\frac{dx}{x+\mu},\ \mu\in[0,1] \qquad (B\cdot5a)$$

and the linear constraint

$$\frac{c\nu_0}{2} \int_0^1 H(x)R(x,x)\,\frac{dx}{\nu_0 - x} = 1 \qquad (B\cdot5b)$$

Having established $H(\mu)$ for $\mu\in[0,1]$, we can extend the definition by allowing μ in Eq (B · 1) to become the general complex variable z:

$$H(z) = \frac{1+z}{(\nu_0 + z)\sqrt{1 - c - \frac{1}{3}c\ell}} \exp\{-\frac{1}{\pi}\int_0^1 \tan^{-1}[\frac{\pi c x R(x,x)}{2\lambda(x)}]\frac{dx}{x+z}\} \qquad (B\cdot6)$$

Except for the pole at $z = -\nu_0$, $H(z)$ is a function analytic in the complex plane cut from −1 to 0 along the real axis The function $H(z)$ can thus be used to factor $\Lambda(z)$ as

$$\Lambda(z) = \frac{1}{H(z)H(-z)} \qquad (B\cdot7)$$

We can also deduce that

$$\frac{1}{H(z)} = 1 - \frac{c}{2} z \int_0^1 H(x)R(x,x)\,\frac{dx}{x+z},\ z\notin[-1,0] \qquad (B\cdot8)$$

Since

$$H(\infty) = \frac{1}{\sqrt{1 - c - \frac{1}{3}c\ell}}, \qquad (B\cdot9)$$

Small can derive from Eq (B · 8) the moments relation

$$\sqrt{1 - c - \frac{1}{3}c\ell} = 1 - \frac{c}{2}(\alpha_0 + \ell\alpha_1), \qquad (B\cdot10)$$

where the moments are defined by

$$\alpha_\beta = \int_0^1 H(\mu)\mu^\beta d\mu \qquad (B\cdot11)$$

Chandrasekhar[5] has also found that

$$\alpha_o^2 + 2\alpha_1^2 = \frac{4}{c}(\alpha_o - 1) \quad \text{(B-12)}$$

and

$$b\hat{q}^2 = \hat{c}^2 + \ell, \quad \text{(B-13)}$$

where

$$\hat{c} = \frac{c\ell\alpha_1}{2 - c\alpha_o} \text{ and } \hat{q} = \frac{2(1-c)}{2 - c\alpha_o}. \quad \text{(B-14)}$$

The half-range orthogonality relations of Kuščer and McCormick[10] can be listed in the H-function notation as follows:

$$\int_0^1 \phi(\nu,\mu)[\phi(\nu',\mu) + \frac{c\nu'}{2}\hat{c}]\mu H(\mu)d\mu = H(\nu)N(\nu)\delta(\nu - \nu'), \ \nu, \nu' \in (0,1), \quad \text{(B-15)}$$

$$\int_0^1 \phi(\nu_o,\mu)[\phi(\nu',\mu) + \frac{c\nu'}{2}\hat{c}]\ \mu H(\mu)d\mu = 0, \ \nu' \in (0,1), \quad \text{(B-16)}$$

$$\int_0^1 \phi(\nu,\mu)[\phi(\nu_o,\mu) + \frac{c\nu_o}{2}\hat{c}]\ \mu H(\mu)d\mu = 0, \ \nu \in (0,1), \quad \text{(B-17)}$$

$$\int_0^1 \phi(\nu_o,\mu)[\phi(\nu_o,\mu) + \frac{c\nu_o}{2}\hat{c}]\ \mu H(\mu)\,d\mu = H(\nu_o)N(\nu_o), \quad \text{(B-18)}$$

$$\int_0^1 \phi(-\nu_o,\mu)[\phi(\nu_o,\mu) + \frac{c\nu_o}{2}\hat{c}]\mu H(\mu)d\mu = \frac{c\nu_o}{4H(\nu_o)}[1 - \ell\nu_o^2 + 2\nu_o\hat{c}], \quad \text{(B-19)}$$

$$\int_0^1 \phi(-\nu_o,\mu)[\phi(\nu',\mu) + \frac{c\nu'}{2}\hat{c}]\ \mu H(\mu)\,d\mu = \frac{c\nu_o\nu'}{2(\nu_o + \nu')H(\nu_o)}[1 - \ell\nu_o\nu' + \hat{c}(\nu_o + \nu')],$$

$$\nu' \in (0,1), \quad \text{(B-20)}$$

$$\int_0^1 \phi(-\nu,\mu)[\phi(\nu_o,\mu) + \frac{c\nu_o}{2}\hat{c}]\ \mu H(\mu)d\mu = \frac{c\nu\nu_o}{2(\nu + \nu_o)\ H(\nu)}[1 - \ell\nu\nu_o + \hat{c}(\nu_o + \nu)],$$

$$\nu \in (0,1), \quad \text{(B-21)}$$

$$\int_0^1 \phi(-\nu,\mu)[\phi(\nu',\mu) + \frac{c\nu'}{2}\hat{c}]\ \mu H(\mu)\,d\mu = \frac{c\nu\nu'}{2(\nu + \nu')H(\nu)}[1 - \ell\nu\nu' + \hat{c}(\nu + \nu')],$$

$$\nu,\nu' \in (0,1), \quad \text{(B-22)}$$

$$\int_0^1 [\phi(\nu',\mu) + \frac{c\nu'}{2} \hat{c}]\,\mu H(\mu) d\mu = \nu' \hat{q},\ \nu' \in (0,1), \tag{B-23}$$

$$\int_0^1 [\phi(\nu_0,\mu) + \frac{c\nu_0}{2} - \hat{c}]\,\mu H(\mu)\, d\mu = \nu_0 \hat{q}, \tag{B-24}$$

$$\int_0^1 \phi(\nu',\mu)\, \mu H(\mu)\, d\mu = \nu'(1 - c)\hat{q}^{-1},\ \nu' \in (0,1), \tag{B-25}$$

$$\int_0^1 \phi(\nu_0,\mu)\, \mu H(\mu)\, d\mu = \nu_0(1 - c)\hat{q}^{-1}, \tag{B-26}$$

$$\int_0^1 \phi(\nu',\mu) H(\mu)\, d\mu = 1 - (1 - c)\nu' \hat{c}\hat{q}^{-1},\ \nu' \in (0,1), \tag{B-27}$$

and

$$\int_0^1 \phi(\nu_0,\mu) H(\mu)\, d\mu = 1 - (1 - c)\nu_0 \hat{c}\hat{q}^{-1}. \tag{B-28}$$

Here

$$N(\nu) = \nu [\{ \lambda(\nu) \}^2 + \{ \frac{c\nu}{2} \pi R(\nu,\nu) \}^2] \tag{B-29}$$

and

$$N(\nu_0) = \frac{c\nu_0^3}{2} R(\nu_0,\nu_0) [\frac{cR(\nu_0,\nu_0)}{\nu_0(\nu_0^2 - 1)} - \frac{(1 - c)R(\hat{C}\nu_0,\nu_0)}{\nu_0 R(\nu_0,\nu_0)}]. \tag{B-30}$$

RESUMO

São utilizadas as soluções elementares da equação de transporte de neutrons monoenergeticas para espalhamento linearmente anisotropico em conjunto com os principios de invariância de Chandrasekhar na resolução do problema de Milne para dois semi espaços adjacentes

REFERENCES

1. BURKART, A. R. *Aplication of invariance principles to critical problems in reflected reactors.* Raleigh, N. C., 1975. (Ph. D. thesis)

2. CASE, K. M. Elementary solutions of the transport equation and their applications. *Annls Phys.*, Paris, 9:1-23, 1960.

3. ______ & ZWEIFEL, P. F. *Linear transport theory.* Reading, Mass., Addison-Wesley, 1967.

4. CHANDRASEKHAR, S. The angular distribution of the radiation at the interface of two adjoining media. *Can. J. Phys.*, Ottawa, 29:14-20, 1951.

5. ______ *Radiative transfer.* London, Oxford University Press, 1950.

6. ISHIGURO, Y. & MAIORINO, J. R. Two half-space Milne problem in two-group neutron-transport theory. To be published in *Trans. Am. nucl. Soc.*, Hinsdale, Ill, 1975.

7 KUSZELL, A The critical problems for multilayer slab systems *Acta phys pol*, Warszawa, 20:567 89, 1961

8 McCORMICK, N J Neutron transport for anisotropic scattering in adjacent hal spaces 1 Theory *Nucl Sci Engng*, New York, 37 243 51, 1969

9 ——— & DOYAS, R J Neutron transport for anisotropic scattering in adjacent half-spaces 2 Numerical results *Nucl Sci Engng*, New York, 37 252-61, 1969

10 ——— & KUŠČER, I Half space neutron transport with linearly anisotropic scattering *J math Phys*, Lancaster, Pa, 6.1939 45, 1965

11 MENDELSON, M R & SUMMERFIELD, G C One-speed neutron transport in two adjacent half-spaces *J math Phys*, Lancaster, Pa, 5:668-74, 1964

12 MULLIKIN, T W Radiative transfer in finite homogeneous atmospheres with anisotropic scattering 1 Linear singular equations *Astrophys J*, Chicago, 139:379-95, 1964

13 PAHOR, S & ZWEIFEL, P F Invariant imbedding and Case eigenfunctions *J math Phys*, Lancaster, Pa, 10:581-9, 1969

14 SIEWERT, C E & BURKART, A R On the critical reactor problem for a reflected slab To be published in *Nucl Sci Engng*, New York