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ABSTRACT

The inverse problem in one-speed transport equation in plane geometry is solved by using an alternative development of the solution recently presented by Case.

I – INTRODUCTION

In a recent paper, Case[1] solved the inverse problem based on the one-speed transport equation in plane geometry. Here we wish to give an alternative development of that solution and then to use the same method to solve the analogous problem in spherical geometry.

II – PLANE SYMMETRY

We consider, for $c < 1$, the infinite-medium problem defined by

$$\mu \frac{\partial}{\partial x} \Psi(x,\mu) + \Psi(x,\mu) = c \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{2}\right) P_\ell(\mu) f_\ell \int_{-1}^{1} \Psi(x,\mu') P_\ell(\mu') d\mu' + \frac{1}{2}\,\delta(x) \tag{1}$$

and $\Psi(\pm\infty,\mu) = 0$. If we multiply Eq. (1) by $P_\ell(\mu)$ and integrate over μ, we find

$$(1 - cf_\ell)(2\ell+1)\Psi_\ell(x) = \delta(x)\,\delta_{0,\ell} - (\ell+1)\Psi'_{\ell+1}(x) - \ell\Psi'_{\ell-1}(x) \quad , \tag{2}$$

where

$$\Psi_\ell(x) = \int_{-1}^{1} \Psi(x,\mu) P_\ell(\mu)\, d\mu \tag{3}$$

and $f_0 = 1$. If we now multiply Eq. (2) for $\ell = 0$, by $x^{2\alpha}$, $\alpha = 0, 1, 2, \ldots$, and integrate over x, we can write

$$(1 - c) \int_{-\infty}^{\infty} x^{2\alpha}\Psi_0(x)dx = \delta_{\alpha,0} + (2\alpha) \int_{-\infty}^{\infty} x^{2\alpha-1}\Psi_1(x)dx \quad . \tag{4}$$

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which, for $\alpha = 0$, yields

$$\frac{1}{1-c} = 2 \int_0^\infty \Psi_0(x)dx \quad . \tag{5}$$

Here $\Psi_0(x)$ is the flux. From Eq. (2) we see that

$$\Psi_1(x) = -\left(\frac{1}{1-cf_1}\right)\frac{1}{3}\left[\, 2\Psi_2'(x) + \Psi_0'(x) \,\right] \quad , \tag{6}$$

which can be used in Eq. (4) to obtain

$$(1-c)M_{2\alpha} - \frac{(2\alpha)\,(2\alpha-1)}{3(1-cf_1)} M_{2\alpha-2} = \frac{(2\alpha)\,(2\alpha-1)}{3(1-cf_1)} \int_{-\infty}^{\infty} x^{2\alpha-2}\Psi_2(x)dx \quad , \tag{7}$$

where the moments of the flux are defined by

$$M_{2\alpha} = \int_0^\infty x^{2\alpha}\Psi_0(x)dx \quad . \tag{8}$$

Since

$$\int_{-\infty}^{\infty} \Psi_\ell(x)dx = 0 \quad , \quad \ell > 0 \quad , \tag{9}$$

we can solve Eq. (7), for $\alpha = 1$, to obtain

$$\frac{1}{1-cf_1} = 3(1-c)^2 \int_0^\infty x^2\,\Psi_0(x)dx \quad . \tag{10}$$

It is clear that for $\alpha > 1$ we can continue the process of using Eq. (2) in Eq. (7) and integrating by parts to find all of the coefficients f_ℓ in terms of moments of the flux. To be specific we list

$$\frac{1}{1-c} = 2\int_0^\infty \Psi_0(x)dx \quad , \tag{11a}$$

$$\frac{1}{1-cf_1} = (1-c)^2 \;\, 3 \int_0^\infty x^2\,\Psi_0(x)dx \quad , \tag{11b}$$

$$\frac{1}{1-cf_2} = (1-c)^2\,(1-cf_1)^2\left(\frac{15}{16}\right) \int_0^\infty x^4\,\Psi_0(x)dx - \frac{5}{4(1-c)} \quad , \tag{11c}$$

$$\frac{1}{1-cf_3} = (1-c)^2 (1-cf_1)^2 (1-cf_2)^2 \frac{35}{288} \int_0^\infty x^6 \Psi_0(x)dx$$

$$- (1-c)(1-cf_1)(1-cf_2)^2 \frac{175}{144} \int_0^\infty x^4 \Psi_0(x)dx$$

$$- (1-c)(1-cf_2) \frac{35}{9} \int_0^\infty x^2 \Psi_0(x)dx - \frac{28}{27}\left(\frac{1}{1-cf_1}\right) , \tag{11d}$$

and

$$\frac{1}{1-cf_4} = (1-c)^2 (1-cf_1)^2 (1-cf_2)^2 (1-cf_3)^2 \frac{35}{4096} \int_0^\infty x^8 \Psi_0(x)dx$$

$$- (1-c)(1-cf_1)(1-cf_2)^2 (1-cf_3)^2 \frac{245}{1536} \int_0^\infty x^6 \Psi_0(x)dx$$

$$- (1-c)(1-cf_2)(1-cf_3)^2 \frac{245}{192} \int_0^\infty x^4 \Psi_0(x)dx$$

$$- \frac{(1-c)(1-cf_3)}{1-cf_1} [27(1-cf_1) + 28(1-cf_3)] \frac{7}{48} \int_0^\infty x^2 \Psi_0(x)dx$$

$$- \frac{[27(1-cf_1) + 28(1-cf_3)]^2}{(1-cf_1)^2 (1-cf_2)} \frac{1}{720} . \tag{11e}$$

III – SPHERICAL SYMMETRY

Considering now a point source in an infinite medium, with $c < 1$, we can write

$$\mu \frac{\partial}{\partial r} \Psi(r,\mu) + \frac{(1-\mu^2)}{r} \frac{\partial}{\partial \mu} \Psi(r,\mu) + \Psi(r,\mu) = c \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{2}\right) P_\ell(\mu) f_\ell \int_{-1}^{1} \Psi(r,\mu') P_\ell(\mu') d\mu' ,$$

$$r > 0 , \tag{12}$$

and the boundary conditions $\Psi(\infty,\mu) = 0$ and

$$\lim_{r \to 0} 4\pi r^2 \Psi(r, \mu) = \delta(1 - \mu) \quad . \tag{13}$$

On multiplying Eq. (12) by $P_\ell(\mu)$ and integrating over μ, we find

$$(1 - cf_\ell)(2\ell + 1)\Psi_\ell(r) = -(\ell + 1)\Psi'_{\ell+1}(r) - \ell\Psi'_{\ell-1}(r)$$

$$- \frac{1}{r}\left[(\ell + 2)(\ell + 1)\Psi_{\ell+1}(r) - \ell(\ell - 1)\Psi_{\ell-1}(r)\right] \quad . \tag{14}$$

If we multiply Eq. (14), for $\ell = 0$, by $r^{2\alpha+2}$ and integrate over r, we can write

$$(1 - c)\int_0^\infty r^{2\alpha+2}\Psi_0(r)dr = \frac{\delta_{\alpha,0}}{4\pi} + 2\alpha\int_0^\infty r^{2\alpha+1}\Psi_1(r)dr \quad , \tag{15}$$

where $\Psi_0(r)$ is the flux. We note that Eq. (13) has been used to evaluate the limit of $r^{2\alpha+2}\Psi_1(r)$ as r tends to zero. For $\alpha = 0$, Eq. (15) yields

$$\frac{1}{1 - c} = 4\pi \int_0^\infty r^2\Psi_0(r)dr \quad . \tag{16}$$

Considering $\alpha > 0$, we can use Eq. (14), for $\ell = 1$, in Eq. (15) and integrate by parts to obtain

$$(1 - c)\int_0^\infty r^{2\alpha+2}\Psi_0(r)dr = \frac{2\alpha}{3(1 - cf_1)}\left[(2\alpha + 1)\int_0^\infty r^{2\alpha}\Psi_0(r)dr\right.$$

$$\left. + 4(\alpha - 1)\int_0^\infty r^{2\alpha}\Psi_2(r)dr\right] \quad , \quad \alpha \geq 1 \quad , \tag{17}$$

which, for $\alpha = 1$, yields

$$\frac{1}{1 - cf_1} = (1 - c)^2\, 2\pi \int_0^\infty r^4\Psi_0(r)dr. \tag{18}$$

It is clear that we can continue to use Eq. (14) and Eq. (17) to develop expressions for the coefficients f_ℓ. We give some explicit results.

$$\frac{1}{1 - c} = 4\pi \int_0^\infty r^2\Psi_0(r)dr \quad , \tag{19a}$$

$$\frac{1}{1-cf_1} = (1-c)^2\, 2\pi \int_0^\infty r^4 \Psi_0(r)dr \quad , \tag{19b}$$

$$\frac{1}{1-cf_2} = (1-c)^2\,(1-cf_1)^2\,\frac{3}{8}\pi \int_0^\infty r^6 \Psi_0(r)dr - \frac{5}{4(1-c)} \quad , \tag{19c}$$

$$\frac{1}{1-cf_3} = (1-c)^2\,(1-cf_1)^2\,(1-cf_2)^2\,\frac{5}{144}\pi \int_0^\infty r^8 \Psi_0(r)dr$$

$$- (1-c)\,(1-cf_1)\,(1-cf_2)^2\,\frac{35}{72}\pi \int_0^\infty r^6 \Psi_0(r)dr$$

$$- (1-c)\,(1-cf_2)\,\frac{70}{27}\,\pi \int_0^\infty r^4 \Psi_0(r)dr - \frac{28}{27}\left(\frac{1}{1-cf_1}\right) \quad , \tag{19d}$$

and

$$\frac{1}{1-cf_4} = (1-c)^2\,(1-cf_1)^2\,(1-cf_2)^2\,(1-cf_3)^2\,\frac{35}{18432}\pi \int_0^\infty r^{10} \Psi_0(r)dr$$

$$- (1-c)\,(1-cf_1)\,(1-cf_2)^2\,(1-cf_3)^2\,\frac{35}{768}\,\pi \int_0^\infty r^8 \Psi_0(r)dr$$

$$- (1-c)\,(1-cf_2)\,(1-cf_3)^2\,\frac{49}{96}\,\pi \int_0^\infty r^6 \Psi_0(r)dr$$

$$- \frac{(1-c)\,(1-cf_3)}{1-cf_1}\,[27(1-cf_1) + 28(1-cf_3)]\,\frac{7}{72}\,\pi \int_0^\infty r^4 \Psi_0(r)dr$$

$$- \frac{[27(1-cf_1) + 28(1-cf_3)]^2}{(1-cf_1)^2\,(1-cf_2)}\;\frac{1}{720} \tag{19e}$$

It is clear that Eqs. (11) and (19) can readily be obtained either one from the other since we have the point-to-plane transformation:

$$\Psi_{0,pt}(r) = -\frac{1}{2r}\frac{d}{dr}\Psi_{0,p\ell}(r) \quad . \tag{20}$$

Finally we note that the results developed here can readily be generalized to the multigroup case.

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RESUMO

A resolução do problema inverso baseado na equação de transporte em geometria plana e de uma velocidade é feita utilizando-se uma alternativa ao método de Case.

RÉSUMÉ

On a fait la résolution du problème inverse basé dans l'équation de transport à une vélocité en géométrie plaine, en utilizant une alternative de la méthode de Case.

NOTA

Este trabalho foi iniciado no IEA pelo Professor Charles E. Siewert durante seu estágio na CEN em julho de 1976 e posteriormente terminado na Universidade Estadual da Carolina do Norte, onde é professor associado.

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