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# THE ELECTRICAL CONDUCTIVITY OF AN INTERACTING ELECTRON GAS

David Y. Kojima\*

## ABSTRACT

A manybody theory by the propagator method developed by Montroll and Ward for the equilibrium statistical mechanics, is reformulated to describe the electrical conductivity for an electron gas system containing impurity. The theory includes electron-impurity interaction to the infinite order and electron-electron interaction to the first order exchange effect. The propagator used by Montroll, and Ward is separated into two propagators, each of which satisfies either Bloch or Schrodinger equation, to utilize the perturbation method. Correct counting of graphs are presented. Change in the relaxation time due to the electron-electron interaction is explicitly shown and compared with recent works.

## 1 - INTRODUCTION

It has been assumed for a long time that the electron-electron (e-e) interaction did not have a great effect on the electrical conductivity and many works were reported with this assumption. Kohn<sup>(9)</sup> pointed out that the effect of the e-e interaction only existed through the break down of the Galilean translational symmetry as a result of introduction of fixed impurities. Langer has reported in his series of papers<sup>(11,12,13)</sup> the importance of the e-e interaction through the electronic screening effect on impurities. The frequency dependence of the relaxation function by the random phase approximation has been calculated by W. Gotze and P. Wolfie<sup>(4)</sup> for a frequency range from zero to values greater than twice the Fermi energy  $E_F$ , using the memory function technique developed by H. Mori<sup>(16,17)</sup>. The technique appears to be powerful, but the introduction of the e-e interaction is not clear. One might have to take into consideration the locality of the external electric field, when the e-e interaction is included because the coulomb interaction is a long range one, while the wave number of an electric field with a frequency  $\omega \sim 2E_F$  is of the order of  $10^4 \sim 10^5$  Å. Ting, Yang and Quinn<sup>(18)</sup> (T.Y.Q.) recently developed a theory which closely resembled Langer's. They introduced the center-of-mass coordinates of an electron gas system to show explicitly the points made by Kohn. They showed that the electron coupling was brought about only through the center-of-mass variable of the system. However, the theory, as well as Langer's, left the renormalization factors of the Fermi level uncalculated.

The propagator method developed for the equilibrium statistical mechanics by E. Montroll and J. Ward<sup>(14)</sup>, and later by Isihara<sup>(5)</sup>, was first utilized by Montroll and Ward<sup>(15)</sup> to describe the electrical conductivity. They, however, developed the theory only to describe two component Boltzmann gas system. The method of counting similar graphs presented by them is, therefore, good for the Boltzmann gas, of which components are countable particles. The use of the distribution function of an equilibrium system to the linear response theory is established in the expression of the Kubo formula<sup>(10)</sup>. We shall develop the propagator method to include the first order exchange effect explicitly, in chapter II, and show that the effect only appears in the expression of the relaxation time, in chapter IV. The time correlated distribution function in the momentum representation  $b(p_1, t_1, p'_1, t'_1)$  is also developed in the process of the formalism in chapter II.

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## II - FORMALISM

The grand ensemble average of a macroscopic observable of a system is defined as; the sum over the number of particles  $N$  of the multiples of the fugacity  $z$  to the power  $N$ , the partition function  $Z_N$  and the partition function average of the observable, divided by the grand partition function  $\Xi$ , under the same volume and temperature. Thus the grand ensemble average of the electrical conductivity is written as

$$\sigma_{\mu\nu}^{(N)}(\omega) = \frac{1}{\Xi} \sum_{N=1}^{\infty} z^N Z_N \sigma_{\mu\nu}^{(N)}(\omega), \quad (2-1)$$

where  $\sigma_{\mu\nu}^{(N)}(\omega)$  is the partition function average of the electrical conductivity as it is given by the Kubo formula for the  $N$  charge carriers:

$$\sigma_{\mu\nu}^{(N)}(\omega) = \int_0^{\infty} dt e^{-i\omega t} \int_0^{\beta} d\lambda \lim_{V \rightarrow \infty} \frac{1}{V} \text{Tr}^{(N)} \{ \rho_0 J_{\nu}^{(N)}(-i\lambda\hbar) J_{\mu}^{(N)}(t) \}, \quad (2-2)$$

where superfix  $(N)$  represents the total number of particles of the system in the volume  $V$ ,  $\rho_0$ , the distribution function defined as  $\rho_0 = \exp(-\beta H)/Z_N$ , and the current operator,  $J_{\mu}^{(N)}(t) = \exp(i\hbar^{-1}tH) J_{\mu}^{(N)} \times \exp(-i\hbar^{-1}tH)$ , respectively. One must take the thermodynamic limit in the above expression as the volume  $V$  is brought to infinity. The conductivity is given as the response of the system to the external electric field:  $\vec{E} = \vec{E}_0 \exp(i\omega t)$ . The grand ensemble average is no longer a function of the total number of the particles, but of the fugacity. It is thus necessary to introduce a supplementary equation which relates the total number with the fugacity, that is

$$N = \frac{\partial \ln \Xi}{\partial \ln z} \Big|_{\beta, V} \quad (2-3)$$

The Eq. (2-3) relates the Fermi momentum  $p_F$ , defined as  $z = \exp(\beta p_F^2 / 2m)$  where  $\beta = 1/kT$  and  $m$  is an electron mass under consideration, with the absolute Fermi momentum defined as

$$N = \frac{p_0^3}{3 \pi^2 \hbar^3} V. \quad (2-4)$$

A ratio  $p_F / p_0$  obtained from Eqs. (2-3) and (2-4) serves as the renormalization factor. The Eq. (2-1), together with Eq. (2-3), thus, determine the conductivity as the function of the number-density of the electron gas, usually by making use of the absolute Fermi momentum  $p_0$ .

The total Hamiltonian  $\mathcal{H}_0$ , the system in the absence of the external field can be described as follows,

$$\mathcal{H} \equiv H + \psi, \quad (2-5-a)$$

$$\psi \equiv \frac{1}{2} \sum_i^N \phi(\vec{r}_i - \vec{r}_j), \quad (2-5-b)$$

$$H \equiv \sum_{j=1}^N H(j) \equiv \sum_{j=1}^N \left[ \frac{p_j^2}{2m} + \sum_{\alpha} V(\vec{r}_j - \vec{R}_{\alpha}) \right], \quad (2-5-c)$$

where  $\phi(\vec{r}_i - \vec{r}_j)$  is the coulombic repulsive potential between the  $i$ -th and  $j$ -th electrons and  $V(\vec{r}_j - \vec{R}_{\alpha})$  is the potential between the  $j$ -th electron at  $\vec{r}_j$  and  $\alpha$ -th impurity atom located at  $\vec{R}_{\alpha}$ . The total Hamiltonian in Eq. (2-5-a) is for the  $N$  electrons, thus Eq. (2-2) is described with this  $\mathcal{H}$ . The electrical conductivity  $\sigma_{\mu\nu}(\omega)$  can be separated as

$$\sigma_{\mu\nu}(\omega) = \sigma_{\mu\nu}^{[0]}(\omega) + \sigma_{\mu\nu}^{[1]}(\omega) + \sigma_{\mu\nu}^{[2]}(\omega) + \dots \quad (2-6)$$

in accordance with the number of the e-e interactions involved, thus the first term in the right hand side is that for the ideal gas, and the second for the first order exchange graphs, and so on. Each term in Eq (2-6) is still the function of  $p_F$ , unless renormalized.

The Kubo formula, Eq.(2.2), is reexpressed, when no magnetic field is present, as

$$\sigma_{\mu\nu}^{(N)}(\omega) = \frac{1}{Z_N V} \sum_{i,j} \frac{e_i e_j}{m_i m_j} \int_0^{\infty} d\tau e^{-i\omega\tau} \int_0^{\beta} d\lambda$$

$$\times \text{Tr}^{(N)} \left[ \exp(-(\beta - \lambda + i\hbar^{-1}\tau)\mathcal{H}) p_{i\nu} \exp(-(\lambda - i\hbar^{-1}\tau)\mathcal{H}) p_{j\nu} \right], \quad (2-7)$$

here  $e_i$  and  $m_i$  are the  $i$ -th electronic charge and mass, and  $p_{i\nu}$  is the  $\nu$ -th component of the  $i$ -th momentum in the operator form. The above expression is now rewritten by making use of propagators, as was done by Montroll and Ward.

$$\sigma_{\mu\nu}^{(N)}(\omega) = \frac{1}{Z_N V} \sum_{i,j} \frac{e_i e_j}{m_i m_j} \int_0^{\infty} d\lambda e^{-i\omega\tau} \int_0^{\beta} d\lambda \int dp^{(1)} dp^{(2)} p_{i\nu}^{(1)} p_{j\nu}^{(2)}$$

$$\times K(p^{(1)}, \beta + i\hbar^{-1}\tau; p^{(2)}, \lambda) K(p^{(2)}, \lambda, p^{(1)}, i\hbar^{-1}\tau), \quad (2-8)$$

where

$$p^{(1)} = (p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, \dots, p_N^{(1)}),$$

$$Z_N = \sum_n \exp(-\beta E_n) = \int dp^{(1)} K(p^{(1)}, p^{(1)}, 0),$$

and

$$K(p^{(1)}, s_1; p^{(2)}, s_2) = \sum_n \Psi_n(p^{(1)}) \Psi_n(p^{(2)}) \exp(-\beta E_n), \quad (2-9)$$

where  $\Psi_n(p^{(1)})$  is a Fourier transform of a wave function of N-particle system with H, satisfying the antisymmetric relation of the exchange of any two variables. In this form, the current is no longer the operator and counting similar graphs becomes possible. One must notice here that Eq.(2-9) is the propagator with which particles at  $p^{(2)}$  at a complex reciprocal temperature  $s_2$  propagate to  $p^{(1)}$  at  $s_1$ . The two propagators in Eq.(2-8) can be rewritten by introducing two more propagators, it becomes

$$\begin{aligned} & K(p^{(1)}, \beta + i\hbar^{-1}\tau; p^{(2)}, \lambda) K(p^{(2)}, \lambda; p^{(1)}, i\hbar^{-1}\tau) \\ &= \int dp^{(3)} dp^{(4)} K(p^{(1)}, i\hbar^{-1}\tau; p^{(3)}, 0) K(p^{(3)}, \beta; p^{(2)}, \lambda) \\ & \quad \times K(p^{(2)}, \lambda; p^{(4)}, 0) K(p^{(4)}, 0; p^{(1)}, i\hbar^{-1}\tau). \end{aligned} \quad (2-10)$$

The second and the third propagators are the Green's functions of Bloch equations, and the first and the fourth, those of Schroedinger equations. The fourth one is the complex conjugate form, thus follows the c.c. form of the Schroedinger equation. These two are identical to the Feynman propagators<sup>(2)</sup>, and follow the same condition of the time flow. The Green's functions of the two equations satisfy the following:

$$\frac{\partial K(r^{(1)}, \beta_1; r^{(2)}, \beta_2)}{\partial \beta_1} + H_1 K = \delta(r^{(1)} - r^{(2)}) \quad (2-11)$$

with conditions:

$$\begin{aligned} K &= \delta(r^{(1)} - r^{(2)}) & \text{if } \beta_1 \rightarrow \beta_2 + 0 \\ &= 0 & \text{if } \beta_1 < \beta_2, \end{aligned}$$

and

$$\frac{\hbar}{i} \frac{\partial K(r^{(1)}, i\hbar^{-1}t_1; r^{(2)}, i\hbar^{-1}t_2)}{\partial t_1} - H_1 K = -\frac{\hbar}{i} \delta(t_1 - t_2) \delta(r^{(1)} - r^{(2)}), \quad (2-12)$$

with conditions:

$$\begin{aligned} K &= \delta(r^{(1)} - r^{(2)}) & \text{if } t_1 \rightarrow t_2 + 0 \\ &= 0 & \text{if } t_1 < t_2, \end{aligned}$$

where  $\delta(r^{(1)} - r^{(2)})$  consists of  $\delta$ - functions of the Slater determinant with an appropriate weight factor. It is possible to combine these two types of propagators at any order of the perturbation series. Introducing Eq.(2-8) into Eq.(2-1), the grand ensemble average of the electrical conductivity yields

$$\sigma_{\mu\nu}(\omega) = \frac{e^2}{m^2} \int e^{-i\omega t} dt \int_0^\beta d\lambda \int dp_1^{(1)} dp_1^{(2)} p_1^{(1)} p_1^{(2)} b(p_1^{(1)}, p_1^{(2)}), \quad (2-13-b)$$

where

$$\begin{aligned}
 b(\rho_1^{(1)}, \rho_1^{(2)}) &= \frac{1}{V \Xi_N} \sum_N z^N \int \prod_{i=2}^N \frac{\pi}{\pi} d\rho_i^{(1)} d\rho_i^{(2)} \\
 &\times K(\rho^{(1)}, \beta + i\hbar^{-1}\tau; \rho^{(2)}, \lambda) K(\rho^{(2)}, \lambda; \rho^{(1)}, i\hbar^{-1}\tau) \\
 &= b^{[0]}(\rho_1^{(1)}, \rho_1^{(2)}) + b^{[1]} + b^{[2]} + \dots
 \end{aligned} \tag{2-13b}$$

The function  $b(\rho^{(1)}, \rho^{(2)})$  is the singlet distribution function, which can be expanded also according to the order of the e-e interaction as is expressed in Eq.(2-13-b).

a) The zeroth order contribution in the e-e interaction ( $\beta = 0$ ). When electrons do not interact among themselves, but with impurities, it is well known that N-particle trace in Eq.(2-2) can be rewritten in terms of one-particle trace. This term was extensively investigated by Chester and Thellung<sup>(1)</sup> in 1959 and by many others. We shall review this briefly by means of the propagator method with emphasis on the counting of the graphs. The singlet distribution function without the e-e interaction is given as

$$\begin{aligned}
 b^{[0]}(\rho_1^{(1)}, \rho_1^{(2)}) &= \frac{1}{V \Xi_0} \sum_N z^N \int \prod_{i=2}^N \frac{\pi}{\pi} d\rho_i^{(1)} d\rho_i^{(2)} \\
 &\times K^{[0]}(\rho^{(1)}, \beta + i\hbar^{-1}\tau; \rho^{(2)}, \lambda) K^{[0]}(\rho^{(2)}, \lambda; \rho^{(1)}, i\hbar^{-1}\tau),
 \end{aligned} \tag{2-14}$$

and

$$\Xi_0 = \sum_N z^N Z_1^{[0]} = \sum_N z^N \int d\rho^{(1)} K^{[0]}(\rho^{(1)}, \beta; \rho^{(1)}, 0), \tag{2-15}$$

where the superfix [0] means the absence of the e-e interaction. Each Kernel in Eq.(2-14) consists of the Slater determinant carrying  $1/N!$  as the norm and it becomes now

$$b^{[0]}(\rho_1^{(1)}, \rho_1^{(2)}) = \frac{1}{V \Xi_0} \sum_N \frac{z^N}{[(N-1)!]^2} \int \prod_{i=2}^N \frac{\pi}{\pi} d\rho_i^{(1)} d\rho_i^{(2)} \tag{2-16}$$

the two corresponding determinant of one-body propagators,

where the one-body propagator is defined with an eigen function  $\phi_n(r_j^{(k)})$  of the Hamiltonian  $H_k(j)$  in Eq.(2-5) and the eigen value  $E_n$ , as

$$K_1(\rho_i^{(1)}, s_1; \rho_j^{(2)}, s_2) = \sum_n \phi_n(\rho_i^{(1)}) \phi_n^\dagger(\rho_j^{(2)}) \exp(-(s_1 - s_2)E_n). \tag{2-17}$$

The multiple of the two determinants consists of  $(N!)^2$  term, and the variables in question,  $\rho_1^{(1)}$  and  $\rho_1^{(2)}$ , are distributed in various ways as the parts of clusters. Obviously from the assumption

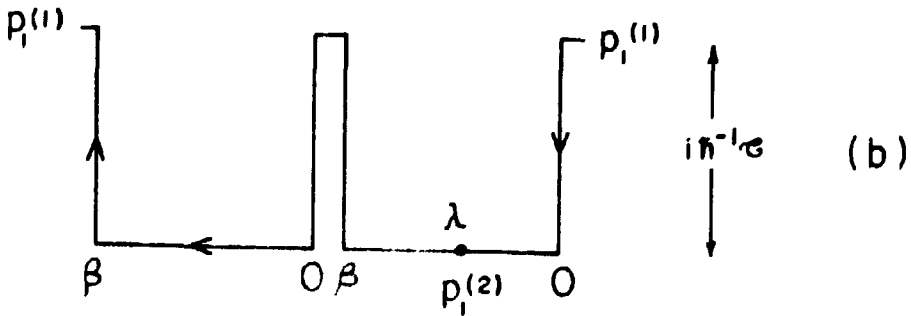
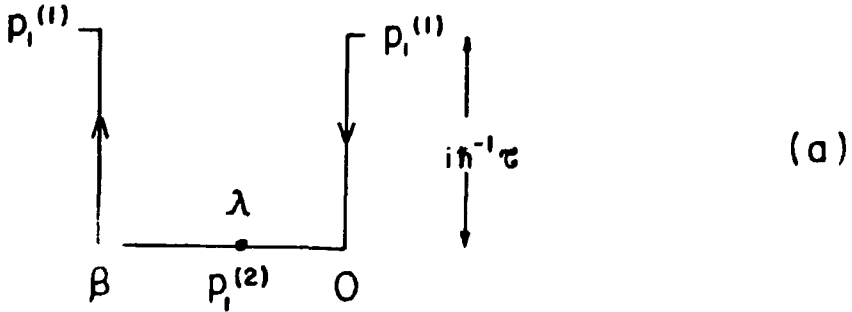


of a homogeneous system, when the two variables are placed in separate clusters, the contribution is zero. It is enough to consider only cases with the two variables in the same cluster. A set of clusters,  $A_Q$ , is defined in which the two variables are distributed in various ways unintegrated. The method of counting graphs is the same as the case without time variables. One cycle of 1-toron, however, consists of a time development from  $i\hbar^{-1}\tau$  at  $p_1^{(1)}$  to zero at an arbitrary coordinate  $p_1^{(3)}$ , a temperature development through a relay point  $(p_1^{(2)}, \lambda)$  to a point  $(p_1^{(4)}, \beta)$ , and finally from zero to  $i\hbar^{-1}\tau$  at  $p_1^{(1)}$  with the reciprocal temperature  $\beta$ . This process is shown in Fig. 1-a, in which the complex temperature flows from right to left. Summits of graphs always belong to the set  $p^{(1)}$ , while planes, to the set  $p^{(2)}$  and  $\lambda$ , the reciprocal temperature integral of zero to  $\beta$ . The Figs. 1-b and 1-c stand for the following clusters ( in this case, 2-torons).

$$(1-b) = \int dp_1^{(1)} K_1(p_1^{(1)}, \beta + i\hbar^{-1}\tau; p_1^{(1)}, i\hbar^{-1}\tau)$$

$$\times K_1(p_1^{(1)}, \beta + i\hbar^{-1}\tau; p_1^{(2)}, \lambda) K_1(p_1^{(2)}, \lambda; p_1^{(1)}, i\hbar^{-1}\tau)$$

$$= K_1(p_1^{(1)}, 2\beta + i\hbar^{-1}\tau; p_1^{(2)}, \lambda) K_1(p_1^{(2)}, \lambda; p_1^{(1)}, i\hbar^{-1}\tau),$$



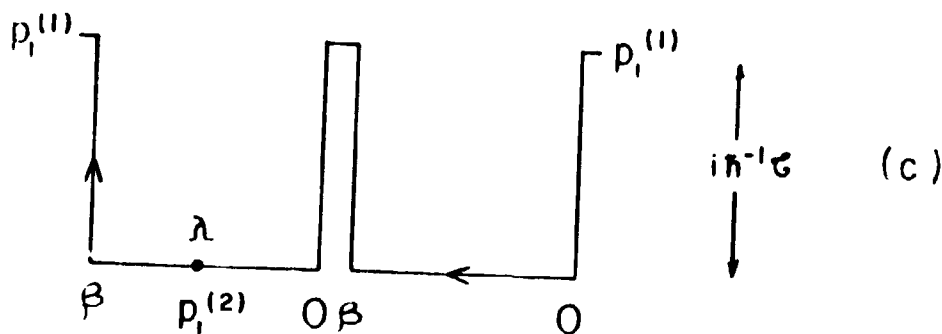


Figure 1 – Graphs appearing in the ideal gas contribution. The graph (a) represents 1-toron with complex reciprocal temperature. The graphs (b) and (c) represent two different 2-torons.

similarly

$$(1-c) \Rightarrow K_1(p_1^{(1)}, \beta + i\hbar^{-1}\tau; p_1^{(2)}, \lambda) K_1(p_1^{(2)}, \beta + \lambda; p_1^{(1)}, i\hbar^{-1}\tau). \quad (2-19)$$

Extending this to the  $\ell$ -toron case, one can obtain for  $A_\ell$  that

$$A_\ell \equiv (-)^{\ell-1} z^\ell \sum_{s=0}^{\ell-1} K_1(p_1^{(1)}, (\ell-s)\beta + i\hbar^{-1}\tau; p_1^{(2)}, \lambda) \times K_1(p_1^{(2)}, s\beta + \lambda, p_1^{(1)}, i\hbar^{-1}\tau) \quad (2-20)$$

where  $(-)^{\ell-1}$  appears because of the Slater determinant, since  $\ell$ -toron exchanges  $(\ell-1)$  times their positions, and  $z^\ell$  comes from  $z^N$ , leaving  $z^{N-\ell}$  for the rest of the configurations. First, the  $N$  particles are separated into  $\ell$  and  $N-\ell$ , in such a way that  $\ell$  particles from a  $\ell$ -toron, with  $p_1^{(1)}$  and  $p_1^{(2)}$  in it. Addressing the two variables in the  $\ell$ -toron, one finds  $\ell-1$  available spaces for each set of the variables. Those spaces are filled with the rest of the variables in the sets,  $p_1^{(1)}$  and  $p_1^{(2)}$ , and integrated, so that characteristic differences no longer exist. The number of ways of filling the spaces are

$$[(N-1)(N-2) \dots (N-1 - (\ell-1))]^2 = \left[ \frac{(N-1)!}{(N-\ell)!} \right]^2,$$

The above argument only describes one of the elements in  $A_\ell$  in Eq.(2-20), however,  $A_\ell$  is the collection of  $\ell$ -torons, with different configurations of  $p_1^{(1)}$  and  $p_1^{(2)}$ . Thus the first part concludes

$$\left[ \frac{(N-1)!}{(N-\ell)!} \right]^2 \times A_\ell. \quad (2-21-a)$$

The rest of the variables form arbitrary number of both same and different toron clusters, and the same configurations appear  $[(N - \ell)!]^2$  times after integration of all variables. All the different configurations of forming the clusters out of  $N - \ell$  variables are given as

$$\sum_t \frac{(b_t^0)^{s_t}}{s_t!} (z^t)^{s_t}, \quad (2-21-b)$$

with a condition

$$\sum_t t s_t = N - \ell,$$

where  $b_t^0$  is defined as

$$b_t^0 = \frac{(-)^{t-1} (t-1)!}{t!} \int K_1(p_1^{(1)}, \tau; p_1^{(1)}, 0) dp_1^{(1)}. \quad (2-22)$$

The detailed derivation of Eq.(2-22) is given in a textbook<sup>(6)</sup> by Ishihara. Collecting all the factors and introducing them into Eq.(2-16), the singlet distribution function of the zeroth order is built up to give

$$\begin{aligned} b^{[0]}(p_1^{(1)}, p_1^{(2)}) &= \frac{1}{V z_0} \sum_{\ell=1}^{\infty} \frac{1}{[(N-\ell)!]^2} \sum_{\ell=1}^{\infty} \left[ \frac{(N-\ell)!}{(N-\ell)!} \right]^2 A_{\ell} \\ &\quad \times [(N-\ell)!]^2 \sum_{t=1}^{\infty} \frac{(b_t z^t)^{s_t}}{s_t!} = \frac{1}{V} \sum_{\ell=1}^{\infty} A_{\ell} \\ &= \frac{1}{V} \sum_{t=1}^{\infty} \sum_{s=0}^{\infty} (-)^{s+t+1} z^{s+t} K_1(p_1^{(1)}, \tau; p_1^{(2)}, \lambda) \\ &\quad K_1(p_1^{(2)}, \tau; p_1^{(1)}, i\hbar^{-1}\tau). \end{aligned} \quad (2-23)$$

b) The first order contribution.

There are two types of graphs appearing as the first order of the e-e interaction, as the partition function is constructed. One of the two is removed by the effect of the positive background, that is, types of graphs characterized by  $u(q=0) = 0$ , where  $u(q)$  is the Fourier transform of the coulombic potential between two electrons. When the perturbation method is applied to the right side of Eq.(2-10), it is easier to treat each kernel (propagator) in the real space and later transform it to the momentum space. Each kernel is expanded in terms of  $v(q)$  and the same kernel used in Eq. (2-14). The kernel involves the impurity-electron interaction as it is. Since both the time and reciprocal temperature kernels possess the same structure, these are recombined after the proper perturbation process at any order of perturbation. Of course, the directions of the times must be chosen correctly. After all the above processes, the singlet distribution function is expressed as

$$\begin{aligned}
b(p_1^{(1)}, p_2^{(2)}) &= \frac{1}{V \bar{\epsilon}_0 \bar{\epsilon}_1 \dots} \sum_{N=1}^{\infty} N^2 z^N \int \prod_{i=2}^N dp_i^{(1)} dp_i^{(2)} \\
&\times \{ \kappa^{[0]}(p^{(1)}, \beta + i\hbar^{-1}\tau; p^{(2)}, \lambda) \kappa^{[0]}(p^{(2)}, \lambda; p^{(4)}, \beta; \vec{q}) \kappa^{[0]}(p^{(4)}, \beta; p^{(1)}, i\hbar^{-1}\tau) \\
&\quad - \int dp^{(4)} d\vec{q} \hbar^{-3} u(q) \Sigma \\
&\quad \times \{ \int_{\lambda}^{\beta + i\hbar^{-1}\tau} d\beta' \kappa^{[0]}(p^{(1)}, \beta + i\hbar^{-1}\tau; p^{(2)}, \lambda) \kappa^{[0]}(p^{(2)}, \lambda; p^{(4)}, \beta'; \vec{q}) \kappa^{[0]}(p^{(4)}, \beta'; p^{(1)}, i\hbar^{-1}\tau) \\
&\quad + \int_{i\hbar^{-1}\tau}^{\lambda} d\beta' \kappa^{[0]}(p^{(1)}, \beta + i\hbar^{-1}\tau; p^{(4)}, \beta'; \vec{q}) \kappa^{[0]}(p^{(4)}, \beta'; p^{(2)}, \lambda) \kappa^{[0]}(p^{(2)}, \lambda; p^{(1)}, i\hbar^{-1}\tau) \} + \\
&\quad \alpha(u^2(q)) \}, \tag{2-24}
\end{aligned}$$

where

$$u(q) = \int \phi(\vec{r}_{ij}) \exp(-i\vec{r}_{ij} \cdot \frac{\vec{q}}{\hbar}) d\vec{r}_{ij}, \tag{2-25}$$

$$\begin{aligned}
\kappa(p^{(1)}, s_1; p^{(2)}, s_2) &= \frac{1}{\hbar^{2N}} \int \kappa(r^{(1)}, s_1; r^{(2)}, s_2) \exp(-\frac{i}{\hbar} \\
&\times (\vec{r}^{(1)} \cdot \vec{p}^{(1)} - \vec{r}^{(2)} \cdot \vec{p}^{(2)})) d\vec{r}^{(1)} d\vec{r}^{(2)}. \tag{2-26}
\end{aligned}$$

The summation in Eq.(2-24) together with  $\kappa^{[0]}(p^{(2)}, \lambda; p^{(4)}, \beta; \vec{q})$  represent  $N!(2!(N-2)!)$  Slater determinants, in each of which a pair of interaction momentum  $\pm \vec{q}$  is placed without repetition in two arbitrary columns of the set,  $p^{(4)}$ . Thus in  $\kappa(p^{(2)}, \lambda; p^{(4)}, \beta; \vec{q})$  one finds

$$p_1^{(4)}, p_2^{(4)}, \dots, p_i^{(4)} + \vec{q}, p_{i+1}^{(4)}, \dots, p_{j-1}^{(4)}, p_j^{(4)} - \vec{q}, \dots, p_N^{(4)}.$$

The higher order terms with respect to  $u(q)$  are also important to form the grand partition function and the grand ensemble average, and cannot be neglected, although in Eq.(2-24) only the first order is shown for the purpose of abbreviation. The first term in Eq.(2-24) is that for the noninteracting electrons, and it produces

$$\frac{1}{V \bar{\epsilon}_0 \bar{\epsilon}_1 \dots} b^{[0]}(p_1^{(1)}, p_1^{(2)}) \bar{\epsilon}_0.$$

From the first order term, two different categories of collections of graphs appear, other than the separation of the pair variables. The first category (i) is that the pair appears in a toron with the e-e interaction, and the second (ii), is that the pair and the e-e interaction,  $u(q)$ , are placed in different torons. For (i), one sees that the independent clusters, which are separated from  $p_1^{(1)}$ ,  $p_1^{(2)}$  and  $u(q)$  yields the grand partition function of the ideal gas,  $\Xi_0$ , and it is multiplied by  $b^{[1]}(p_1^{(1)}, p_1^{(2)})$ . For (ii), the clusters minus the cluster with the pair make a group which does not involve  $u(q)$  and a toron which forms the first order exchange interaction so as to create finally  $\sum b_t^{[1]} z^t$ , where  $b_t^{[1]}$  is an irreducible cluster of the first order. The first order term in Eq.(2-24) will, therefore, results the following form:

$$\frac{1}{V \Xi_0 \Xi_{\dots}} \{ b^{[1]}(p_1^{(1)}, p_1^{(2)}) \Xi_0 + b^{[0]} \Xi_0 \sum_{t=2}^{\infty} b_t^{[1]} z^t \}. \tag{2-28}$$

The second order term in the perturbation will also produce similar terms. Considering these higher order together, one obtains Eq.(2-13-b). To obtain the explicit form of  $b^{[1]}(p_1^{(1)}, p_1^{(2)})$ , it is thus sufficient to observe the first appearance of the term in Eq.(2-24) for the category (i). Now define  $C_{\mathcal{L}}$  for this purpose as given in Eq.(2-29), which is expressed only by the one-body kernel and one e-e interaction,  $u(q)$ . It contains all the possibilities of constructing  $\mathcal{L}$ -torons with the momentum pair and  $u(q)$  placed in various locations, as was done to construct  $A_{\mathcal{L}}$ .

$$\begin{aligned} C_{\mathcal{L}} = & (-i)^{\mathcal{L}} \int \frac{d\vec{q}}{h^3} u(q) c d p_1'' d p_2'' \sum_{v,s,t \geq 1} \frac{v+s+t+u\mathcal{L}}{t u \geq 0} \left[ \int_{\lambda} \frac{\beta + i\hbar^{-1}\tau}{d\beta'} \right. \\ & \times \{ (p_1, X; (p_1'' - \vec{q}), \beta') (p_1', Y; p_1', \lambda) (p_1', R; (p_2'' + \vec{q}), \beta') (p_2', W; p_1', i\hbar^{-1}\tau) \\ & (p_1, X; (p_1'' - \vec{q}), \beta') (p_1', \varphi; (p_2'' + \vec{q}), \eta) (p_2', Y; p_1', \lambda) (p_1', W'; p_1', i\hbar^{-1}\tau) \\ & + (p_1, X; p_1', \lambda) (p_1', R; (p_1'' - \vec{q}), \beta') (p_1', \tau; \beta; (p_2'' + \vec{q}), 0) (p_2', W; p_1', i\hbar^{-1}\tau) \} \\ & + \int_{i\hbar^{-1}\tau}^{\lambda} d\beta' \{ (p_1, X; (p_1'' - \vec{q}), \beta') (p_1', \varphi + \beta'; p_1', \lambda) (p_1', \varphi + \lambda; (p_2'' + \vec{q}), \beta') (p_2', W; p_1', i\hbar^{-1}\tau) \\ & + (p_1, X; (p_1'' - \vec{q}), \beta') (p_1', \varphi; (p_2'' + \vec{q}), 0) (p_2', \tau; \beta + \beta'; p_1', \lambda) (p_1', W'; p_1', i\hbar^{-1}\tau) \\ & + (p_1, X; p_1', \lambda) (p_1', \varphi + \lambda; (p_1'' - \vec{q}), \beta') (p_1', \varphi; \beta; (p_2'' + \vec{q}), 0) (p_2', W; p_1', i\hbar^{-1}\tau) \} \}. \end{aligned}$$

with

$$X = v\beta + i\hbar^{-1}\tau, W = \varphi + \beta', W' = \varphi + \lambda, Y = \varphi + \beta'$$

$$R = S\beta + \lambda, \vec{p}_1 = p_1^{(1)} \text{ and } \vec{p}_1 = p_1^{(2)}, \tag{2-29}$$

where (t) in the sum-restriction is either t or t', and each bracket represents the one-body kernel defined in Eq.(2-17). The first, second, fourth and fifth terms are schematically shown in Fig. 2-a through 2-d for the case of  $\ell=2$ . The third and sixth start from  $\ell=3$ . The thick lines in Fig.2 represents the region where  $\beta'$  integral variable associated with  $\mu(q)$  moves, and the wavy horizontal lines,  $\mu(q)$  itself, respectively. In the first and fourth graphs, the pair is partitioned by the wavy lines but the second and third have no such structure. These four graphs can not be combined in a simple manner in general. As an example of constructing the general case of  $C_q$  we only show schematically the graph corresponding to the first term in  $C_q$  in Fig. 3. Following the same technique as derivation of  $b^{[0]}(p_1^{(1)}, p_1^{(2)})$ , one can obtain the following result:

$$b^{[1]}(p_1^{(1)}, p_1^{(2)}) = -\frac{1}{V} \sum_{\ell=2}^{\infty} C_{\ell} \tag{2-30}$$

Here, besides the distribution of the momentum pair, one must take care of two more variables, which belong to the set,  $p^{(4)}$ , in Eq.(2-24) and which play roles of the entrance and exit of the interaction variable q. The second category (ii) is easily investigated following the same technique, but it will not be presented here.

By introducing Eq.(2-23) into Eq.(2-13-a) and breaking the kernels to represent it by the one-body trace, one obtains well known formula of the electrical conductivity with-out the e-e interaction.

$$\sigma_{\mu\nu}^{[0]}(\omega) = \frac{e^2}{V m^2} \int_0^{\infty} d\tau e^{-i\omega\tau} \int_0^{\beta} d\lambda \text{Tr}^{(1)} \{ p_{\mu}(\tau) f(H) p_{\nu}(-i\hbar^{-1}\tau) (1-f(H)) \}, \tag{2-31}$$

where  $f(H)$  is the Fermi distribution function with the Hamiltonian  $H(j)$ . Similarly by introducing Eq.(2-30) and Eq.(2-29) into Eq.(2-13-a) and performing the same procedure, the first order contribution to the conductivity,  $\sigma_{\mu\nu}^{[1]}(\omega)$  yields

$$\begin{aligned} \sigma_{\mu\nu}^{[1]}(\omega) = & \frac{e^2}{V m^2} \int_0^{\infty} d\tau e^{-i\omega\tau} \int_0^{\beta} d\lambda \int dp_1^{(1)} dp_2^{(1)} \int d\vec{q} h^{-3} u(q) \left[ \int_{\lambda}^{\beta+i\hbar^{-1}\tau} \{ \right. \\ & | P_{\mu}(\tau + i\beta'\hbar) | b \rangle \langle c | P_{\nu}(i\hbar(\beta' - \lambda)) | d \rangle + \langle a | f_{-} p_{\nu}(i\hbar(\beta' - \lambda)) P_{\mu}(\tau + i\beta'\hbar) | b \rangle \langle cd \rangle \\ & \left. + \langle a | P_{\mu}(\tau + i\beta'\hbar) p_{\nu}(i\hbar(\beta' - \lambda)) f | b \rangle \langle cd \rangle + \int_{i\hbar^{-1}\tau}^{\lambda} d\beta' \{ \right. \\ & \langle a | P_{\mu}(\tau + i\beta'\hbar) | b \rangle \langle c | f p_{\nu}(i\hbar(\beta' - \lambda)) f | d \rangle + \langle a | P_{\mu}(\tau + i\beta'\hbar) p_{\nu}(i\hbar(\beta' - \lambda)) f_{-} | b \rangle \langle cd \rangle \\ & \left. + \langle a | f p_{\nu}(i\hbar(\beta' - \lambda)) P_{\mu}(\tau + i\beta'\hbar) | b \rangle \langle cd \rangle \right], \tag{2-32} \end{aligned}$$

where  $| \alpha \rangle \equiv | p_1^{(2)} \rangle$ ,  $| b \rangle \equiv | p_1^{(1)} - q \rangle$ ,  $| c \rangle \equiv | p_1^{(1)} \rangle$ ,  $| d \rangle \equiv | p_1^{(2)} + q \rangle$ ,

$$f \equiv f(H), f_{-} \equiv 1 - f(H), \langle cd \rangle = \langle c | f | d \rangle \text{ and } P_{\mu}(t) = f_{-} p_{\mu}(t) f.$$

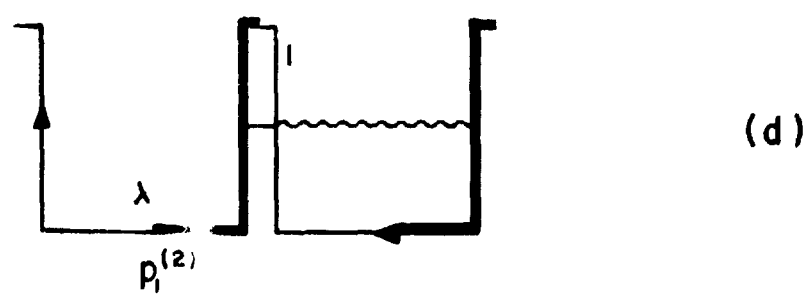
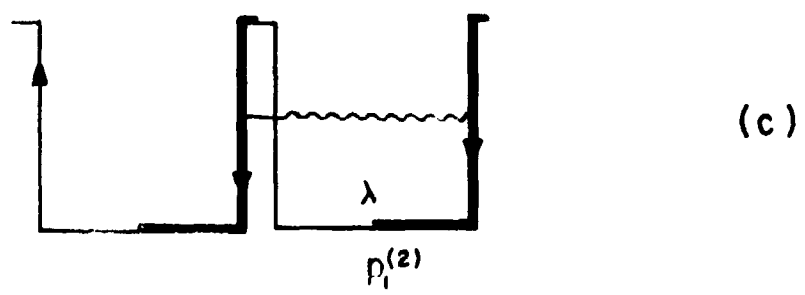
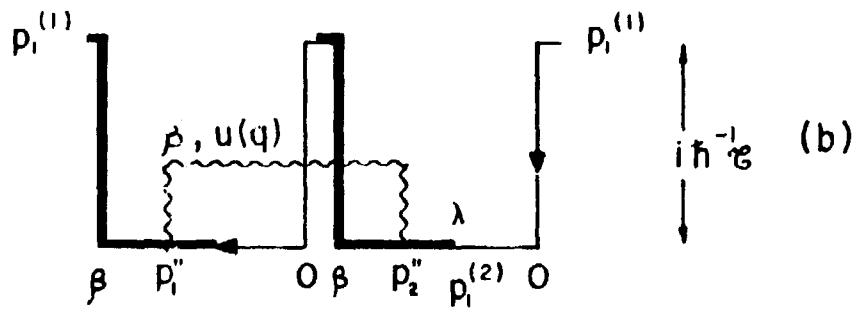
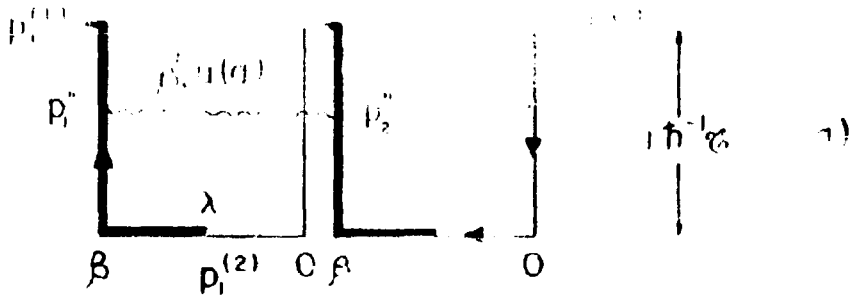


Figure 2 - Graphs for the first order exchange contributions in the case of  $\ell = 2$

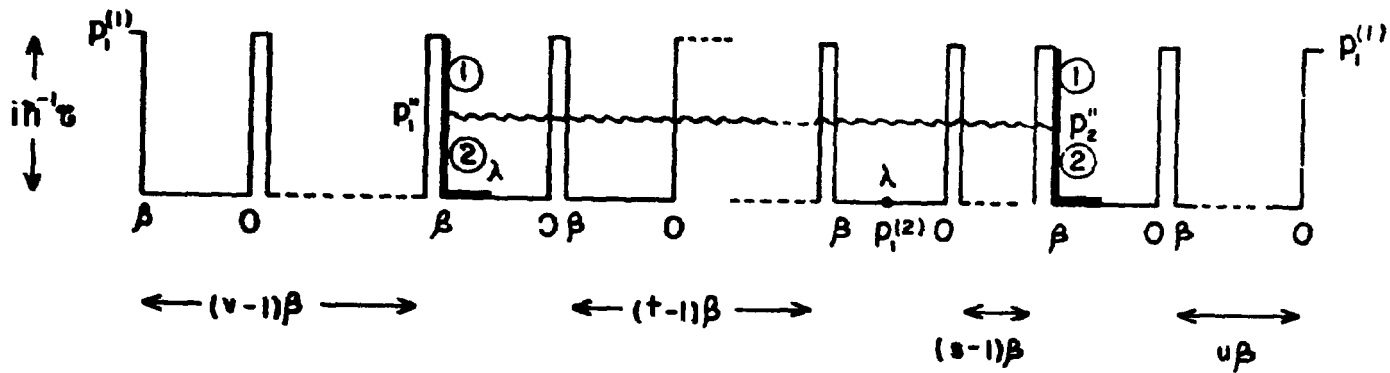


Figure 3 – The graphical representation of the first term in Eq.(2-29). The numbers, 1 and 2, in the graph represent  $\beta + i\hbar^{-1}\tau - \beta$  and  $\beta' - \lambda$ , respectively. The summand parameter  $t$  starts with zero, yet  $(t - 1)\beta$  appears because boxes on the both sides of the  $(t - 1)\beta$  boxes can become identical one.



The above expression is most general and can not be expressed in terms of the one-body trace.

### III - THE CONDUCTIVITY OF THE ELECTRON PLASMA (NO-IMPURITY)

When impurity effect is not introduced, the whole theory must be reduced to that for the ideal gas even with the presence of the e-e interaction. Although the result is obvious it is worthwhile to show how it is derived in our theory. The Hamiltonian H here consists of only the kinetic energy and commutes with the momentum operator. The kernels for Eq.(2-23) are those for the ideal gas, then

$$K_1(p_1^{(1)}, s_1; p_1^{(2)}, s_2) = \delta^{(3)}(p_1^{(1)} - p_1^{(2)}) \exp\{-\beta(p_1^{(1)})^2/2m\} \quad (3.1)$$

As a result one obtains

$$\sigma_{\mu\nu}^{[0]}(\omega) = \delta_{\mu\nu} \frac{e^2}{i\omega m} \frac{p_F^3}{2 \times 3\pi^2 \hbar^3} \quad (3.2)$$

The first order contribution is given as

$$\begin{aligned} \sigma_{\mu\nu}^{[1]}(\omega) &= -\frac{e^2}{m^2} \frac{\beta^2}{i\omega} \frac{1}{\hbar^6} \int d\vec{q} u(q) \int d\vec{p} \{ p_\mu(p_\nu + q_\nu) f(p) [1 - f(p)] \\ &\quad \times f(\vec{p} + \vec{q}) [1 - f(\vec{p} + \vec{q})] + p_\mu p_\nu [1 - f(p)][1 - 2f(p)] f(p) f(\vec{p} + \vec{q}) \} \\ &= \sigma_{\mu\nu}^{[0]}(\omega) \cdot \frac{3m e^2}{\pi p_F \hbar} \end{aligned} \quad (3.3)$$

The grand partition function necessary to this order can be found elsewhere<sup>(7)</sup>, and the operation in Eq.(2-3) gives

$$\frac{N}{V} = \frac{\rho_0^3}{6\pi^2} = \frac{\rho_F^3}{6\pi^2} (1 + 3s + \dots), \quad (3.4)$$

and then

$$\rho_0 = \rho_F (1 + 3s + \dots)^{1/3}, \quad (3.5)$$

where

$$s \equiv \frac{m e^2}{\pi p_F \hbar} = \frac{1}{\pi} \left[ \frac{4\pi}{9} \right]^{1/3} r_s \left\{ \frac{\rho_0}{\rho_F} \right\} \equiv s_0 \left[ \frac{\rho_0}{\rho_F} \right].$$

Adding Eq.(3-2) with Eq (3-3) and introducing Eq (3-5), one obtains

$$\sigma_{\mu\nu}(\omega) = \frac{e^2 n}{i\omega m} \quad (3-6)$$

where  $n = p_0^3/6\pi^2 \hbar^3$ . Throughout the formalism the electronic spin factor was neglected.

#### IV – IMPURITY EFFECT ON THE CONDUCTIVITY

The procedure of solving Eq.(2-31) is complicated for finite frequencies but it is possible in the case that the frequencies are much smaller than the Fermi energy as was done by T.Y.Q.. To make the situation simpler, let us assume the frequency to be zero here, and later combine this result with the plasma conductivity. Chester and Thellung showed the following description of the conductivity:

$$\begin{aligned} \sigma_{\mu\nu} &= \beta \lim_{T \rightarrow \infty} \int_{-T}^T dt \text{Tr} \left\{ \frac{1}{2} \rho_0 [J_\mu(t) J_\nu(0) + J_\mu(0) J_\nu(t)] \right\} \\ &= \frac{\beta}{2} \lim_{T \rightarrow \infty} \int_{-T}^T dt \text{Tr} \left\{ \rho_0 J_\mu(t) J_\nu(0) \right\}, \end{aligned} \quad (4-1)$$

where  $\rho_0$  is either  $\exp(\beta[\mu N - H])/Z_0$  or  $\exp(-\beta H)/Z_N^0$ , and  $B(t) = \exp(i\hbar^{-1} t H) B \exp(-i\hbar^{-1} t H)$ . The Eq (4-1) can be obtained with an assumption that  $\sigma_{\mu\nu} = \sigma_{\nu\mu}$ . With this formula one notices some slight change in the complex reciprocal temperature in the propagator expression. In Eqs.(2-23) and (2-29), the reciprocal temperatures,  $\beta + i\hbar^{-1} \tau$ ,  $\nu\beta + i\hbar^{-1} \tau$  and  $t\beta + i\hbar^{-1} \tau$  are replaced by  $\beta$ ,  $\nu\beta$  and  $t\beta$ , all the  $\lambda$ 's, by  $i\hbar^{-1} \tau$  and  $i\hbar^{-1} \tau$  in the last kernels, by zero, respectively. The  $\lambda$ -integration must be replaced by  $\beta$ . Because of the lengthy expression the explicit forms are omitted. The expression with operators are now given with the Hamiltonian of one particle in Eq.(2-5-c) as

$$\begin{aligned} \sigma_{\mu\nu}^{[0]} &= \frac{e^2 \beta}{m \hbar^3} \lim_{T \rightarrow \infty} \int_{-T}^T dt \int d\vec{p} \langle \vec{p} | f(H)(1 - f(H)) p_\mu(0) p_\nu(t) | \vec{p} \rangle \\ \sigma_{\mu\nu}^{[1]} &= \frac{e^2 \beta}{m \hbar^3} \lim_{T \rightarrow \infty} \int_{-T}^T dt \int d\vec{p} \int d\vec{p}' \int d\vec{q} \int d\vec{q}' \int_0^\beta d\beta' \\ &\times \{ \langle \vec{p}' | P_\mu(i\hbar\beta') | \vec{p} - \vec{q} \rangle \langle \vec{p} | p_\nu(t) | \vec{p} + \vec{q} \rangle + \langle \vec{p}' | f_- p_\nu(t) P_\mu(i\hbar\beta') | \vec{p} - \vec{q} \rangle \langle \vec{p} | f | \vec{p}' + \vec{q} \rangle \\ &- \langle \vec{p}' | P_\mu(i\hbar\beta') p_\nu(t) f | \vec{p} - \vec{q} \rangle \langle \vec{p} | f | \vec{p}' + \vec{q} \rangle \} + \frac{i}{\hbar} \int_0^\beta d\tau \\ &\times \{ \langle \vec{p}' | P_\mu(-\tau) | \vec{p} - \vec{q} \rangle \langle \vec{p} | f p_\nu(t - \tau) - p_\nu(t - \tau) f | \vec{p}' + \vec{q} \rangle \\ &+ \langle \vec{p}' | P_\mu(-\tau) p_\nu(t - \tau) - p_\nu(t - \tau) P_\mu(-\tau) | \vec{p} - \vec{q} \rangle \langle \vec{p} | f | \vec{p}' + \vec{q} \rangle \} \end{aligned} \quad (4-3)$$

where  $f$ ,  $f_-$ ,  $P_\mu(t)$  are defined in Eq.(2-33). The first three terms can be combined by making use of the fact that<sup>(3)</sup>

$$\frac{\partial}{\partial \mu \nu} f' = \int_0^\beta d\beta' P_\mu(i\hbar\beta) = \int_0^\beta d\beta' f(H) p_\mu (-i\hbar\beta') (1 - f(H))' \quad (4-4-a)$$

where

$$f' = [1 + \exp(\beta(H - E_F - \vec{p} \cdot \vec{u}))]^{-1}. \quad (4-4-b)$$

with  $u$  being a c-number. One then obtains

$$\sigma_{\mu\nu}^{[1]} = \frac{e^2 \beta}{m \hbar^3} \lim_{T \rightarrow \infty} \int_{-T}^T dt \int d\vec{p} d\vec{p}' d\vec{q} u(q) \left[ \frac{\partial}{\partial u_\mu} \langle \vec{p}' | f' | \vec{p} - \vec{q} \rangle \right. \\ \left. x \langle \vec{p} | f' | \vec{p}_\nu(t) f' | \vec{p}' + \vec{q} \rangle + \frac{i}{\hbar} \int_0^t d\tau \text{ (the same as Eq.(4-3))} \right] \quad (4-5)$$

To obtain the lowest order in impurity density,  $n_i$ , it is only necessary to maintain the correct Hamiltonian in the places of  $p_\nu(t)$ , and the rest of the Hamiltonians are approximated by those for the free particles. The second part of the right side of Eq.(4-5) then disappears and Eqs.(4-2) and (4-5) are reexpressed to give the following simple forms:

$$\sigma_{\mu\nu}^{[0]} = \frac{e^2 \beta}{m \hbar^3} \lim_{T \rightarrow \infty} \int_{-T}^T d\vec{p} (1 - f_0(p)) f_0(p) p_\mu \int_{-T}^T d\vec{p}' p_\nu(t) |\vec{p}\rangle \langle \vec{p}'| \quad (4-6-a)$$

$$\sigma_{\mu\nu}^{[1]} = \frac{e^2 \beta}{m \hbar^3} \lim_{T \rightarrow \infty} \int_{-T}^T d\vec{p} d\vec{q} u(q) \frac{\partial}{\partial u_\mu} [f_0(\vec{p} + \vec{q}) [1 - f_0(p)] f_0(p)] \int_{-T}^T d\vec{p}' p_\nu(t) |\vec{p}\rangle \langle \vec{p}'| \quad (4-6-b)$$

Assuming that the impurities are distributed at random in the system, and that only binary scattering is important, the time integral in Eq.(4-6) can be given as

$$\lim_{T \rightarrow \infty} \int_{-T}^T d\vec{p}' p_\nu(t) |\vec{p}\rangle \langle \vec{p}'| = p_\mu \tau(p), \quad (4-7-a)$$

where

$$\frac{1}{\tau(p)} = \frac{(2\pi)^4 n_i}{(2\pi)^6 \hbar^4} \int d\vec{k} |\mathcal{V}(\vec{k})|^2 \delta(E_{\vec{p}+\vec{k}} - E_{\vec{p}}) [1 - \cos(\vec{p}, \vec{p} + \vec{k})] \quad (4-7-b)$$

Substituting Eq.(4-7-a) into Eq. (4-6) and recognizing that a possible value of  $p$  is only at  $p_F$  and that  $\tau(p)$  is a smooth function of  $p$  at least near  $p_F$ , then  $\tau(p)$  can be factored out of the integration, as  $\tau(p_F)$ . The final expression is identical to the case without impurities except that instead of  $1/\omega$ ,  $\tau(p_F)$  is introduced. Thus,

$$\sigma_{\mu\nu} = \delta_{\mu\nu} \frac{e^2 n}{m} \tau(p_F) \left[ \frac{\eta_F}{p_0} \right]^3 (1 + 3s + \dots) \\ = \delta_{\mu\nu} \frac{e^2 n}{m} \tau(p_F). \quad (4-8)$$

Here, the first order exchange effect only appears through the value of  $p_F$  in the expression of the relaxation time. The addition of relaxation times from various sources is given as

$$\frac{1}{\tau_{\text{tot}}} = \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3} + \dots$$

and if we recognize that  $1/\omega$  is a part of the relaxation time mechanism, the Eq (3-6) may be added to Eq.(4-4) to obtain

$$\sigma_{\mu\nu} \approx \delta_{\mu\nu} \frac{e^2 n}{m} \frac{(p_F)}{1 + i\omega \tau(p_F)} \quad (4-9)$$

It is important to recognize that  $\tau(p_F)$  also depends on the frequency  $\omega$ , in reality, but it may be neglected if  $\omega$  is much smaller than  $E_F$ . Thus the Drude formula for the low frequencies is obtained. The ideal gas part of the T.Y.Q.'s result has the opposite sign to our result. Our result is the consequence of the definition of the external electric field. It, therefore, does not contradict with the classical result. Gotze and Woffle obtained the frequency dependence of the relaxation time quantitatively, and showed that for the small frequencies, the dependence is minor.

The relaxation time at the Fermi momentum is given, by integrating the radial part of the momentum variable, by

$$\begin{aligned} \frac{1}{\tau(p_F)} &= \frac{n_i m p_0}{(2\pi)^2 [\rho_i / \rho_0]^2 [1 + 3s + \dots]} \int d\Omega \sigma_{p_F}(\theta) (1 - \cos \theta) \\ &= \frac{n_i m p_F}{(2\pi)^2} \int d\Omega \sigma_{p_F}(\theta) (1 - \cos \theta), \end{aligned} \quad (4-10)$$

where

$$V(k) = \int V(r) \exp\left\{ \frac{i}{\hbar} \vec{k} \cdot \vec{r} \right\} d\vec{r} \sigma_{p_F}(\theta) = \left| \frac{V(\vec{p}_F - \vec{p}_F)}{\hbar^2} \right|^2$$

Here, the scattering amplitude,  $\sigma_{p_F}(\theta)$ , depends on  $p_F$  as well as the coefficient in front of the inverse of the relaxation time.

## V - DISCUSSION

We have performed explicit calculations of the e-e interaction for the electrical conductivity to the lowest order of impurity density by making use of quantum statistical mechanics, and also have investigated the  $p_F$ -dependency of the electrical conductivity. The final expression is simple and only the relaxation time is purely  $p_F$ -dependent, and its form is the same as that for the ideal gas relaxation time with impurity. The rest of the conductivity expression does not depend on  $p_F$  but on the real number density of the electron gas. It must be pointed out, however, that although the e-e interaction changes the magnitude of the relaxation time, it is not only because of the screening effect as was pointed out by Langer and others but also because of the decrease of the Fermi momentum, which makes electrons at the Fermi surface less active. To include the screening effect due to the electron cloud, one has to involve the chain diagrammatic expression in between an impurity and an electron of interest: thus  $V(k)$  will be modified approximately by

$$\frac{V(k)}{1 + u(k)\lambda_j(k)} \quad (5-1)$$

where  $\lambda_j(k)$ , the eigenvalues of the electron propagator, can be found in the same book by Isihara. The appearance of this sort of screening effect may be independent of the counting method of graphs which we have investigated.

Assuming only  $j=0$  term with small momentum is important, Eq.(5-1) becomes

$$\frac{4\pi e^2 \hbar^2 Z}{\kappa^2 + k^2} \cdot \kappa^2 = \xi r_s p_0^2 \left[ \frac{p_F}{p_0} \right], \quad (5-2)$$

where  $\xi = 4 [2\pi^2/3]^{2/3}/\pi^2$ ,

and  $Z$ , the number of excess valence of the impurity atom.

Using Eq.(5-2) the relaxation time has the form

$$\frac{1}{\tau(p_F)} = \frac{n_i m p_0}{\pi} \left[ \frac{4\pi e^2 Z}{p_0^2} \right]^2 \left( \frac{p_0}{p_F} \right)^3 \left[ \frac{A}{1+A} - 1 + \ln \frac{1+A}{A} \right], \quad (5-3)$$

with  $A = \xi (r_s/4) (p_0/p_F)$ .

The ratio  $\tau(p_0)$  to  $\tau(p_F)$  is approximately given as

$$\frac{\tau(p_0)}{\tau(p_F)} \sim \left[ \frac{p_0}{p_F} \right]^3. \quad (5-4)$$

The screening effect buffers the impurity effect so as to make the relaxation time in Eq.(5-3) larger. On the other hand, the effects of the e-e interaction makes  $p_F$  decrease with increase of  $r_s$  so that, as can be seen from Eq.(5-4), the real relaxation time decreases slightly from the ideal gas relaxation time. These two separate effects, thus, play different roles on the relaxation time. Langer did not mention this point although he referred to the same screening effect, since the renormalization of the Fermi level was not performed explicitly. Using Eq.(4-10), the conductivity can be written as

$$\sigma_{\nu} = \delta_{\mu\nu} \frac{e^2 p_F^2}{\pi m^2 n_i} [1 + 3s + O(s^2)] \left[ \int_0^\pi \sin\theta d\theta |\nu(\vec{p}_F - \vec{p}_F')|^2 (1 - \cos(\vec{p}_F, \vec{p}_F')) \right]^{-1}.$$

Comparing this with the result by Langer, one observes that the term,  $(1 + 3s + O(s^2))$ , is not included in his expression. This comes from the fact that the Fermi momentum  $k_F$  in their theory, which differs from the absolute Fermi momentum, is not identical to our  $p_F$ . The Fermi momentum which we use is directly connected with the fugacity or the chemical potential. The theory by T.Y.Q. is the extension of Langer's. They have the final expression of the conductivity with an effective mass  $m^* = p_0/u_g$ , where  $u_g$  is the group velocity at the Fermi level, and with the effective relaxation time  $\tau^*$ , which also contains  $u_g$ . Reexpression of their result shows that the group velocity goes away from the expression of the conductivity. The result is given in their notation as

$$\sigma = \frac{n_e e^2}{m^*} \tau^* = \frac{n_e e^2}{m} \sigma,$$

$$\frac{1}{\tau} = \frac{n_e p_0 m}{(2\pi)^2} \left[ 1 - \frac{m}{p_0} \frac{\partial \Sigma(p_0, 0)}{\partial p_0} \right]^{-2} \int d\Omega |u(\theta)|^2 (1 - \cos\theta), \quad (5-5)$$

where  $p_0$  is the absolute Fermi momentum.

Comparing this expression with ours, Eq.(4-10), our renormalization may be identified as

$$\left\{ \frac{p_F}{p_0} \right\}^2 [1 + 3s + \dots] \Leftrightarrow \left[ 1 - \frac{m}{p_0} \frac{\partial \Sigma(p_0, 0)}{\partial p_0} \right]^2.$$

It is not clear in their notation whether or not  $|u(\theta)|^2$  depends on  $p_F$ . According to Langer, the corresponding part does. When the classical equation of motion for the center-of-mass degree of freedom is used to obtain the relaxation time i.e. the center-of-mass relaxation time, the momentum appearing in the expression must be the Fermi momentum  $p_F$  which involves the quantum effects and the e-e interaction effects.

Before closing our discussion we would like to comment a few points concerning our result. The dependence of the relaxation time and the conductivity on the Fermi momentum  $p_F$  would not change even with the inclusion of the higher order e-e interactions, apart from the argument concerning screening effects. If so, the effective mass appearing in the conductivity is mainly due to the band effect. This may be true since the e-e interaction effect only appears through the impurity potential with  $p_F$ . The Fermi momentum up to the third order ring diagram contribution (except the third order exchange) was recently calculated by the author and A. Isihara<sup>(9)</sup>. In the metallic region, the ratio,  $p_F/p_0$ , decreases from unity by 20 ~ 50 per cent.

#### Appendix I

First few terms in Eq.(2-16) can be expressed as

$$b^{[0]}(p_1^{(1)}, p_1^{(2)}) = \frac{1}{V \bar{z}_0} \sum_N D_N \quad (I-1)$$

and

$$D_1 = A_1$$

$$D_2 = (zI_1 A_1 + A_2 + B_2)$$

$$D_3 = (A_3 + I_1 A_2 z + (I_1 - I_2) z^2 A_1 / 2 - B_3 + zI_1 B_2)$$

$$D_4 = \dots$$

.

.

.

, where  $A_q$  is defined in Eq.(2-20), and

$$B_q \equiv \sum_{s=1}^q K_1(p_1^{(1)}, s\beta; p_1^{(1)}, 0) K_1(p_1^{(2)}, (q-s)\beta; p_1^{(2)}, 0) \quad (I-2)$$

$$I_Q \equiv \int K_1(p_1^{(1)}, \varrho; p_1^{(1)}, 0) dp_1^{(1)} \equiv (-)^{\lambda-1} \mathcal{R} b_Q^0 \tag{1-3}$$

Substituting Eqs.(1-2), (1-3) and (2-20), into (1-1) one will observe following procedure:

$$\begin{aligned} b^0 &= \frac{1}{\Xi_0} \cdot [ A_1(1 + zI_1 + \frac{1}{2} (I_1^2 - I_2)z^2 + \dots) \\ &+ A_2(1 + zI_1 + \dots) \\ &+ A_3(1 + \dots) + \dots ] \\ &= \frac{1}{\Xi_0} \sum A_Q (1 + zI_1 + \frac{1}{2!} I_1^2 z^2 + \dots - \frac{1}{2} I_2 z^2 + \dots) \\ &= \frac{1}{\Xi_0} \sum_Q A_Q \exp(zI_1 - z^2 I_2/2 + \dots) \\ &= \sum_Q A_Q \end{aligned}$$

The final expression is the same as Eq.(2-23).

### Appendix II

As a example of construction of  $C_Q$ , let us make the first term in Eq.(2-29). For this purpose see Fig. 4.

To arrive at  $p_2''$  from  $p_1^{(1)}$ , the complex reciprocal temperature moves a path

$$-i; \hbar^{-1}\tau - \nu\beta + i\hbar^{-1}\tau - i\hbar^{-1}\tau + \lambda + 2 = \nu\beta + \beta' - i\hbar^{-1}\tau$$

Therefore one has the kernel  $K_1(p_2', \nu\beta + \beta'; p_1^{(1)}, i\hbar^{-1}\tau)$ . To arrive at  $p_1^{(2)}$  from  $p_2''$ , the path is

$$i; -i\hbar^{-1}\tau + (s-1)\beta + i\hbar^{-1}\tau - i\hbar^{-1}\tau + \lambda = s\beta - \beta' + \lambda$$

,where s starts from one, then obtains  $K_1(p_1^{(2)}, \nu\beta + \lambda; p_2' - \nu\beta')$ . To arrive at  $p_1'$  from there,

$$\beta - \lambda + i\hbar^{-1}\tau - i\hbar^{-1}\tau + (t-1)\beta + i\hbar^{-1}\tau - i\hbar^{-1}\tau + \lambda + 2 = \nu\beta + \beta' - \lambda$$

then  $K_1(p_1, \nu\beta + \beta'; p_1^{(2)}, \lambda)$ , here t starts with zero, yet (t-1) appears because boxes on the both sides of (t-1) $\beta$  boxes can become identical one. For the last part, the path is

$$i\hbar^{-1}\tau + (\nu-1)\beta - i\hbar^{-1}\tau + 1 = \nu\beta + i\hbar^{-1} - \beta'$$

Thus the corresponding kernel is  $K_j(p_1^{(1)}, \nu\beta + i\hbar^{-1}\tau; p_1^{(2)} - q, \beta')$

A convention was used in which whenever  $p_1^{(i)}$  appears in the left part of the kernel either  $\pm q$  was attached. Several important rules of the construction of graphs are given below:

- 1) The reciprocal temperature must always be overall positive
- 2) The imaginary part never exceed  $i\hbar^{-1}\tau$ .
- 3) The position of  $(p_1^{(2)}, \lambda)$  can be chosen at any position in the bottom parts of the boxes.
- 4) The total number of  $\beta$  distributed in the kernel must have altogether  $l$
- 5) The positions of  $p_1^{(1)}$  are always fixed at the both corners of the graph.
- 6) The two thick lines representing the range of the  $\beta'$ -integration appears in any positions in the graph, but the pair must appear in the same range.

Exactly same method can be used to construct the second order exchange contribution.

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#### RESUMO

Uma teoria de muitos corpos por meio do método dos propagadores, desenvolvido por Montroll e Ward para a mecânica estatística no equilíbrio, é reformulada para descrever a condutividade elétrica de um gás de elétrons contendo impurezas. A teoria inclui interações elétron-impureza até ordem infinita e interações elétron-elétron até efeitos de troca de primeira ordem. O propagador usado por Montroll e Ward é separado em dois propagadores, cada um dos quais satisfaz a equação de Bloch ou de Schrödinger, para utilizar o método perturbativo. A contagem correta dos diagramas é apresentada. A mudança no tempo de relaxação ocasionada pela interação elétron-elétron é mostrada explicitamente e comparada com trabalhos recentes.

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