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ABSTRACT

Three problems involving two media in plane geometry are solved numerically in two-group neutron transport theory for isotropic scattering: two slabs with an incident flux, the critical problem for reflected slab reactors, and the cell problem. Each problem is reduced to a set of regular integral equations for the coefficients of the Case expansions and solved iteratively. Numerical results are reported for all problems.

1 INTRODUCTION

The two-group neutron transport equation for isotropic scattering in plane geometry has been studied by many researchers in the singular eigenfunction expansion method[4]. The first work was reported, soon after the introduction of the method, by Zelazny and Kuszell[22], but their completeness arguments were not quite conclusive. Some years later Siewert and Shieh[18], following the work of Siewert and Zweifel[19] on a special case of the multigroup model, proved rigorously the full-range completeness and orthogonality theorems and analysed the discrete spectrum. Some attempts were made to solve half-space[12,14,15] and slab[1,6] problems but it was not until the half-range completeness and orthogonality theorems were established[3,17,20] that these problems were solved in a concise manner. Half-space problems[17] are solved in terms of and H matrix that can be obtained numerically by a rapidly converging iterative scheme and slab problems[11] can be converted to systems of regular integral equations for the expansion coefficients which can then be solved by numerical iterations.

Problems involving two media, however, have remained unsolved in two-group theory, though in the one-group model some problems have been solved using the two-media orthogonality relations[13] and by other methods[2,16]. The difficulty is in that the use of the full-range and half-range orthogonality relations does not remove all singularities that are inherent in the Case method and that the numerical solution of the resulting singular integral equations involves numerical differentiations. Further, two-media orthogonality relations have not been found in two-group theory. Jauho and Rajamäki[10] studied two-media problems in multi-group theory but they did not report any numerical results and it appears rather difficult to obtain numerical results based on their analysis. The first numerical results for two-media problems were reported by Ishiguro and Maiorino[9] using a method based on the half-range orthogonality relations and invariance principles. Their method, however, is applicable only to two-half-space problems. Thus, a general systematic method to solve various two- or multi-slab problems has been lacking and many model problems in transport theory have remained unsolved.

In a recent paper[8] Ishiguro proposed a method of this kind and reported some numerical solutions in one-group theory.

The purpose of this paper is to show that two-media problems in two-group neutron transport theory for isotropic scattering can be converted, using the method of Ref. 8, to a set of regular integral equations for the coefficients of the Case expansions and solved numerically by a standard iterative method and to report numerical results for some model problems based on exact theory. We shall consider three problems, two slabs with an incident flux, the critical problem for reflected slab reactors, and the cell problem, but we would like first to summarize the method of regularization and the basic theory.

THE METHOD OF REGULARIZATION

The method to derive a set of regular integral equations for the expansion coefficients from the set of singular integral equations that results from boundary and interface conditions can be summarized in the following steps[8]:

1 At an interface separate the continuity condition into two equations, one for $\mu\epsilon(0,1)$, the other for $\mu\epsilon(-1,0)$.

2a To the $\mu\epsilon(0,1)$ equation apply the half-range orthogonality relations for the right-side medium.

b In the $\mu\epsilon(-1,0)$ equation change μ to $-\mu$ and then apply the orthogonality relations for the left-side medium.

3a If any singularity remains in step 2a, consider the interface (or boundary) condition for $\mu > 0$ at the left-side boundary of the left-side medium and generate the same singularity, subtract the result from the equation in step 2a and remove the singularity.

b For step 2b consider the right-side interface of the right-side medium and generate the same singularity from the $\mu < 0$ equation.

4 If sigularities remain in step 3 repeat the process, generating the same singularities at different interfaces.

Although the equation for a discrete coefficient is always found to be regular, we apply to this equation the same operations as those applied to the equation for the corresponding continuum coefficient, since the convergence of iterations is sometimes faster and the discrete and continuum coefficients are obtained in the same form. We not that for a symmetric geometry the right and left interfaces are equivalent.

SOLUTION OF THE TRANSPORT EQUATION

The two-group neutron transport equation for isotropic scattering can be written as

$$\mu \frac{\partial}{\partial x} \underset{\sim}{I}(x,\mu) + \underset{\sim}{\Sigma}\,\underset{\sim}{I}(x,\mu) = \underset{\sim}{Q} \int_{-1}^{1} \underset{\sim}{I}(x,\mu')\, d\mu' , \tag{1}$$

where the space variable x is measured in units of the mean-free-path for group 2 neutrons. As in previous works[9,11,17], we assume that the scattering matrix $\underset{\sim}{Q}$ is neither diagonal nor triangular and that det $\underset{\sim}{Q} \neq 0$, and introduce a matrix $\underset{\sim}{P}$ defined as

$$\underset{\sim}{P} = \begin{bmatrix} \sqrt{q_{21}/q_{12}} & 0 \\ 0 & 1 \end{bmatrix} .$$

where q_{ij} are the elements of Q. Then the solution of Eq. (1) is given by

$$I(x,\mu) = P^{-1}\,\Psi(x,\mu)\,, \tag{3}$$

where $\Psi(x,\mu)$ is the solution of

$$\mu\frac{\partial}{\partial x}\Psi(x,\mu) + \Sigma\,\Psi(x,\mu) = C\int_{-1}^{1}\Psi(x,\mu')\,d\mu'\,, \tag{4}$$

with the symmetrized scattering matrix given by $C = PQP^{-1}$ and the elements of Σ are $\Sigma_{11} = \sigma$, $\Sigma_{12} = \Sigma_{21} = 0$, and $\Sigma_{22} = 1$.

The general solution of Eq (4) can be written[17,18] as

$$\Psi(x,\mu) = \sum_{i=1}^{\kappa}\left[A(\nu_i)\,\Phi(\nu_i,\mu)\exp(-x/\nu_i) + A(-\nu_i)\,\Phi(-\nu_i,\mu)\exp(x/\nu_i)\right]$$

$$+ \int_{①}\left[A_1^{(1)}(\nu)\,\Phi_1^{(1)}(\nu,\mu)\exp(-x/\nu) + A_2^{(1)}(\nu)\,\Phi_2^{(1)}(\nu,\mu)\exp(-x/\nu)\right]d\nu$$

$$+ \int_{②} A^{(2)}(\nu)\,\Phi^{(2)}(\nu,\mu)\exp(-x/\nu)\,d\nu\,, \tag{5}$$

where A's are expansion coefficients to be determined by the boundary condition once a specific problem is considered, discrete eigenvalues $\pm\nu_i$ are the zeros of det $\Lambda(z)$ with

$$\Lambda(z) = I - z\int_{-1}^{1} K(z,\mu)\,d\mu\,C\,, \tag{6}$$

κ (either 1 or 2)[18] is the number of pairs of the discrete eigenvalues, and the eigenfunctions can be written as

$$\Phi(\pm\nu_i,\mu) = \nu_i K(\nu_i,\pm\mu)\,C\,U(\nu_i)\,, \tag{7a}$$

$$\Phi_\alpha^{(1)}(\nu,\mu) = \{\nu K(\nu,\mu)\,C + \delta(\nu-\mu)\,\lambda(\nu)\}\,U_\alpha^{(1)}(\nu)\,,\quad \alpha = 1,2\,,$$

$$\nu\epsilon\,\text{Region } ① = (-1/\sigma, 1/\sigma)\,, \tag{7b}$$

and

$$\Phi^{(2)}(\nu,\mu) = \{\nu K(\nu,\mu)\,C + \delta(\nu-\mu)\,\lambda(\nu)\}\,U^{(2)}(\nu)\,,$$

$$\nu\epsilon\,\text{Region } ② = (-1,-1/\sigma)\cup(1/\sigma,1)\,.$$

Here

$$\underset{\sim}{K}(\xi,\mu) = \begin{bmatrix} \dfrac{P}{\sigma\xi-\mu} & 0 \\ 0 & \dfrac{P}{\xi-\mu} \end{bmatrix} , \quad \underset{\sim}{\delta}(\nu,\mu) = \begin{bmatrix} \delta(\sigma\nu-\mu) & 0 \\ 0 & \delta(\nu-\mu) \end{bmatrix} , \tag{8a,b}$$

$$\underset{\sim}{U}_1^{(1)}(\nu) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underset{\sim}{U}_2^{(1)}(\nu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9a,b}$$

$$\underset{\sim}{U}^{(2)}(\nu) = \begin{bmatrix} -\lambda_{12}(\nu) \\ \lambda_{11}(\nu) \end{bmatrix} \qquad \underset{\sim}{U}(\nu_i) = \begin{bmatrix} -\Lambda_{12}(\nu_i) \\ \Lambda_{11}(\nu_i) \end{bmatrix} \tag{9c,d}$$

and

$$\underset{\sim}{\lambda}(\nu) = \underset{\sim}{I} - \nu\int_{-1}^{1} \underset{\sim}{K}(\nu,\mu)\, d\mu \underset{\sim}{C} \tag{10}$$

with $\underset{\sim}{I}$ being the 2x2 identify matrix.

The full-range amd half-range completeness and orthogonality theorems regarding the solution given by Eq. (5) have been established[3,17,18,20].

Although the solution has been used in previous works[9,11,17] in the form of Eq. (5), we write the general solution in a more compact form as

$$\underset{\sim}{\Psi}(x,\mu) = \sum_{i=1}^{k} [\, A(\nu_i)\, \underset{\sim}{\Phi}(\nu_i,\mu) \exp(-x/\nu_i) + A(-\nu_i)\, \underset{\sim}{\Phi}(-\nu_i,\mu) \exp(x/\nu_i) \,]$$
$$+ \int_0^1 \underset{\sim}{\Phi}(\nu,\mu)\, \underset{\sim}{A}(\nu) \exp(-x/\nu)\, d\nu + \int_0^1 \underset{\sim}{\Phi}(-\nu,\mu)\, \underset{\sim}{A}(-\nu) \exp(x/\nu)\, d\nu \ , \tag{11}$$

where the discrete eigenfunctions are the same as in Eq. (7a), the continuum eigenfunction is a 2x2 matrix defined as

$$\underset{\sim}{\Phi}(\nu,\mu) = \nu \underset{\sim}{K}(\nu,\mu)\, \underset{\sim}{C} + \underset{\sim}{\delta}(\nu,\mu)\, \underset{\sim}{\lambda}(\nu) \quad , \quad \nu\epsilon(-1\,,1) \quad , \tag{12}$$

and $\underset{\sim}{A}(\pm\nu)$ are two-vector expansion coefficients. We not that the expansion given in Eq. (11) is not the general solution of Eq. (4) if $\underset{\sim}{A}(\pm\nu)$ are arbitrary for $\nu\epsilon(1/\sigma,1)$. However, as later equations show, $\underset{\sim}{A}(\pm\nu)$ are always found, in our formalism, to be proportional to $\underset{\sim}{U}^{(2)}(\nu)$ for $\nu\epsilon(1/\sigma,1)$ and thus, considering Eqs. (5), (7), and (9), we can write Eq. (5) in the more compact form of Eq. (11). We shall always separate positive and negative eigenvalues, as in Eq. (11), and use the symbols ν, ξ, and η to denote positive eigenvalues.

THE H MATRIX

The $\underset{\sim}{H}$ matrix introduced in Ref. 17 plays a principal role in the half-range orthogonality theorem and has been discussed in detail in Ref. 20. We list some of the equations it satisfies for use in our problems.

The $\underset{\sim}{H}$ matrix satisfies the integral equations

$$\underset{\sim}{\tilde{H}}(z)\,\underset{\sim}{\Lambda}(z) = \underset{\sim}{I} + z\int_0^1 \underset{\sim}{\tilde{H}}(\mu)\,\underset{\sim}{\Theta}(\mu)\,\frac{d\mu}{\mu - z}\,\underset{\sim}{C}\,, \qquad z \in (0\,,1)\,, \tag{13a}$$

and

$$\nu_i \int_0^1 \underset{\sim}{\tilde{H}}(\mu)\,\underset{\sim}{\Theta}(\mu)\,\frac{d\mu}{\nu_i - \mu}\,\underset{\sim}{C}\,\underset{\sim}{U}(\nu_i) = \underset{\sim}{U}(\nu_i)\,, \qquad i = 1,\cdots,\kappa, \tag{13b}$$

where

$$\underset{\sim}{\Theta}(\mu) = \begin{bmatrix} \Theta(\mu) & 0 \\ 0 & 1 \end{bmatrix}, \quad \Theta(\mu) = 1 \text{ for } \mu \epsilon\, (0,1/\sigma) \text{ and } \Theta(\mu) = 0 \text{ otherwise.} \tag{14}$$

To calculate the $\underset{\sim}{H}$ matrix numerically we can use equation

$$\underset{\sim}{H}(z) = \underset{\sim}{I} + z\underset{\sim}{H}(z)\underset{\sim}{C}\int_0^1 \underset{\sim}{\tilde{H}}(\mu)\,\underset{\sim}{\Theta}(\mu)\,\frac{d\mu}{\mu + z}\,, \qquad z \notin (-1,0)\,. \tag{15}$$

The dispersion matrix $\underset{\sim}{\Lambda}(z)$ can be factored in terms of the $\underset{\sim}{H}$ matrix as

$$\underset{\sim}{H}(-z)\,\underset{\sim}{C}\,\underset{\sim}{\tilde{H}}(z)\,\underset{\sim}{\Lambda}(z) = \underset{\sim}{C}\,, \qquad z \notin (-1,1)\,. \tag{16}$$

If we let $z \to \nu \pm 0$ in Eq. (13a) we can find

$$\underset{\sim}{\tilde{H}}(\nu)\,\underset{\sim}{\lambda}(\nu) = \underset{\sim}{I} + \nu\int_0^1 \underset{\sim}{\tilde{H}}(\mu)\,\underset{\sim}{\Theta}(\mu)\,\frac{P}{\mu - \nu}\,d\mu\,\underset{\sim}{C}\,, \qquad \nu \in (0,1)\,. \tag{17}$$

Since the existence of a unique solution of these equations has been established[3,20] we shall use them freely in our problem: for example, we have from Eq. (15)

$$z\int_0^1 \underset{\sim}{\tilde{H}}(\mu)\,\underset{\sim}{\Theta}(\mu)\,\frac{d\mu}{\mu + z}\,\underset{\sim}{k} = \underset{\sim}{C}^{-1}\underset{\sim}{k} - \underset{\sim}{C}^{-1}\underset{\sim}{H}^{-1}(z)\,\underset{\sim}{k}\,, \qquad z \in (-1,0) \tag{18a}$$

and from Eq. (17)

$$\nu\int_0^1 \underset{\sim}{\tilde{H}}(\mu)\,\underset{\sim}{\Theta}(\mu)\,\frac{P}{\nu - \mu}\,d\mu\,\underset{\sim}{k} = \underset{\sim}{C}^{-1}\underset{\sim}{k} - \underset{\sim}{\tilde{H}}(\nu)\,\underset{\sim}{\lambda}(\nu)\,\underset{\sim}{C}^{-1}\,\underset{\sim}{k}\,, \qquad \nu \in (0,1) \tag{}$$

for an arbitrary 2x2 matrix $\underset{\sim}{k}$. We shall call these equations collectively the H equations.

HALF-RANGE ORTHOGONALITY AND RELATED INTEGRALS

Half-range orthogonality relations of the eigenfunctions are given in Ref. 17. However, since we use a different form to write the solution, we redefine the adjoint functions.

The discrete adjoint vector is the same as in Ref. 17:

$$\underset{\sim}{\Theta}(\nu_j,\mu) = \nu_j \underset{\sim}{K}(\nu_j,\mu)\, \underset{\sim}{h}(\mu)\, \underset{\sim}{H}^{-1}(\nu_j)\, \underset{\sim}{C}\,\underset{\sim}{U}(\nu_j) \quad , \quad \nu_j > 1 \text{ or } \nu_j = i\,|\nu_j| \quad . \tag{19a}$$

We define the continuum adjoint matrix as

$$\underset{\sim}{\Theta}(\nu,\mu) = [\; \nu\, \underset{\sim}{K}(\nu,\mu)\, \underset{\sim}{h}(\mu)\, \underset{\sim}{H}^{-1}(\nu)\, \underset{\sim}{C} + \underset{\sim}{\delta}(\nu,\mu)\, \underset{\sim}{\lambda}(\nu) \;]\; \underset{\sim}{W}(\nu) \quad , \quad \nu \in (0,1) \quad , \tag{19b}$$

where the symmetric matrix

$$W(\nu) = \begin{bmatrix} N_{22}(\nu) & -N_{21}(\nu) \\ -N_{12}(\nu) & N_{11}(\nu) \end{bmatrix} \Theta(\nu) + \underset{\sim}{U}^{(2)}(\nu)\, \underset{\sim}{\tilde{U}}^{(2)}(\nu)\,[\,1-\Theta(\nu)\,] \tag{20}$$

is the same matrix as was used in Ref. 11 and

$$\underset{\sim}{h}(\mu) = \begin{bmatrix} H_{11}(\mu/\sigma) & H_{12}(\mu/\sigma) \\ H_{21}(\mu) & H_{22}(\mu) \end{bmatrix} \qquad \mu \in (0,1) \quad , \tag{21}$$

with H_{ij} being the elements of the $\underset{\sim}{H}$ matrix.

With these adjoint functions, the orthogonality relations can be written as

$$\int_0^1 \underset{\sim}{\tilde{\Theta}}(\nu_i,\mu)\, \underset{\sim}{\Phi}(\nu_j,\mu)\, \mu d\mu = N(\nu_i)\, \delta_{ij} \quad , \tag{22a}$$

$$\int_0^1 \underset{\sim}{\tilde{\Theta}}(\nu_i,\mu)\, \underset{\sim}{\Phi}(\nu,\mu)\, \mu d\mu = \underset{\sim}{0} \quad , \tag{22b}$$

$$\int_0^1 \underset{\sim}{\tilde{\Theta}}(\nu,\mu)\, \underset{\sim}{\Phi}(\nu_i,\mu)\, \mu d\mu = \underset{\sim}{0} \quad ,$$

$$\int_0^1 \tilde{O}(\nu,\mu)\,\Phi(\nu',\mu)\,A(\nu')\,\mu d\mu = N(\nu)\,A(\nu)\,\delta(\nu-\nu') \quad , \tag{22d}$$

where $A(\nu)$ in the last formula is an arbitrary two-vector and the N functions in Eqs. (20) and (22a,d) are given explicitly in Refs. 11 and 17.

Since we shall need various half-range integrals of the product of eigenfunction and adjoint function, we summarize some of these formulas here. To simplify the notation we let

$$\tilde{X}(\nu_i) = \tilde{U}(\nu_i)\,C\,\tilde{H}^{-1}(\nu_i)\,C^{-1} \tag{23a}$$

and

$$\tilde{X}(\nu) = \tilde{W}(\nu)\,C\,\tilde{H}^{-1}(\nu)\,C^{-1} \quad . \tag{23b}$$

When the eigenfunction and adjoint belong to the same medium, we can evaluate the following integrals, using the H equations, to obtain

$$\int_0^1 \tilde{O}(\nu_i,\mu)\,\Phi(-\nu_j,\mu)\,\mu d\mu = \frac{\nu_i\nu_j}{\nu_i+\nu_j}\,\tilde{X}(\nu_i)\,H^{-1}(\nu_j)\,C\,U(\nu_j) \quad , \tag{24a}$$

$$\int_0^1 \tilde{O}(\nu_i,\mu)\,\Phi(-\nu,\mu)\,\mu d\mu = \frac{\nu_i\nu}{\nu_i+\nu}\,\tilde{X}(\nu_i)\,H^{-1}(\nu)\,C \quad , \tag{24b}$$

$$\int_0^1 \tilde{O}(\nu,\mu)\,\Phi(-\nu_i,\mu)\,\mu d\mu = \frac{\nu_i\nu}{\nu+\nu_i}\,\tilde{X}(\nu)\,H^{-1}(\nu_i)\,C\,U(\nu_i) \quad , \tag{24c}$$

and

$$\int_0^1 \tilde{O}(\nu,\mu)\,\Phi(-\nu',\mu)\,\mu d\mu = \frac{\nu\nu'}{\nu+\nu'}\,\tilde{X}(\nu)\,H^{-1}(\nu')\,C \quad . \tag{24d}$$

If the eigenfunction and adjoint belong to different media the integral of their product is more involved. All integrals can be performed, however, if we decompose the K matrix as

$$K(\xi,\mu) = \frac{P}{\sigma\xi-\mu}\,k_1 + \frac{P}{\xi-\mu}\,k_2 \tag{25a}$$

with

$$\mathbf{k}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{k}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \tag{25b,c}$$

and use the $\mathbf{H}$ equations, e.g., Eqs. (18). Since these formulas are rather lengthy and since the later equations for the three problems show most of them clearly, we report here only one, the simplest, of them:

$$\int_0^1 \mu \tilde{\mathbf{\Theta}}_1(\nu_i,\mu)\, \mathbf{G}\, \mathbf{\Phi}_2(-\eta_j,\mu)\, d\mu$$

$$= \nu_i \tilde{\mathbf{X}}_1(\nu_i) \left[\frac{\sigma_1\eta_j}{\sigma_2\eta_j + \sigma_1\nu_i} \mathbf{H}_1^{-1}(\sigma_2\eta_j/\sigma_1)\, \mathbf{k}_1 + \frac{\eta_j}{\eta_j+\nu_i} \mathbf{H}_1^{-1}(\eta_j)\, \mathbf{k}_2 \right] \mathbf{G}\, \mathbf{C}_2 \mathbf{U}_2(\eta_j), \tag{26}$$

where $\mathbf{G}$ is a diagonal 2x2 matrix and the subscripts are used to refer to the media.

We not that, among the various integrals involving eigenfunction and adjoint, only the following two are singular after integration over μ:

$$\int_0^1 \mu \tilde{\mathbf{\Theta}}_i(\nu,\mu) \int_0^1 \mathbf{\Phi}_j(\eta,\mu)\, \mathbf{A}_j(\eta)\, d\eta\, d\mu \quad , \quad \nu,\eta \in (0,1) \quad , \quad i \neq j \; , \tag{27}$$

and

$$\int_0^1 \mu \tilde{\mathbf{\Theta}}_i(\nu,\mu)\, \mathbf{E}(\nu) \int_0^1 \mathbf{\Phi}_i(\nu',\mu)\, \mathbf{A}_i(\nu')\, d\nu' d\mu \quad , \quad \nu,\nu' \in (0,1) \; , \tag{28}$$

where $\mathbf{E}(\nu)$ is a 2x2 matrix.

Here we notice a difference between one-group and two-group theories in that in one-group theory the integral corresponding to Eq. (28) is regular since it reduces to one corresponding to Eq. (22d). Finally the following integral is of interest:

$$\int_0^1 \tilde{\mathbf{\Theta}}(\xi,\mu)\, \mu\, d\mu = \xi \tilde{\mathbf{X}}(\xi) \left\{ \mathbf{I} - \mathbf{C}\tilde{\mathbf{H}}_0 \right\} \mathbf{\Sigma} \; , \tag{29}$$

where $\mathbf{H}_0$ is a moment of the $\mathbf{H}$ matrix

$$\mathbf{H}_0 = \int_0^1 \mathbf{\Theta}(\mu)\, \mathbf{H}(\mu)\, d\mu \; . \tag{30}$$

2 – THE TWO-SLAB PROBLEM

We consider a slab of thickness α_1 of medium 1 ($0 \leq x \leq \alpha_1$) adjacent to another of thickness α_2 of medium 2 ($\alpha_1 \leq x \leq \gamma$, $\gamma = \alpha_1 + \alpha_2$) irradiated on the $x = 0$ surface by a flux of neutrons $\underset{\sim}{f}(\mu)$, $\mu \epsilon (0,1)$.

We write the solutions of Eq. (4) as

$$\underset{\sim}{\Psi}_1(x,\mu) = \sum_{i=1}^{k_2} [\, A_1(\nu_i)\,\underset{\sim}{\Phi}_1(\nu_i,\mu)\exp(-x/\nu_i)$$

$$+ A_1(-\nu_i)\,\underset{\sim}{\Phi}_1(-\nu_i,\mu)\exp\{-(\alpha_1 - x)/\nu_i\}\,]$$

$$+ \int_0^1 [\, \underset{\sim}{\Phi}_1(\nu,\mu)\,\underset{\sim}{A}_1(\nu)\exp(-x/\nu) + \underset{\sim}{\Phi}_1(-\nu,\mu)\,\underset{\sim}{A}_1(-\nu)\exp\{-(\alpha_1 - x)/\nu\} d\nu \;,$$

$$0 \leqslant x \leqslant \alpha_1 \;, \tag{31}$$

and

$$\underset{\sim}{\Psi}_2(x,\mu) = \sum_{i=1}^{k_2} [\, A_2(\eta_i)\,\underset{\sim}{\Phi}_2(\eta_i,\mu)\exp\{-(x-\alpha_1)/\eta_i\}$$

$$+ A_2(-\eta_i)\,\underset{\sim}{\Phi}_2(-\eta_i,\mu)\exp\{-(\gamma - x)/\eta_i\}\,]$$

$$+ \int_0^1 [\, \underset{\sim}{\Phi}_2(\eta,\mu)\,\underset{\sim}{A}_2(\eta)\exp\{-(x-\alpha_1)/\eta\}$$

$$+ \underset{\sim}{\Phi}_2(-\eta,\mu)\,\underset{\sim}{A}_2(-\eta)\exp\{-(\gamma - x)/\eta\}\,]d\eta \;,\quad \alpha_1 \leqslant x \leqslant \gamma \;, \tag{32}$$

subject to the conditions

$$\underset{\sim}{\Psi}_1(0,\mu) = \underset{\sim}{P}_1\,\underset{\sim}{f}(\mu) \;,\quad \mu \in (0,1) \;, \tag{33a}$$

$$\underset{\sim}{\Psi}_2(\gamma,-\mu) = \underset{\sim}{0} \;,\quad \mu \in (0,1) \;, \tag{33b}$$

and

$$\underset{\sim}{\Psi}_1(\alpha_1,\mu) = \underset{\sim}{G}\,\underset{\sim}{\Psi}_2(\alpha_1,\mu) \;,\quad \mu \in (-1,1) \;. \tag{33c}$$

We assume, considering the data sets for our calculations, that the groups are similarly ordered for both media and thus the matrix $\underset{\sim}{G}$ is diagonal and given by $\underset{\sim}{G} = \underset{\sim}{P}_1\,\underset{\sim}{P}_2^{-1}$.

The conditions at outer boundaries, Eqs. (33a,b), result in the equations

$$\sum_{i=1}^{k_1} A_1(\nu_i)\, \underset{\sim}{\Phi}_1(\nu_i,\mu) + \int_0^1 \underset{\sim}{\Phi}_1(\nu,\mu)\, \underset{\sim}{A}_1(\nu)\, d\nu = \underset{\sim}{P}_1\, \underset{\sim}{f}(\mu)$$

$$- \sum_{i=1}^{k_2} A_1(-\nu_i)\, \underset{\sim}{\Phi}_1(-\nu_i,\mu)\, E_1(\nu_i) - \int_0^1 \underset{\sim}{\Phi}_1(-\nu,\mu)\, \underset{\sim}{A}_1(-\nu)\, E_1(\nu)\, d\nu \quad ,$$

$$\mu \in (0,1) \quad , \tag{34}$$

and

$$\sum_{i=1}^{k_2} A_2(-\eta_i)\, \underset{\sim}{\Phi}_2(\eta_i,\mu) + \int_0^1 \underset{\sim}{\Phi}_1(\eta,\mu)\, \underset{\sim}{A}_2(-\eta)\, d\eta$$

$$= - \sum_{i=1}^{k_2} A_2(\eta_i)\, \underset{\sim}{\Phi}_2(-\eta_i,\mu)\, E_2(\eta_i) - \int_0^1 \underset{\sim}{\Phi}_2(-\eta,\mu)\, \underset{\sim}{A}_2(\eta)\, E_2(\eta)\, d\eta \quad ,$$

$$\mu \in (0,1) \quad , \tag{35}$$

and we write the interface condition, Eq. (33c), in two equations

$$\sum_{i=1}^{k_1} A_1(-\nu_i)\, \underset{\sim}{\Phi}_1(\nu_i,\mu) + \int_0^1 \underset{\sim}{\Phi}_1(\nu,\mu)\, \underset{\sim}{A}_1(-\nu)\, d\nu$$

$$= - \sum_{i=1}^{k_1} A_1(\nu_i)\, \underset{\sim}{\Phi}_1(-\nu_i,\mu)\, E_1(\nu_i) - \int_0^1 \underset{\sim}{\Phi}_1(-\nu,\mu)\, \underset{\sim}{A}_1(\nu)\, E_1(\nu)\, d\nu$$

$$+ \sum_{i=1}^{k_2} \underset{\sim}{G}\, [\, A_2(\eta_i)\, \underset{\sim}{\Phi}_2(-\eta_i,\mu) + A_2(-\eta_i)\, \underset{\sim}{\Phi}_2(\eta_i,\mu)\, E_2(\eta_i) \,]$$

$$+ \int_0^1 \underset{\sim}{G}\, [\, \underset{\sim}{\Phi}_2(-\eta,\mu)\, \underset{\sim}{A}_2(\eta) + \underset{\sim}{\Phi}_2(\eta,\mu)\, \underset{\sim}{A}_2(-\eta)\, E_2(\eta) \,]\, d\eta \quad ,$$

$$\mu \in (0,1) \quad , \tag{36}$$

and

$$\sum_{i=1}^{k_2} A_2(\eta_i)\, \underset{\sim}{\Phi}_2(\eta_i,\mu) + \int_0^1 \underset{\sim}{\Phi}_2(\eta,\mu)\, \underset{\sim}{A}_2(\eta)\, d\eta$$

$$= \sum_{i=1}^{k_1} \underset{\sim}{G}^{-1}\, [\, A_1(\nu_i)\, \underset{\sim}{\Phi}_1(\nu_i,\mu)\, E_1(\nu_i) + A_1(-\nu_i)\, \underset{\sim}{\Phi}_1(-\nu_i,\mu) \,]$$

$$+ \int_0^1 \underset{\sim}{G}^{-1}\, [\, \underset{\sim}{\Phi}_1(\nu,\mu)\, \underset{\sim}{A}_1(\nu)\, E_1(\nu) + \underset{\sim}{\Phi}_1(-\nu,\mu)\, \underset{\sim}{A}_1(-\nu) \,]\, d\nu$$

$$- \sum_{i=1}^{k_2} A_2(\eta_i)\, \Phi_2(\eta_i,\mu)\, E_2(\eta_i)$$

$$- \int_0^1 \underset{\sim}{\Phi}_2(\eta,\mu)\, \underset{\sim}{A}_2(\eta)\, E_2(\eta)\, d\eta \quad , \quad \mu \in (0,1) \ , \tag{37}$$

where $E_i(\xi) = \exp(\alpha_i/\xi)$.

Our aim is to derive a set of regular integral equations for the expansion coefficients so that the coefficients can be found numerically by a standard iterative method.

If we apply the half-range orthogonality theorem to Eq. (34), i.e., multiply Eq. (34) by $\mu\Theta_1(\xi,\mu)$, $\xi = \nu_i$ or $\nu\epsilon(0,1)$, we obtain

$$A_1(\nu_i) = A_1^0(\nu_i) - \nu_i N_1^{-1}(\nu_i)\, \underset{\sim}{\widetilde{X}}_1(\nu_i)\, \underset{\sim}{Y}_2(\nu_i) \ , \tag{38a}$$

and

$$\underset{\sim}{A}_1(\nu) = \underset{\sim}{A}_1^0(\nu) - \nu N_1^{-1}(\nu)\, \underset{\sim}{\widetilde{X}}_1(\nu)\, \underset{\sim}{Y}_1(\nu) \ , \tag{38b}$$

and in the same way we obtain from Eq. (35)

$$A_2(-\eta_i) = -\eta_i N_2^{-1}(\eta_i)\, \underset{\sim}{\widetilde{X}}_2(\eta_i)\, \underset{\sim}{Y}_2(\eta_i) \ , \tag{39a}$$

and

$$\underset{\sim}{A}_2(-\eta) = -\eta N_2^{-1}(\eta)\, \underset{\sim}{\widetilde{X}}_2(\eta)\, \underset{\sim}{Y}_2(\eta) \ , \tag{39b}$$

where

$$A_1^0(\nu_i) = N_1^{-1}(\nu_i) \int_0^1 \underset{\sim}{\widetilde{\Theta}}_1(\nu_i,\mu)\, \underset{\sim}{P}_1\, \underset{\sim}{f}(\mu)\, \mu d\mu \ , \tag{40a}$$

$$\underset{\sim}{A}_1^0(\nu) = N_1^{-1}(\nu) \int_0^1 \underset{\sim}{\widetilde{\Theta}}_1(\nu,\mu)\, \underset{\sim}{P}_1\, \underset{\sim}{f}(\mu)\, \mu d\mu \ , \tag{40b}$$

$$\underset{\sim}{Y}_1(\xi) = \sum_{i=1}^{k_1} \frac{\nu_i}{\nu_i+\xi}\, \hat{H}_1^{-1}(\nu_i)\, \underset{\sim}{C}_1\, \underset{\sim}{U}_1(\nu_i)\, A_1(-\nu_i)\, E_1(\nu_i)$$

$$+ \int_0^1 \frac{\nu}{\nu+\xi}\, \hat{H}_1^{-1}(\nu)\, \underset{\sim}{C}_1\, \underset{\sim}{A}_1(-\nu)\, E_1(\nu)\, d\nu \ ,$$

$$\underset{\sim}{Y}_2(\xi) = \sum_{i=1}^{k_2} \frac{\eta_i}{\eta_i + \xi} \underset{\sim}{H}_2^{-1}(\eta_i)\, \underset{\sim}{C}_2\, \underset{\sim}{U}_2(\eta_i)\, A_2(\eta_i)\, E_2(\eta_i)$$

$$+ \int_0^1 \frac{\eta}{\eta + \xi} \underset{\sim}{H}_2^{-1}(\eta)\, \underset{\sim}{C}_2\, \underset{\sim}{A}_2(\eta)\, E_2(\eta)\, d\eta \quad . \tag{42}$$

Next we apply the orthogonality theorem for medium 1 to Eq. (36) to isolate the coefficients on the left side. After integrating over μ, the $A_2(-\eta)$ term remains to be principal-value integrals for $\xi = \nu$. Following the method of regularization summarized in Section 1, we multiply Eq. (35) by

$$\mu \underset{\sim}{\tilde{Q}}_1(\xi,\mu)\, \underset{\sim}{G} \begin{bmatrix} E_2(\sigma_1\xi/\sigma_2) & 0 \\ 0 & E_2(\xi) \end{bmatrix} \tag{43}$$

and integrate over $\mu \epsilon (0,1)$. We find on the left side the same singular integrals, but with different exponential functions, as from Eq. (36). Then subtracting this result from the previous equation, we obtain equations with removable singularities

$$A_1(-\nu_i) = \nu_i\, N_1^{-1}(\nu_1)\, \underset{\sim}{\tilde{X}}_1(\nu_i)\, \underset{\sim}{Y}_3(\nu_i) \quad , \tag{44a}$$

and

$$A_1(-\nu) = \nu N_1^{-1}(\nu)\, \underset{\sim}{\tilde{X}}_1(\nu)\, \underset{\sim}{Y}_3(\nu) \tag{44b}$$

where

$$\underset{\sim}{Y}_3(\xi) = -\sum_{i=1}^{k_1} \frac{\nu_i}{\nu_i + \xi} \underset{\sim}{H}_1^{-1}(\nu_i)\, \underset{\sim}{C}_1\, \underset{\sim}{U}_1(\nu_i)\, A_1(\nu_i)\, E_1(\nu_i) - \int_0^1 \frac{\nu}{\nu + \xi} \underset{\sim}{H}_1^{-1}(\nu)\, \underset{\sim}{C}_1\, \underset{\sim}{A}_1(\nu)\, E_1(\nu)\, d\nu$$

$$+ \sum_{i=1}^{k_2} \Big[\frac{\sigma_1\eta_i}{\sigma_2\eta_i + \nu_1\xi} \underset{\sim}{H}_1^{-1}(\sigma_2\eta_i/\sigma_1)\, \underset{\sim}{k}_1 \{1 - E_2(\sigma_1\xi/\sigma_2)\, E_2(\eta_i)\}$$

$$+ \frac{\eta_i}{\eta_i + \xi} \underset{\sim}{H}_1^{-1}(\eta_i)\, \underset{\sim}{k}_2 \{1 - E_2(\xi)\, E_2(\eta_i)\}\Big]\, \underset{\sim}{G}\, \underset{\sim}{C}_2\, \underset{\sim}{U}_2(\eta_i)\, A_2(\eta_i)$$

$$+ \sum_{i=1}^{k_2} \Big[\frac{\sigma_1\eta_i}{\sigma_2\eta_i - \sigma_1\xi} \underset{\sim}{H}_1^{-1}(\sigma_2\eta_i/\sigma_1)\, \underset{\sim}{k}_1 \{E_2(\eta_i) - E_2(\sigma_1\xi/\sigma_2)\}$$

$$+ \frac{\eta_i}{\eta_i - \xi} \underset{\sim}{H}_1^{-1}(\eta_i)\, \underset{\sim}{k}_2 \{E_2(\eta_i) - E_2(\xi)\}\Big]\, \underset{\sim}{G}\, \underset{\sim}{C}_2\, \underset{\sim}{U}_2(\eta_i)\, A_2(\eta_i)$$

$$+ \int_0^1 \left[\frac{\sigma_1 \eta}{\sigma_2 \eta + \sigma_1 \xi} \underset{\sim}{H}_1^{-1} (\sigma_2 \eta/\sigma_1) \underset{\sim}{k}_1 \{1 - E_2 (\sigma_1 \xi/\sigma_2) E_2 (\eta) \} \right.$$

$$\left. + \frac{\eta}{\eta + \xi} \underset{\sim}{H}_1^{-1} (\eta) \underset{\sim}{k}_2 \{1 - E_2 (\xi) E_2 (\eta)\} \right] \underset{\sim}{G} \underset{\sim}{C}_2 \underset{\sim}{A}_2 (\eta) \, d\eta$$

$$+ \int_0^1 \underset{\sim}{C}_1 \left[\frac{\sigma_1 \eta}{\sigma_2 \eta - \sigma_1 \xi} \underset{\sim}{\tilde{H}}_1 (\sigma_2 \eta/\sigma_1) \underset{\sim}{C}_1^{-1} \underset{\sim}{\tilde{\lambda}}_1 (\sigma_2 \eta/\sigma_1) \underset{\sim}{k}_1 \{E_2 (\eta) - E_2 (\sigma_1 \xi/\sigma_2) \} \right.$$

$$\left. + \frac{\eta}{\eta - \xi} \underset{\sim}{\tilde{H}}_1 (\eta) \underset{\sim}{C}_1^{-1} \underset{\sim}{\tilde{\lambda}}_1 (\eta) k_2 \{E_2 (\eta) - E_2 (\xi)\} \right] \underset{\sim}{G} \underset{\sim}{C}_2 \underset{\sim}{A}_2 (-\eta) \, d\eta$$

$$+ \int_0^1 \underset{\sim}{C}_1 \left[\frac{\sigma_2 \eta}{\sigma_1 \xi - \sigma_2 \eta} \tilde{H}_1 (\sigma_2 \eta/\sigma_1) \underset{\sim}{k}_1 \{E_2 (\eta) - E_2 (\sigma_1 \xi/\sigma_2)\} \right.$$

$$\left. + \frac{\eta}{\xi - \eta} \underset{\sim}{\tilde{H}}_1 (\eta) \underset{\sim}{k}_2 \{E_2 (\eta) - E_2 (\xi) \} \right] \underset{\sim}{Q}_2 (\eta) \underset{\sim}{G} \underset{\sim}{\lambda}_2 (\eta) \underset{\sim}{A}_2 (-\eta) \, d\eta \ . \tag{45}$$

In the same way we first multiply Eq. (37) by $\mu \underset{\sim}{Q}_2 (\xi, \mu)$, $\xi = \eta_i$ or $\eta \epsilon (0,1)$, and integrate over $\mu \epsilon (0,1)$, next multiply Eq. (34) by

$$\mu \underset{\sim}{\tilde{Q}}_2 (\xi, \mu) \underset{\sim}{G}^{-1} \begin{bmatrix} E_1 (\sigma_2 \xi/\sigma_1) & 0 \\ 0 & E_1 (\xi) \end{bmatrix} \tag{46}$$

and integrate over μ, and then subtract between the two results to obtain

$$A_2 (\eta_i) = A_2^0 (\eta_i) + \eta_i N_2^{-1} (\eta_i) \underset{\sim}{\tilde{X}}_2 (\eta_i) \underset{\sim}{Y}_4 (\eta_i) \ , \tag{47a}$$

and

$$\underset{\sim}{A}_2 (\eta) = \underset{\sim}{A}_2^0 (\eta) + \eta N_2^{-1} (\eta) \underset{\sim}{\tilde{X}}_2 (\eta) \underset{\sim}{Y}_4 (\eta), \tag{47b}$$

where

$$A_2^0 (\eta_i) = N_2^{-1} (\eta_i) \int_0^1 \underset{\sim}{\tilde{Q}}_2 (\eta_i, \mu) \underset{\sim}{G}^{-1} \begin{bmatrix} E_1 (\sigma_2 \eta_i/\sigma_1) & 0 \\ 0 & E_1 (\eta_i) \end{bmatrix} \underset{\sim}{P}_1 \underset{\sim}{f} (\mu) \, \mu d\mu \ , \tag{48a}$$

$$\underset{\sim}{A}_2^0 (\eta) = N_2^{-1} (\eta) \int_0^1 \underset{\sim}{\tilde{Q}}_2 (\eta, \mu) \underset{\sim}{G}^{-1} \begin{bmatrix} E_1 (\sigma_2 \eta/\sigma_1) & 0 \\ 0 & E_1 (\eta) \end{bmatrix} \underset{\sim}{P}_1 \underset{\sim}{f} (\mu) \, \mu d\mu \ ,$$

$$Y_4(\xi) = -\sum_{i=1}^{k_2} \frac{\eta_i}{\eta_i+\xi} \underset{\sim}{H}_2^{-1}(\eta_i)\underset{\sim}{C}_2\underset{\sim}{U}_2(\eta_i)A_2(-\eta_i)E_2(\eta_i) - \int_0^1 \frac{\eta}{\eta+\xi}\underset{\sim}{H}_2^{-1}(\eta)\underset{\sim}{C}_2\underset{\sim}{A}_2(-\eta)E_2(\eta)d\eta$$

$$+\sum_{i=1}^{k_1}\Big[\frac{\sigma_2\nu_i}{\sigma_1\nu_i+\sigma_2\xi}\underset{\sim}{H}_2^{-1}(\sigma_1\nu_i/\sigma_2)\underset{\sim}{k}_1\{1-E_1(\nu_i)E_1(\sigma_2\xi/\sigma_1)\}$$

$$+\frac{\nu_i}{\nu_i+\xi}\underset{\sim}{H}_2^{-1}(\nu_i)\underset{\sim}{k}_2\{1-E_1(\nu_i)E_1(\xi)\}\Big]\underset{\sim}{G}_1\underset{\sim}{C}_1\underset{\sim}{U}_1(\nu_i)A_1(-\nu_i)$$

$$+\sum_{i=1}^{k_1}\Big[\frac{\sigma_2\nu_i}{\sigma_1\nu_i-\sigma_2\xi}\underset{\sim}{H}_2^{-1}(\sigma_1\nu_i/\sigma_2)\underset{\sim}{k}_1\{E_1(\nu_i)-E_1(\sigma_2\xi/\sigma_1)\}$$

$$+\frac{\nu_i}{\nu_i-\xi}\underset{\sim}{H}_2^{-1}(-\nu_i)\underset{\sim}{k}_2\{E_1(\nu_i)-E^1(\xi)\}\Big]\underset{\sim}{G}^{-1}\underset{\sim}{C}_1\underset{\sim}{U}_1(\nu_i)A_1(\nu_i)$$

$$+\int_0^1\Big[\frac{\sigma_2\nu}{\sigma_1\nu+\sigma_2\xi}\underset{\sim}{H}_2^{-1}(\sigma_1\nu/\sigma_2)\underset{\sim}{k}_1\{1-E_1(\nu)E_1(\sigma_2\xi/\sigma_1)\}$$

$$+\frac{\nu}{\nu+\xi}\underset{\sim}{\tilde{H}}_2^{-1}(\nu)\underset{\sim}{k}_2\{1-E_1(\xi)\}\Big]\underset{\sim}{G}^{-1}\underset{\sim}{C}_1\underset{\sim}{A}_1(-\nu)\,d\nu$$

$$+\int_0^1\underset{\sim}{C}_2\Big[\frac{\sigma_2\nu}{\sigma_1\nu-\sigma_2\xi}\underset{\sim}{\tilde{H}}_2(\sigma_1\nu/\sigma_2)\underset{\sim}{C}_2^{-1}\underset{\sim}{\tilde{\lambda}}_2(\sigma_1\nu/\sigma_2)\underset{\sim}{k}_1\{E_1(\nu)-E_1(\sigma_2\xi/\sigma_1)\}$$

$$+\frac{\nu}{\nu-\xi}\underset{\sim}{\tilde{H}}_2(\nu)\underset{\sim}{C}_2^{-1}\underset{\sim}{\tilde{\lambda}}_2(\nu)\underset{\sim}{k}_2\{E_1(\nu)-E_1(\xi)\}\Big]\underset{\sim}{G}^{-1}\underset{\sim}{C}_1\underset{\sim}{A}_1(\nu)\,d\nu$$

$$+\int_0^1\underset{\sim}{C}_2\Big[\frac{\sigma_1\nu}{\sigma_2\xi-\sigma_1\nu}\underset{\sim}{\tilde{H}}_2(\sigma_1\nu/\sigma_2)\underset{\sim}{k}_1\{E_1(\nu)-E_1(\sigma_2\xi/\sigma_1)\}$$

$$+\frac{\nu}{\xi-\nu}\underset{\sim}{\tilde{H}}_2(\nu)\underset{\sim}{k}_2\{E_1(\nu)-E_1(\xi)\}\Big]\underset{\sim}{\Theta}_1(\nu)\underset{\sim}{G}^{-1}\underset{\sim}{\lambda}_1(\nu)\underset{\sim}{A}_1(\nu)\,d\nu\ . \qquad (49)$$

Equations (38), (39), (44), and (47) are our final equations for the coefficients. All singularities are removed in terms of the exponential function and, therefore, numerical iterations can be performed in a standard manner. It is clear from these equations that, as was mentioned before, the continuum coefficients for Region (2) are proportional to $\underset{\sim}{U}^{(2)}$.

We note that if we let $\sigma_2 \to 0$ all terms in Y_1 except the first two vanish and Eq. (44), together with Eq. (38), reduces to the case of a single slab. Similarly, in the limit $\sigma_1 \to 0$ Eqs. (39) and (47) reduce to the case of a single slab of medium 2.

3 – THE CRITICAL PROBLEM

The critical problem for bare reactors has been solved by Kriese, Siewert, and Yener[11]. We consider here the critical problem for reflected slab reactors, a typical textbook problem in diffusion theory. The core of multiplying medium 1 extends from $-\alpha$ to $+\alpha$ surrounded by infinite reflectors of non-multiplying medium 2. We assume that both media are specified and, thus, our aim is to determine the value of α such that non-trivial solutions exist.

We write the solutions of Eq. (4) as

$$\begin{aligned}\underset{\sim}{\Psi}_1(x,\mu) = &\sum_{i=1}^{k_1} A_1(\nu_i)\,\underset{\sim}{\Phi}_1(\nu_i,\mu)\exp\{-(x+\alpha)/\nu_i\}\\ &+\sum_{i=1}^{k_1} A_1(\nu_i)\,\underset{\sim}{\Phi}_1(-\nu_i,\mu)\exp\{-(\alpha-x)/\nu_i\}\\ &+\int_0^1 \underset{\sim}{\Phi}_1(\nu,\mu)\,\underset{\sim}{A}_1(\nu)\exp\{-(x+\alpha)/\nu\}d\nu\\ &+\int_0^1 \underset{\sim}{\Phi}_1(-\nu,\mu)\,\underset{\sim}{A}_1(\nu)\exp\{-(\alpha-x)/\nu\}d\nu\,. \end{aligned} \tag{50}$$

and

$$\begin{aligned}\underset{\sim}{\Psi}_2(x,\mu) = &\sum_{i=1}^{k_2} A_2(\eta_i)\,\underset{\sim}{\Phi}_2(\eta_i,\mu)\exp\{-(x-\alpha)/\eta_i\}\\ &+\int_0^1 \underset{\sim}{\Phi}_2(\eta,\mu)\,\underset{\sim}{A}_2(\eta)\exp\{-(x-\alpha)/\eta\}d\eta\,. \end{aligned} \tag{51}$$

The symmetry condition and the condition for $|x|\to\infty$ are already incorporated in the solutions and we consider hereafter only $x \geqslant 0$. The remaining continuity condition at $x=\alpha$ can be written in two equations for $\mu\epsilon(0,1)$,

$$\begin{aligned}&\sum_{i=1}^{k_1} A_1(\nu_i)\,\underset{\sim}{\Phi}_1(\nu_i,\mu) + \int_0^1 \underset{\sim}{\Phi}_1(\nu,\mu)\,\underset{\sim}{A}_1(\nu)\,d\nu\\ &\quad = -\sum_{i=1}^{k_1} A_1(\nu_i)\,\underset{\sim}{\Phi}_1(-\nu_i,\mu)\,E(\nu_i) - \int_0^1 \underset{\sim}{\Phi}_1(-\nu,\mu)\,\underset{\sim}{A}_1(\nu)\,E(\nu)\,d\nu\\ &\quad + \sum_{i=1}^{k_2} A_2(\eta_i)\underset{\sim}{G}\,\underset{\sim}{\Phi}_2(-\eta_i,\mu) + \int_0^1 \underset{\sim}{G}\,\underset{\sim}{\Phi}_2(-\eta,\mu)\,\underset{\sim}{A}_2(\eta)\,d\eta\,, \end{aligned} \tag{52}$$

and

$$\sum_{i=1}^{k_2} A_2(\eta_i)\,\underset{\sim}{\Phi}_2(\eta_i,\mu) + \int_0^1 \underset{\sim}{\Phi}_2(\eta,\mu)\,\underset{\sim}{A}_2(\eta)\,d\eta$$

$$= \sum_{i=1}^{k_1} A_1(\nu_i)\,\underset{\sim}{G}^{-1}\,\underset{\sim}{\Phi}_1(\nu_i,\mu)\,E(\nu) + \sum_{i=1}^{k_1} A_1(\nu_i)\,\underset{\sim}{G}^{-1}\,\underset{\sim}{\Phi}_1(-\nu_i,\mu)$$

$$+ \int_0^1 \underset{\sim}{G}^{-1}\,\underset{\sim}{\Phi}_1(\nu,\mu)\,\underset{\sim}{A}_1(\nu)\,E(\nu)\,d\nu + \int_0^1 \underset{\sim}{G}^{-1}\,\underset{\sim}{\Phi}_1(-\nu,\mu)\,\underset{\sim}{A}_1(\nu)\,d\nu \;, \qquad (53)$$

where $E(\xi) = \exp(-2\alpha/\xi)$. For the moment we assume that α is a given constant and multiply Eq. (52) by $\mu\underset{\sim}{\tilde{\Phi}}_1(\xi,\mu)$, $\xi = \nu_i$ or $\nu\epsilon(0,1)$, and integrate over $\mu\epsilon(0,1)$ to obtain equations for the coefficients

$$A_1(\nu_1)\left\{1 + \frac{1}{2}\,\nu_1 N_1^{-1}(\nu_1)\,\underset{\sim}{\tilde{X}}_1(\nu_1)\,\underset{\sim}{H}_1^{-1}(\nu_1)\,\underset{\sim}{C}_1\,\underset{\sim}{U}_1(\nu_1)\,E_1(\nu_1)\right\}$$

$$= \nu_1 N_1^{-1}(\nu_1)\,\underset{\sim}{\tilde{X}}_1(\nu_1)\left\{\underset{\sim}{Y}_1(\nu_1) - (\kappa-1)\frac{\nu_2}{\nu_1+\nu_2}\,\underset{\sim}{H}_1^{-1}(\nu_2)\,\underset{\sim}{C}_1\,\underset{\sim}{U}_1(\nu_2)\,E(\nu_2)\,A_1(\nu_2)\right\}, \qquad (54)$$

$$\underset{\sim}{A}_1(\nu) = \nu N_1^{-1}(\nu)\,\underset{\sim}{\tilde{X}}_1(\nu)\left\{\underset{\sim}{Y}_1(\nu) - \sum_{i=1}^{k_1}\frac{\nu_i}{\nu_i+\nu}\,\underset{\sim}{H}_1^{-1}(\nu_i)\,\underset{\sim}{C}_1\,\underset{\sim}{U}_1(\nu_i)\,E(\nu_i)\,A_1(\nu_i)\right\}, \qquad (55a)$$

and, if $\kappa_1 = 2$,

$$A_1(\nu_2)\left\{1 + \frac{1}{2}\,\nu_2 N_1^{-1}(\nu_2)\,\underset{\sim}{\tilde{X}}_1(\nu_2)\,\underset{\sim}{H}_1^{-1}(\nu_2)\,\underset{\sim}{C}_1\,\underset{\sim}{U}_1(\nu_2)\,E(\nu_2)\right\}$$

$$= \nu_2 N_1^{-1}(\nu_2)\,\underset{\sim}{\tilde{X}}_1(\nu_2)\left\{\underset{\sim}{Y}_1(\nu_2) - \frac{\nu_1}{\nu_1+\nu_2}\,\underset{\sim}{H}_1^{-1}(\nu_1)\,\underset{\sim}{C}_1\,\underset{\sim}{U}_1(\nu_1)\,E(\nu_1)\,A_1(\nu_1)\right\} \;, \qquad (55b)$$

where

$$\underset{\sim}{Y}_1(\xi) = \sum_{i=1}^{k_2}\left\{\frac{\sigma_1\eta_i}{\sigma_2\eta_i+\sigma_1\xi}\,\underset{\sim}{H}_1^{-1}(\sigma_2\eta_i/\sigma_1)\,\underset{\sim}{k}_1 + \frac{\eta_i}{\eta_i+\xi}\,\underset{\sim}{H}_1^{-1}(\eta_i)\,\underset{\sim}{k}_2\right\}\underset{\sim}{G}\underset{\sim}{C}_2\,\underset{\sim}{U}_2(\eta_i)\,A_2(\eta_i)$$

$$+ \int_0^1 \left\{\frac{\sigma_1\eta}{\sigma_2\eta+\sigma_1\xi}\,\underset{\sim}{H}_1^{-1}(\sigma_2\eta/\sigma_1)\,\underset{\sim}{k}_1 + \frac{\eta}{\eta+\xi}\,\underset{\sim}{H}_1^{-1}(\eta)\,\underset{\sim}{k}_2\right\}\underset{\sim}{G}\underset{\sim}{C}_2\,\underset{\sim}{A}_2(\eta)\,d\eta$$

$$- \int_0^1 \frac{\nu}{\nu+\xi}\,\underset{\sim}{H}_1^{-1}(\nu)\,\underset{\sim}{C}_1\,\underset{\sim}{A}_1(\nu)\,E(\nu)\,d\nu \;. \qquad (56)$$

Similarly, we multiply Eq. (53) by $\mu \tilde{O}_2(\xi,\mu)$, $\xi = \eta_i$ or $\eta \epsilon (0,1)$, and integrate over μ to isolate the coefficients in the left-side expansion. The $\tilde{A}_1(\nu)$ term on the right side remains to be singular. Next we multiply Eq. (52) by

$$\mu \tilde{O}_2(\xi,\mu)\, \underset{\sim}{G}^{-1} \begin{bmatrix} E(\sigma_2\xi/\sigma_1) & 0 \\ 0 & E(\xi) \end{bmatrix} \tag{57}$$

and integrate over $\mu \epsilon (0,1)$. On the left side we find the same singular integrals, with different exponential functions, as in the previous equation. All other terms are regular. Then, subtracting the last equation from the previous one, we obtain equations with removable singularities

$$\begin{aligned} A_2(\eta_i) & \{ 1 - \frac{1}{2}\eta_i N_2^{-1}(\eta_i)\, \tilde{X}_2(\eta_i)\, \underset{\sim}{H}_2^{-1}(\eta_i)\, \underset{\sim}{E}(\eta_i)\, \underset{\sim}{C}_2\, \underset{\sim}{U}_2(\eta_i) \} \\ & - \eta_i N_2^{-1}(\eta_i)\, \tilde{X}_2(\eta_i) \{ \underset{\sim}{Y}_2(\eta_i) + \sum_{j=1}^{k_2} \frac{\eta_i}{\eta_i + \eta_j}(1 - \delta_{ij})\, \underset{\sim}{H}_2^{-1}(\eta_j)\, \underset{\sim}{E}(\eta_i)\, C_2 U_2(\eta_j) \\ & A_2(\eta_j) \} \end{aligned} \tag{58a}$$

and

$$A_2(\eta) - \eta N_2^{-1}(\eta)\, \tilde{X}_2(\eta) \{ \underset{\sim}{Y}_2(\eta) + \sum_{i=1}^{k_2} \frac{\eta_i}{\eta_i + \eta} \underset{\sim}{H}_2^{-1}(\eta_i)\, \underset{\sim}{E}(\eta)\, \underset{\sim}{C}_2 \underset{\sim}{U}_2(\eta_i)\, A_2(\eta_i) \} \tag{58b}$$

where

$$\underset{\sim}{E}(\xi) = \begin{bmatrix} E(\sigma_2\xi/\sigma_1) & 0 \\ 0 & E(\xi) \end{bmatrix} \tag{59}$$

and

$$\begin{aligned} Y_2(\xi) - & \sum_{i=1}^{k_1} [\frac{\sigma_2\nu_i}{\sigma_1\nu_i + \sigma_2\xi} H_2^{-1}(\sigma_1\nu_i/\sigma_2)\, \underset{\sim}{k}_1 \{ 1 - E(\nu_i)\, E(\sigma_2\xi/\sigma_1) \} \\ & + \frac{\nu_i}{\nu_i + \xi} H_2^{-1}(\nu)\, k_2 \{ 1 - E(\nu_i)\, E(\xi) \}]\, \underset{\sim}{G}^{-1}\, \underset{\sim}{C}_1\, \underset{\sim}{U}_1(\nu_i) A_1(\nu_i) \\ & + \sum_{i=1}^{k_1} [\frac{\sigma_2\nu_i}{\sigma_1\nu_i - \sigma_2\xi} H_2^{-1}(-\sigma_1\nu_i/\sigma_2)\, k_1 \{ E(\nu_i) - E(\sigma_2\xi/\sigma_1) \} \end{aligned}$$

$$+ \frac{\nu_i}{\nu_i - \xi} \underset{\sim}{H}_2^{-1}(-\nu_i)\, \underset{\sim}{k}_2 \{E(\nu_i) - E(\xi)\}\,]\, \underset{\sim}{G}^{-1}\, \underset{\sim}{C}_1\, \underset{\sim}{U}_1(\nu_i)\, A_1(\nu_i)$$

$$+ \int_0^1 [\, \frac{\sigma_2 \nu}{\sigma_1 \nu + \sigma_2 \xi} \underset{\sim}{H}_2^{-1}(\sigma_1 \nu/\sigma_2)\, \underset{\sim}{k}_1 \{1 - E(\nu)\, E(\sigma_2 \xi/\sigma_1)\}$$

$$+ \frac{\nu}{\nu + \xi} \underset{\sim}{H}_2^{-1}(\nu)\, \underset{\sim}{k}_2 \{1 - E(\nu)\, E(\xi)\}\,]\, \underset{\sim}{G}^{-1}\, \underset{\sim}{C}_1\, \underset{\sim}{A}_1(\nu)\, d\nu$$

$$+ \int_0^1 \frac{\eta}{\eta + \xi} \underset{\sim}{H}_2^{-1}(\eta)\, \underset{\sim}{E}(\xi)\, \underset{\sim}{C}_2\, \underset{\sim}{A}_2(\eta)\, d\eta$$

$$+ \int_0^1 \underset{\sim}{C}_2\, [\, \frac{\sigma_2 \nu}{\sigma_1 \nu - \sigma_2 \xi} \underset{\sim}{\widetilde{H}}_2(\sigma_1 \nu/\sigma_2)\, \underset{\sim}{C}_2^{-1}\, \underset{\sim}{\widetilde{\lambda}}_2(\sigma_1 \nu/\sigma_2)\, \underset{\sim}{k}_1 \{E(\nu) - E(\sigma_2 \xi/\sigma_1)\}$$

$$+ \frac{\nu}{\nu - \xi} \underset{\sim}{\widetilde{H}}_2(\nu)\, \underset{\sim}{C}_2^{-1}\, \underset{\sim}{\widetilde{\lambda}}_2(\nu)\, \underset{\sim}{k}_2 \{E(\nu) - E(\xi)\} \underset{\sim}{G}^{-1}\, \underset{\sim}{C}_1\, \underset{\sim}{A}_1(\nu)\, d\nu$$

$$+ \int_0^1 \underset{\sim}{C}_2\, [\, \frac{\sigma_1 \nu}{\sigma_2 \xi - \sigma_1 \nu} \underset{\sim}{\widetilde{H}}_2(\sigma_1 \nu/\sigma_2)\, \underset{\sim}{k}_1 \{E(\nu) - E(\sigma_2 \xi/\sigma_1)\}$$

$$+ \frac{\nu}{\xi - \nu} \underset{\sim}{\widetilde{H}}_2(\nu)\, \underset{\sim}{k}_2 \{E(\nu) - E(\xi)\}\,]\, \underset{\sim}{G}^{-1}\, \underset{\sim}{O}_1(\nu)\, \underset{\sim}{\lambda}_1(\nu)\, \underset{\sim}{A}_1(\nu)\, d\nu \;. \tag{60}$$

The condition of criticality can be incorporated as the condition of non-triviality of the solution. If we normalize the solution by taking $A_1(\nu_1) = \exp(a/\nu_1)$, the critical half-thickness of the core is given by

$$a = \frac{\pi}{2} \,|\nu_1|\, + \frac{\nu_1}{2} \ln(N/D) \tag{61}$$

where

$$N = \frac{1}{2} \nu_1\, N_1^{-1}(\nu_1)\, \underset{\sim}{\widetilde{X}}_1(\nu_1)\, \underset{\sim}{H}_1^{-1}(\nu_1)\, \underset{\sim}{C}_1\, \underset{\sim}{U}_1(\nu_1) \tag{62a}$$

and

$$D = 1 - \nu_1\, N_1^{-1}(\nu_1)\, \underset{\sim}{\widetilde{X}}_1(\nu_1) \exp(-a/\nu_1) \{\underset{\sim}{Y}_1(\nu_1) - (\kappa_1 - 1) \frac{\nu_2}{\nu_1 + \nu_2} \underset{\sim}{H}_1^{-1}(\nu_2)$$

$$\underset{\sim}{C}_1\, \underset{\sim}{U}_1(\nu_2)\, E(\nu_2)\, A_1(\nu_2)\} \;. \tag{62b}$$

Equations (55), (58), and (61) are our final equations to be solved by numerical iterations.

THE CASE OF FINITE REFLECTOR

If the thickness of the reflector is finite, the core solution, Eq. (50), is the same but the reflector solution is written as

$$\underset{\sim}{\Psi}_2 (x , \mu) = \sum_{i=1}^{k_2} A_2 (\eta_i) \underset{\sim}{\Phi}_2 (\eta_i , \mu) \exp \{ -(x - \alpha)/\eta_i \}$$

$$+ \sum_{i=1}^{k_2} A_2 (-\eta_i) \underset{\sim}{\Phi}_2 (-\eta_i , \mu) \exp \{ -(\gamma + \alpha - x)/\eta_i \}$$

$$+ \int_0^1 \underset{\sim}{\Phi}_2 (\eta , \mu) \underset{\sim}{A}_2 (\eta) \exp \{ -(x - \alpha)/\eta \} d\eta$$

$$+ \int_0^1 \underset{\sim}{\Phi}_2 (-\eta , \mu) \underset{\sim}{A}_2 (-\eta) \exp \{ -(\gamma + \alpha - x)/\eta \} d\eta \quad , \tag{63}$$

where γ is the reflector thickness (given) and α is the critical half-thickness to be determined.

We write the interface condition, symbolically, as

$$\underset{\sim}{\Psi}_1 (\alpha, -\mu) = \underset{\sim}{G}\, \underset{\sim}{\Psi}_2 (\alpha , -\mu) \quad , \quad \mu \in (0,1) \; , \tag{64a}$$

and

$$\underset{\sim}{\Psi}_2 (\alpha , \mu) = \underset{\sim}{G}^{-1} \underset{\sim}{\Psi}_1 (\alpha , \mu) \quad , \quad \mu \in (0,1) \quad , \tag{64b}$$

and the boundary condition at $x = \alpha + \gamma$ as

$$\underset{\sim}{\Psi}_2 (\alpha + \gamma , -\mu) = \underset{\sim}{0} \quad , \quad \mu \in (0,1). \tag{64c}$$

While in the case of infinite reflector we obtain immediately a regular integral equation for $A_1(\nu)$ and need only one step of regularization for the $\underset{\sim}{A}_2(\eta)$ equation, here the $A_1(\nu)$ equation must be regularized once and that for $\underset{\sim}{A}_2(\eta)$ in two steps, due to the existence of a boundary at $x = \alpha + \gamma$.

The procedure can be summarized as follows. First, we apply to Eq. (64a) the orthogonality relations for medium 1, and obtain equations with the coefficients $A_1(\nu_i)$ and $A_1(\nu)$ isolated on the left side. In the $\underset{\sim}{A}_i(\nu)$ equation the $\underset{\sim}{A}_2(-\eta)$ term remains singular. To remove this singularity we multiply Eq. (64c) by

$$\mu\, \tilde{\underset{\sim}{\Theta}}_1\,(\nu,\mu)\, \underset{\sim}{G} \begin{bmatrix} E_2(\sigma_1\nu/\sigma_2 & 0 \\ 0 & E_2(\nu) \end{bmatrix} \tag{65}$$

where $E_2(\xi) = \exp(-\gamma/\xi)$, and integrate over $\mu\,\epsilon(0,1)$. On the left side we find the same singular integrals, with different exponential functions, as in the previous equation. Subtracting the last equation from the previous one, we obtain equations with removable singularities

$$A_1(\nu_i)\ \{1 + \frac{1}{2}\nu_i N_1^{-1}(\nu_i)\, \tilde{\underset{\sim}{X}}_1(\nu_i)\, \underset{\sim}{H}_1^{-1}(\nu_i)\, \underset{\sim}{C}_1\, \underset{\sim}{U}_1(\nu_i)\, E_1(\nu_i)\}$$

$$= \nu_i N_1^{-1}(\nu_i)\, \tilde{\underset{\sim}{X}}_1(\nu_i)\, \{\underset{\sim}{Y}_1(\nu_i) - \sum_{j=1}^{k_1} \frac{\nu_j}{\nu_i+\nu_j}(1-\delta_{ij})\, \underset{\sim}{H}^{-1}(\nu_j)\, \underset{\sim}{C}_1\, \underset{\sim}{U}_1(\nu_j)\, E_1(\nu_j)$$

$$A_1(\nu_j)\}\ , \tag{66a}$$

and

$$\underset{\sim}{A}_1(\nu) = \nu N_1^{-1}(\nu)\, \tilde{\underset{\sim}{X}}_1(\nu)\, \{\underset{\sim}{Y}_1(\nu) - \sum_{i=1}^{k_1} \frac{\nu_i}{\nu+\nu_i}\, \underset{\sim}{H}_1^{-1}(\nu_i)\, \underset{\sim}{C}_1\underset{\sim}{U}_1(\nu_i)\, E_1(\nu_i)\, A_1(\nu_i)\} \tag{66b}$$

where $E_1(\xi) = \exp(-2\alpha/\xi)$ and $\underset{\sim}{Y}_1(\xi)$ is given in Ap, ndix A.

Similarly, we apply to Eq. (64b) the orthogonality relations for medium 2, and obtain equations with the coefficients $A_2(\eta_i)$ and $A_2(\eta)$ isolated on the left side. In the $\underset{\sim}{A}_2(\eta)$ equation the $\underset{\sim}{A}_1(\nu)$ term is singular. Next we multiply Eq. (64a) by

$$\mu\, \tilde{\underset{\sim}{\Theta}}_2\,(\eta,\mu)\, \underset{\sim}{G}^{-1} \begin{bmatrix} E_1(\sigma_2\eta/\sigma_1) & 0 \\ 0 & E_1(\eta) \end{bmatrix} \tag{67}$$

and integrate over $\mu\,\epsilon(0,1)$. On the left side we find the same singular integrals in the $\underset{\sim}{A}_1(\nu)$ term. However, we obtain new singularities on the right side. Finally multiplying Eq. (64) by

$$\mu\, \tilde{\underset{\sim}{\Theta}}_2\,(\eta,\mu)\, E_2(\eta) \begin{bmatrix} E_1(\sigma_2\eta/\sigma_1) & 0 \\ 0 & E_1(\eta) \end{bmatrix} \tag{68}$$

and integrating over $\mu\,\epsilon(0,1)$, we obtain siqularities to remove the last ones. The following equations are obtained

$$A_2(\eta_i)\ \{1 - \frac{1}{2}\ \eta_i\, N_2^{-1}(\eta_i)\ \tilde{\underset{\sim}{X}}_2(\eta_i)\ \underset{\sim}{H}_2^{-1}\ [1 - E_2^2(\eta_i)]\ \underset{\sim}{E}(\eta_i)\,\underset{\sim}{C}_2\,\underset{\sim}{U}_2(\eta_i)\}$$

$$= \eta_i\, N_2^{-1}(\eta_i)\ \tilde{\underset{\sim}{X}}_2(\eta_i)\ \{\underset{\sim}{Y}_2(\eta_i) + \sum_{j=1}^{k_2} \frac{\eta_j}{\eta_j + \eta_i}\ (1-\delta_{ij})\ \underset{\sim}{H}_2^{-1}(\eta_j)\,[1 - E_2(\eta_j)\,E_2(\eta_i)]$$

$$\underset{\sim}{E}(\eta_i)\,\underset{\sim}{C}_2\,\underset{\sim}{U}_2(\eta_j)\,A_2(\eta_j) + \sum_{j=1}^{k_2} \frac{\eta_i}{\eta_j - \eta_i}\ (1-\delta_{ij})\ \underset{\sim}{H}_2^{-1}(-\eta_j)\,[E_2(\eta_j) - E_2(\eta_i)]$$

$$\underset{\sim}{E}(\eta_i)\,\underset{\sim}{C}_2\,\underset{\sim}{U}_2(\eta_j)\,A_2(-\eta_j) \tag{69a}$$

and

$$\underset{\sim}{A}_2(\eta) = \eta\, N_2^{-1}(\eta)\ \tilde{\underset{\sim}{X}}_2(\eta)\ \{\underset{\sim}{Y}_2(\eta) + \sum_{i=1}^{k_2} \frac{\eta_i}{\eta + \eta_i}\ \underset{\sim}{H}_2^{-1}(\eta_i)\,[1 - E_2(\eta_i)\,E_2(\eta)]$$

$$\underset{\sim}{E}(\eta)\,\underset{\sim}{C}_2\,\underset{\sim}{U}_2(\eta_i)\,A_2(\eta_i) + \sum_{i=1}^{k_2} \frac{\eta_i}{\eta_i - \eta}\ \underset{\sim}{H}_2^{-1}(-\eta_i)\,[E_2(\eta_i) - E_2(\eta)]\ \underset{\sim}{E}(\eta)\,\underset{\sim}{C}_2\,\underset{\sim}{U}_2(\eta_i)$$

$$A_2(-\eta_i)\} \tag{69b}$$

where $\underset{\sim}{E}(\xi) = \begin{bmatrix} E_1(\sigma_2\xi/\sigma_1) & 0 \\ 0 & E_1(\xi) \end{bmatrix}$ and $\underset{\sim}{Y}_2(\xi)$ is given in Appendix A.

Finally, we apply to Eq. (64c) the orthogonality relations for medium 2, and obtain equations for the coefficients $A_2(-\eta_i)$ and $\underset{\sim}{A}_2(-\eta)$

$$A_2(-\eta_i) = \eta_i\, N_2^{-1}(\eta_i)\ \tilde{\underset{\sim}{X}}_2(\eta_i)\ \underset{\sim}{Y}_3(\eta_i)\ , \tag{70a}$$

$$\underset{\sim}{A}_2(\eta) = \eta\, N_2^{-1}(\eta)\ \tilde{\underset{\sim}{X}}_2(\eta)\,\underset{\sim}{Y}_3(\eta)\ , \tag{70b}$$

where $\underset{\sim}{Y}_3(\xi)$ is given in Appendix A.

Using the normalization $A_1(\nu_1) = \exp\ (a/\nu_1)$, the critical half-thickness of the core is obtained from Eq. (66a) in the form of Eq. (61). Final equations are solved by numerical iterations.

I THE CELL PROBLEM

We consider here an infinitely repeating array of two slabs of dissimilar media as a symplified model of flat-plate fuel assemblies and analyse a unit cell consisting of a half-slab of medium 1 ($-a_1 \leq x \leq 0$) and a half-slab of medium 2 ($0 \leq x \leq a_2$) with the condition of symmetry with respect to the boundary surfaces. We assume uniform sources of neutrons in medium 2.

The symmetric solutions can be written as

$$\underset{\sim}{I}_1 (x,\mu) = \underset{\sim}{P}_1^{-1} \; \underset{\sim}{\Psi}_1 (x,\mu) \quad , \quad -a_1 \leq x \leq 0 \;, \tag{71}$$

and

$$\underset{\sim}{I}_2 (x,\mu) = \underset{\sim}{P}_2^{-1} \; \{ \underset{\sim}{\Psi}_2 (x,\mu) + \underset{\sim}{\Psi}_{2p} (x,\mu) \} \quad , \quad 0 \leq x \leq a_2 \;, \tag{72}$$

where

$$\underset{\sim}{\Psi}_1 (x,\mu) = \sum_{i=1}^{k_1} A_1 (\nu_i) [\underset{\sim}{\Phi}_1 (\nu_i,\mu) \exp\{-(x+2a_1)/\nu_i\} + \underset{\sim}{\Phi}_1 (-\nu_i,\mu) \exp(x/\nu_i)]$$

$$+ \int_0^1 [\underset{\sim}{\Phi}_1 (\nu,\mu) \exp\{-(x+2a_1)/\nu\} + \underset{\sim}{\Phi}_1 (-\nu,\mu) \exp(x/\nu)] \underset{\sim}{A}_1 (\nu) \, d\nu \;, \tag{73}$$

$$\underset{\sim}{\Psi}_2 (x,\mu) = \sum_{i=1}^{k_2} A_2 (\eta_i) [\underset{\sim}{\Phi}_2 (\eta_i,\mu) \exp(-x/\eta_i) + \underset{\sim}{\Phi}_2 (-\eta_i,\mu) \exp\{-(2a_2-x)/\eta_i\}]$$

$$+ \int_0^1 [\underset{\sim}{\Phi}_2 (\eta,\mu) \exp(-x/\eta) + \underset{\sim}{\Phi}_2 (-\eta,\mu) \exp\{-(2a_2-x)/\eta\}] \underset{\sim}{A}_2 (\eta) \, d\eta \;, \tag{74}$$

and

$$\underset{\sim}{\Psi}_{2p} (x,\mu) = \{ \underset{\sim}{\Sigma}_2 - 2\underset{\sim}{C}_2 \}^{-1} \underset{\sim}{P}_2 \underset{\sim}{S} \;, \tag{75}$$

with $\underset{\sim}{S}$ being a constant two-vector.

We write the continuity condition in two equations for $\mu \epsilon (0,1)$,

$$\sum_{i=1}^{k_1} A_1 (\nu_i) \underset{\sim}{\Phi}_1 (\nu_i,\mu) + \int_0^1 \underset{\sim}{\Phi}_1 (\nu',\mu) A_1 (\nu') \, d\nu' = \underset{\sim}{G} \underset{\sim}{\Psi}_{2p} (0,\mu)$$

$$- \sum_{i=1}^{k_2} A_1 (\nu_i) \Phi_1 (\nu_i,\mu) E_1 (\nu_i) - \int_0^1 \Phi_1 (-\nu',\mu) \underset{\sim}{A}_1 (\nu') E_1 (\nu') \, d\nu'$$

$$+ \sum_{i=1}^{k_2} A_2(\eta_i)\, \underset{\sim}{G}\,[\,\underset{\sim}{\Phi}_2(-\eta_i,\mu) + \underset{\sim}{\Phi}_2(\eta_i,\mu)\,E_2(\eta_i)\,]$$

$$+ \int_0^1 \underset{\sim}{G}\,[\,\underset{\sim}{\Phi}_2(-\eta',\mu) + \underset{\sim}{\Phi}_2(\eta',\mu)\,E_2(\eta')\,]\,\underset{\sim}{A}_2(\eta')\,d\eta' \quad . \tag{76}$$

and

$$\sum_{i=1}^{k_2} A_2(\eta_i)\,\underset{\sim}{\Phi}_2(\eta_i,\mu) + \int_0^1 \underset{\sim}{\Phi}_2(\eta',\mu)\,\underset{\sim}{A}_2(\eta')\,d\eta = \underset{\sim}{\Psi}_{2P}(0,\mu)$$

$$+ \sum_{i=1}^{k_1} A_1(\nu_i)\,\underset{\sim}{G}^{-1}\,[\,\underset{\sim}{\Phi}_1(\nu_i,\mu)\,E_1(\nu_i) + \underset{\sim}{\Phi}_1(-\nu_i,\mu)\,]$$

$$+ \int_0^1 \underset{\sim}{G}^{-1}\,[\,\underset{\sim}{\Phi}_1(\nu',\mu)\,E_1(\nu') + \underset{\sim}{\Phi}_1(-\nu',\mu)\,]\,\underset{\sim}{A}_1(\nu')\,d\nu'$$

$$- \sum_{i=1}^{k_2} A_2(\eta_i)\,\underset{\sim}{\Phi}_2(-\eta_i,\mu)\,E_2(\eta_i) - \int_0^1 \underset{\sim}{\Phi}_2(-\eta',\mu)\,\underset{\sim}{A}_2(\eta')\,E_2(\eta')\,d\eta' \quad , \tag{77}$$

where $E_i(\xi) = \exp(-2a_i/\xi)$.

In this problem, because we are actually dealing with an infinite array, a straightforward application of the method of regularization requires an infinite number of steps. This is due to the facts that at each step we multiply an equation not only by the adjoint function but also by a matrix of exponential functions, as in Eqs. (43), (46), and (57), and that the integrals of the type that appear in Eq. (28) are singular after integration over μ. In one-group theory, integrals of this type are regular and the regularization is accomplished after a finite number of steps even for an infinite array of multi-slab cells.

However, the series of operations required for our proble can be summed up nicely and we can derive a regularized equation for $\underset{\sim}{A}_1(\nu)$ by the following steps:

1) Multiply Eq. (76) by $\mu\underset{\sim}{\tilde{Q}}_1(\nu,\mu)$ and integrate over μ. On the left side $\underset{\sim}{A}_1(\nu)$ is isolated. On the right side the $\underset{\sim}{\Phi}_2(\eta',\mu)$ term remains sigular.

2) Multiply Eq. (76) by

$$\mu\,\underset{\sim}{\tilde{Q}}_1(\nu,\mu)\begin{bmatrix} \dfrac{E_1(\nu)E_2(\sigma_1\nu/\sigma_2)}{1 - E_1(\nu)E_2(\sigma_1\nu/\sigma_2)} & 0 \\ & \\ 0 & \dfrac{E_1(\nu)E_2(\nu)}{1 - E_1(\nu)E_2(\nu)} \end{bmatrix} \tag{78}$$

and integrate over μ. The $\underset{\sim}{\Phi}_1(\nu',\mu)$ and $\underset{\sim}{\Phi}_1(\eta',\mu)$ terms remain singular.

3) Multiply Eq. (77) by

$$\mu\,\underset{\sim}{\tilde{\Theta}}_1(\nu,\mu)\,\underset{\sim}{G}\begin{bmatrix}\dfrac{E_2(\sigma_1\nu/\sigma_2)}{1-E_1(\nu)E_2(\sigma_1\nu/\sigma_2)} & 0\\ 0 & \dfrac{E_2(\nu)}{1-E_1(\nu)E_2(\nu)}\end{bmatrix} \tag{79}$$

and integrate over μ. Again the $\underset{\sim}{\Phi}_1(\nu',\mu)$ and $\underset{\sim}{\Phi}_2(\eta',\mu)$ terms remain singular.

If we now add three resulting equations on each side we find an equation for $A_1(\nu)$ in which all singularities are removed in terms of exponential functions. Obviously the equation $\underset{\sim}{A}_2(\nu)$ can be regularized similarly:

1) Multiply Eq. (77) by $\mu\underset{\sim}{\tilde{\Theta}}_2(\eta,\mu)$ and integrate over μ.

2) Multiply Eq. (77) by the following and integrate over μ:

$$\mu\,\underset{\sim}{\tilde{\Theta}}_2(\eta,\mu)\begin{bmatrix}\dfrac{E_1(\sigma_2\eta/\sigma_1)E_2(\eta)}{1-E_1(\sigma_2\eta/\sigma_1)E_2(\eta)} & 0\\ 0 & \dfrac{E_1(\eta)E_2(\eta)}{1-E_1(\eta)E_2(\eta)}\end{bmatrix} \tag{80}$$

3) Multiply Eq. (76) by the following and integrate over μ:

$$\mu\,\underset{\sim}{\tilde{\Theta}}_2(\eta,\mu)\,\underset{\sim}{G}^{-1}\begin{bmatrix}\dfrac{E_1(\sigma_2\eta/\sigma_1)}{1-E_1(\sigma_2\eta/\sigma_1)E_2(\eta)} & 0\\ 0 & \dfrac{E_1(\eta)}{1-E_1(\eta)E_2(\eta)}\end{bmatrix} \tag{81}$$

As in previous problems we apply these operations to the equations for the discrete coefficients, too. We obtain the following equations:

$$A_1(\nu_i)\,\{1-\frac{1}{2}\,\nu_i\,N_i^{-1}(\nu_i)\,\underset{\sim}{\tilde{X}}_1(\nu_i)\,\underset{\sim}{H}_i^{-1}(\nu_i)\,\underset{\sim}{J}_1(\nu_i,\nu_i)\,\underset{\sim}{C}_1\,\underset{\sim}{U}_1(\nu_i)\}$$

$$=A_1^0(\nu_i)+\nu_i\,N_i^{-1}(\nu_i)\,\underset{\sim}{\tilde{X}}_1(\nu_i)\,\{\sum_{j=1}^{k_1}(1-\delta_{ij})\,\underset{\sim}{Y}_1(\nu_i,\nu_j)\,A_1(\nu_j)$$

$$+ \int_0^1 \underset{\sim}{Y}_2 (\nu_i,\nu') \underset{\sim}{A}_1 (\nu') d\nu' + \sum_{j=1}^{k_2} \underset{\sim}{Y}_3 (\nu_i,\eta_j) A_2 (\eta_j) + \int_0^1 \underset{\sim}{Y}_4 (\nu_i,\eta') \underset{\sim}{A}_2 (\eta') d\eta' \} , \quad (82a)$$

$$\underset{\sim}{A}_1 (\nu) = \underset{\sim}{A}_1^0 (\nu) + \nu N_1^{-1} (\nu) \underset{\sim}{\tilde{X}}_1 (\nu) \{ \sum_{j=1}^{k_1} \underset{\sim}{Y}_1 (\nu,\nu_j) A_1 (\nu_j) + \int_0^1 \underset{\sim}{Y}_2 (\nu,\nu') \underset{\sim}{A}_1 (\nu') d\nu'$$

$$+ \sum_{j=1}^{k_2} \underset{\sim}{Y}_3 (\nu,\eta_j) A_2 (\eta_j) + \int_0^1 \underset{\sim}{Y}_4 (\nu,\eta') \underset{\sim}{A}_2 (\eta') d\eta' \} , \quad (82b)$$

$$A_2 (\eta_i) \{ 1 - \frac{1}{2} \eta_i N_2^{-1} (\eta_i) \underset{\sim}{\tilde{X}}_2 (\eta_i) \underset{\sim}{H}_2^{-1} (\eta_i) \underset{\sim}{J}_5 (\eta_i,\eta_i) \underset{\sim}{C}_2 \underset{\sim}{U}_2 (\eta_i) \}$$

$$= A_2^0 (\eta_i) + \eta_i N_2^{-1} (\eta_i) \underset{\sim}{\tilde{X}}_2 (\eta_i) \{ \sum_{j=1}^{k_2} (1 - \delta_{ij}) \underset{\sim}{Y}_5 (\eta_i,\eta_j) A_2 (\eta_j)$$

$$+ \int_0^1 \underset{\sim}{Y}_6 (\eta_i,\eta') \underset{\sim}{A}_2 (\eta') d\eta' + \sum_{j=1}^{k_1} \underset{\sim}{Y}_7 (\eta_i,\nu_j) A_1 (\nu_j) + \int_0^1 \underset{\sim}{Y}_8 (\eta_i,\nu') \underset{\sim}{A}_1 (\nu') d\nu' \} , \quad (83a)$$

and

$$\underset{\sim}{A}_2 (\eta) = \underset{\sim}{A}_2^0 (\eta) + \eta N_2^{-1} (\eta) \underset{\sim}{\tilde{X}}_2 (\eta) \{ \sum_{j=1}^{k_2} \underset{\sim}{Y}_5 (\eta,\eta_j) A_2 (\eta_j) + \int_0^1 \underset{\sim}{Y}_6 (\eta,\eta') \underset{\sim}{A}_2 (\eta') d\eta'$$

$$+ \sum_{j=1}^{k_1} \underset{\sim}{Y}_7 (\eta,\nu_j) A_1 (\nu_j) + \int_0^1 \underset{\sim}{Y}_8 (\eta,\nu') \underset{\sim}{A}_1 (\nu') d\nu' \} , \quad (83b)$$

where A_i^0 and $\underset{\sim}{A}_i^0$ are constant terms due to the source, $\underset{\sim}{J}$'s are 2x2 matrices of exponential functions, and $\underset{\sim}{Y}$'s are known vectors and matrices involving the $\underset{\sim}{H}$ matrices and exponential functions, similar to the expressions that appear in Eqs. (45) and (49); we list these functions in Appendix B.

5 – NUMERICAL RESULTS

Computations were performed on an IBM 370/155 computer in double-precision arithmetic using standard Gaussian quadrature sets to represent integrals. Our results reported here are obtained using a 20-point and a 40-point set in the intervals $(0, 1/\sigma)$ and $(1/\sigma, 1)$, respectively. The accuracy of iterative solutions depends on the quadrature sets used. Because of long computation times, we did not use any higher order quadrature sets and the accuracy of our results is generally five or six significant figures, as verified by calculating moments of various order of the equations for the boudary and interface conditions.

CROSS SECTION SETS

Several cross section sets for two group calculations are available in the literature[1,11,17].

However, since in two-media problems the group energies must be compatible, we have generated the cross section sets given in Tables I and II using the XSDRN code[7]. The entire energy range ($0 < E < 15$ MeV) is divided at 0.3 eV (0.2994 eV in the code) to give thermal and fast energy groups:

group 1: $E < 0.3$ eV

group 2: $E \geq 0.3$ eV

This dividing energy may be considered too low for a conventional division of thermal and fast groups. We have selected this value to keep the matrix Q from becoming triangular, since for higher dividing energies the up-scattering cross section becomes quite small. Sets 1-4 are calculated for infinite media. To calculate Set 5 we took from the calculation of Set 5 we took from the calculation of Set 3 the microscopic cross sections for U^{235} and multiplied them by the normal density of uranium. The fission cross sections are taken to be zero for use in the cell problem.

The elements of the matrices $\underset{\sim}{\Sigma}$ and $\underset{\sim}{Q}$ are calculated from the data sets as follows:

$$\sigma = \sigma_1/\sigma_2 \quad , \quad q_{ij} = \{\sigma_{ij} + X_i \bar{\nu}_j \sigma_{fj}\}/2\sigma_2 \ .$$

THE TWO-SLAB PROBLEM

We consider three cases of incident flux

$$\underset{\sim}{f}(\mu) = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \qquad \text{Case 1} \ ,$$

$$\underset{\sim}{f}(\mu) = 3\mu \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \qquad \text{Case 2} \ ,$$

and

$$\underset{\sim}{f}(\mu) = 4\mu^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \qquad \text{Case 3} \ ,$$

and use Sets 1 and 2 for sample calculations.

The scalar fluxes are defined by

$$\begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \int_{-1}^{1} \underset{\sim}{I}(x,\mu)\, d\mu \ .$$

We report in Table III the group 1 scalar flux for Cases 1 and 2 in Figure 1 the scalar fluxes for Case 3. All results are for $\alpha_1 = \alpha_2 = 1$ and we use the notation (Set i, Set j) to denote that Set i is used for medium 1 and Set j for medium 2. The group 2 scalar flux is unchanged to the third digits with the reversal of the media. The number of iterations is about 35 and the computation time for one case is around 12 minutes.

THE CRITICAL PROBLEM

We consider two cases:

Case	Core	Refletor
1	Set 3	Set 1
2	Set 4	Set 1

For Case 2 we considered only the case of infinite reflector, but for Case 1 several reflector thicknesses are considered. Our results for the case of infinite reflector are shown in Table IV together with percent errors of P_N-approximation results. The P_1 approximation gives slightly larger critical sizes but the P_3 approximation is quite good for the cases considered here. We report in Table V our results for finite reflectors, where γ is the reflector thickness in mean-free-path. Figures 2 and 3 show the scalar fluxes for the cases of infinite reflector and Figures 4 and 5 show those for various reflector thicknesses. In Figures 6 through 9 we show angular fluxes at three places, inside the core, at the interface, and in the reflector, for both Cases 1 and 2 with infinite reflector.

The number of iterations is 51 and 39 and the computation time is about 61 and 53 minutes for Cases 1 and 2, respectively, with infinite reflector. The long computation time is due, partly, to the fact that most of the calculation must be performed in complex mode.

We have also considered two cases of fast reactor model using the cross section sets for U^{235}, U^{238}, and Pu^{239} given in Ref. 21. The convergence is quite slow and we have not pursued to obtain results of reportable accuracy.

THE CELL PROBLEM

We use Set 5 for the fuel and Set 1 for the moderator to calculate the thermal disadvantage factor defined as

$$\xi = (\alpha_1/\alpha_2) \int_0^{\alpha_2} \phi_{2t}(x)\,dx \Big/ \int_{-\alpha_1}^{0} \phi_{1t}(x)\,dx$$

where ϕ_{it} is the thermal group scalar flux in medium i.

In the fuel region we take $\nu\sigma_f = 0$ and in the moderator we consider uniform sources of thermal neutrons:

$$\underset{\sim}{S} = \begin{bmatrix} 1/\alpha_2 \\ \\ 0 \end{bmatrix}$$

Figure 10 shows the thermal flux for three fuel thicknesses with $\alpha_2 = 0.2$. We report in Table VI thermal disadvantage factor for several cell sizes and in Figure 11 a comparison between the exact and S-N results of the thermal flux is presented. The S_N results were obtained by the ANISN code[5]. For the exact calculation the number of iterations and the computation time are of the same order as for the two-slab problem. For the S_N calculation the computation time is from 12 seconds (S_2) to 30 seconds (S_{16}) with 20 spatial mesh points in each of the fuel and moderator regions. All S_N results for the disadvantage factor are smaller than our results.

For smaller cell sizes the convergence is faster if the equations for the discrete coefficients are derived simply by applying the orthogonality theorem to Eqs. (76) and (77), i.e., without the steps 2 and 3 applied to the equations for the continuum coefficients. This seems to be due to the factors that appear in the denominator in Eqs. (78-81). The computer program based on Eqs. (82) and (83) can be modified to include this case with an addition of a few statements.

6 – COMMENTS AND CONCLUSIONS

We have shown that problems involving dissimilar media can be analysed numerically in two-group transport theory for isotropic scattering using the exact singular-eigenfunction-expansion method. In principle the method used here can be applied to any multiregion and multimedia problems in plane geometry. However, the computation time (and/or memory requirements) is quite long compared with the P_N and S_N approximations. The computation time can be reduced if single-precision arithmetic and low-order quadrature sets are used. Further reduction is possible if, for example, the $\tilde{Y}$ functionals in the cell problem are stored, since they are independent of the coefficients and can be calculated once and for all: in our calculation they were calculated in every iterative step due to large memory requirements to store them. Further, as was mentioned previously, in some cases the convergence of our solution is quite slow.

Our solution is not practical for routine calculations or parametric surveys. However, since one of the purposes of exact transport theory analysis is to supply standards of comparison of various approximate methods, we believe that our numerical results can serve for this purpose and that our solution and the method of regularization used here are of value in that they facilitate exact analyses of two- or multi-media problems for the first time.

APPENDIX A

The Y functionals that appear in Eqs. (68), (69) and (70) are as follows:

$$
\begin{aligned}
\underset{\sim}{Y}_1(\xi) = & \sum_{i=1}^{k_2} \Big[\frac{\sigma_1\eta_i}{\sigma_2\eta_i+\sigma_1\xi} \underset{\sim}{H}_1^{-1}(\sigma_2\eta_i/\sigma_1)\, \underset{\sim}{k}_1 \,\{1-E_2(\eta_i)\,E_2(\sigma_1\xi/\sigma_2)\} \\
& + \frac{\eta_i}{\eta_i+\xi} \underset{\sim}{H}_1^{-1}(\eta_i)\, \underset{\sim}{k}_2 \,\{1-E_2(\eta_i)\,E_2(\xi)\}\Big] \underset{\sim}{G}\, \underset{\sim}{C}_2\, \underset{\sim}{U}_2(\eta_i)\, A_2(\eta_i) \\
& + \sum_{i=1}^{k_2} \Big[\frac{\sigma_1\eta_i}{\sigma_2\eta_i-\sigma_1\xi} \underset{\sim}{H}_1^{-1}(-\sigma_2\eta_i/\sigma_1)\, \underset{\sim}{k}_1 \,\{E_2(\eta_i)-E_2(\sigma_1\xi/\sigma_2)\} \\
& + \frac{\eta_i}{\eta_i-\xi} \underset{\sim}{H}_1^{-1}(-\eta_i)\, \underset{\sim}{k}_2 \,\{E_2(\eta_i)-E_2(\xi)\}\, \underset{\sim}{G}\, \underset{\sim}{C}_2\, \underset{\sim}{U}_2(\eta_i)\, A_2(-\eta_i) \\
& + \int_0^1 \Big[\frac{\sigma_1\eta}{\sigma_2\eta+\sigma_1\xi} \underset{\sim}{H}_1^{-1}(\sigma_2\eta/\sigma_1)\, \underset{\sim}{k}_1 \,\{1-E_2(\eta)\,E_2(\sigma_1\xi/\sigma_2)\} \\
& + \frac{\eta}{\eta+\xi} \underset{\sim}{H}_1^{-1}(\eta)\, \underset{\sim}{k}_2 \,\{1-E_2(\eta)\,E_2(\xi)\}\Big] \underset{\sim}{G}\, \underset{\sim}{C}_2\, \underset{\sim}{A}_2(\eta)\, d\eta \\
& + \underset{\sim}{C}_1 \int_0^1 \Big[\frac{\sigma_1\eta}{\sigma_2\eta-\sigma_1\xi} \underset{\sim}{\tilde{H}}_1(\sigma_2\eta/\sigma_1)\, \underset{\sim}{C}_1^{-1}\, \underset{\sim}{\tilde{\lambda}}_1(\sigma_2\eta/\sigma_1)\, \underset{\sim}{k}_1 \,\{E_2(\eta)-E_2(\sigma_1\xi/\sigma_2)\} \\
& + \frac{\eta}{\eta-\xi} \underset{\sim}{\tilde{H}}_1(\eta)\, \underset{\sim}{C}_1^{-1}\, \underset{\sim}{\tilde{\lambda}}_1(\eta)\, \underset{\sim}{k}_2 \,\{E_2(\eta)-E_2(\xi)\}\Big] \underset{\sim}{G}\, \underset{\sim}{C}_2\, \underset{\sim}{A}_2(-\eta)\, d\eta \\
& + \underset{\sim}{C}_1 \int_0^1 \Big[\frac{\sigma_2\eta}{\sigma_1\xi-\sigma_2\eta} \underset{\sim}{\tilde{H}}_1(\sigma_2\eta/\sigma_1)\, \underset{\sim}{k}_1 \,\{E_2(\eta)-E_2(\sigma_1\xi/\sigma_2) \\
& + \frac{\eta}{\xi-\eta} \underset{\sim}{\tilde{H}}_1(\eta)\, \underset{\sim}{k}_2 \,\{E_2(\eta)-E_2(\xi)\}\Big] \underset{\sim}{G}\, \underset{\sim}{\Theta}_2(\eta)\, \underset{\sim}{\lambda}_2(\eta)\, \underset{\sim}{A}_2(-\eta)\, d\eta \\
& - \int_0^1 \frac{\nu}{\nu+\xi} \underset{\sim}{H}_1^{-1}(\nu)\, \underset{\sim}{C}_1\, \underset{\sim}{E}_1(\nu)\, \underset{\sim}{A}_1(\nu)\, d\nu \;,
\end{aligned}
$$

$$
Y_2(\xi) = \sum_{i=1}^{k_1} \Big[\frac{\sigma_2\nu_i}{\sigma_1\nu_i+\sigma_2\xi} \underset{\sim}{H}_2^{-1}(\sigma_1\nu_i/\sigma_2)\, \underset{\sim}{k}_1 \,\{1-E_1(\nu_i)\,E_1(\sigma_2\xi/\sigma_1)\}
$$

$$+ \frac{\nu_i}{\nu_i + \xi} \underset{\sim}{H}_2^{-1}(\nu_i)\, \underset{\sim}{k}_2 \{ 1 - E_1(\nu_i) E_1(\xi) \}] \underset{\sim}{G}^{-1} \underset{\sim}{C}_1 \underset{\sim}{U}_1(\nu_i) A_1(\nu_i)$$

$$+ \sum_{i=1}^{k_2} [\frac{\sigma_2 \nu_i}{\sigma_1 \nu_i - \sigma_2 \xi} \underset{\sim}{H}_2^{-1}(-\sigma_1 \nu_i/\sigma_2)\, \underset{\sim}{k}_1 \{ E_1(\nu_i) - E_1(\sigma_2 \xi/\sigma_1) \}$$

$$+ \frac{\nu_i}{\nu_i - \xi} \underset{\sim}{H}_2^{-1}(-\nu_i)\, \underset{\sim}{k}_2 \{ E_1(\nu_i) - E_1(\xi) \}] \underset{\sim}{G}^{-1} \underset{\sim}{C}_1 \underset{\sim}{U}_1(\nu_i) A_1(\nu_i)$$

$$+ \int_0^1 [\frac{\sigma_2 \nu}{\sigma_1 \nu + \sigma_2 \xi} \underset{\sim}{H}_2^{-1}(\sigma_1 \nu/\sigma_2)\, \underset{\sim}{k}_1 \{ 1 - E_1(\nu) E_1(\sigma_2 \xi/\sigma_1) \}$$

$$+ \frac{\nu}{\nu + \xi} \underset{\sim}{H}_2^{-1}(\nu)\, \underset{\sim}{k}_2 \{ 1 - E_1(\nu) E_1(\xi) \}] \underset{\sim}{G}^{-1} \underset{\sim}{C}_1 \underset{\sim}{A}_1(\nu)\, d\nu$$

$$+ \underset{\sim}{C}_2 \int_0^1 [\frac{\sigma_2 \nu}{\sigma_1 \nu - \sigma_2 \xi} \underset{\sim}{\tilde{H}}_2(\sigma_1 \nu/\sigma_2)\, \underset{\sim}{C}_2^{-1} \underset{\sim}{\tilde{\lambda}}_2(\sigma_1 \nu/\sigma_2)\, \underset{\sim}{k}_1 \{ E_1(\nu) - E_1(\sigma_2 \xi/\sigma_1) \}$$

$$+ \frac{\nu}{\nu - \xi} \underset{\sim}{\tilde{H}}_2(\nu)\, \underset{\sim}{C}_2^{-1} \underset{\sim}{\tilde{\lambda}}_2(\nu)\, \underset{\sim}{k}_2 \{ E_1(\nu) - E_1(\xi) \}] \underset{\sim}{G}^{-1} \underset{\sim}{C}_1 \underset{\sim}{A}_1(\nu)\, d\nu$$

$$+ \underset{\sim}{C}_2 \int_0^1 [\frac{\sigma_1 \nu}{\sigma_2 \xi - \sigma_1 \nu} \underset{\sim}{\tilde{H}}_2(\sigma_1 \nu/\sigma_2)\, \underset{\sim}{k}_1 \{ E_1(\nu) - E_1(\sigma_2 \xi/\sigma_1) \}$$

$$+ \frac{\nu}{\xi - \nu} \underset{\sim}{\tilde{H}}_2(\nu)\, \underset{\sim}{k}_2 \{ E_1(\nu) - E_1(\xi) \}] \underset{\sim}{G}^{-1} \underset{\sim}{\Theta}_1(\nu)\, \underset{\sim}{\lambda}_1(\nu)\, \underset{\sim}{A}_1(\nu)\, d\nu$$

$$+ \int_0^1 \frac{\eta}{\eta + \xi} \underset{\sim}{H}_2^{-1}(\eta) \{ 1 - E_2(\eta) E_2(\xi) \} \underset{\sim}{E}(\xi)\, \underset{\sim}{C}_2 \underset{\sim}{A}_2(\eta)\, d\eta$$

$$- \sum_{i=1}^{k_2} \frac{\eta_i}{\eta_i + \xi} \underset{\sim}{H}_2^{-1}(\eta_i)\, E_2(\eta_i)\, \underset{\sim}{C}_2 \underset{\sim}{U}_2(\eta_i)\, A_2(-\eta_i) - \int_0^1 \frac{\eta}{\eta + \xi} \underset{\sim}{H}_2^{-1}(\eta) E_2(\eta) \underset{\sim}{C}_2 \underset{\sim}{A}_2(-\eta)\, d\eta$$

$$+ \underset{\sim}{C}_2 \int_0^1 \frac{\eta}{\eta - \xi} \underset{\sim}{\tilde{H}}_2(\eta)\, \underset{\sim}{C}_2^{-1} \underset{\sim}{\tilde{\lambda}}_2(\eta) \{ E_2(\eta) - E_2(\xi) \} \underset{\sim}{E}(\xi)\, \underset{\sim}{C}_2 \underset{\sim}{A}_2(-\eta)\, d\eta$$

$$+ \underset{\sim}{C}_2 \int_0^1 \frac{\eta}{\xi - \eta} \underset{\sim}{\tilde{H}}_2(\eta) \{ E_2(\eta) - E_2(\xi) \} \underset{\sim}{E}(\xi)\, \underset{\sim}{\Omega}_2(\eta)\, \underset{\sim}{\lambda}_2(\eta)\, \underset{\sim}{A}_2(-\eta)\, d\eta \ ,$$

$$\underset{\sim}{Y}_1(\xi) = \sum_{i=1}^{k_2} \frac{\eta_i}{\eta_i + \xi} \underset{\sim}{H}_2^{-1}(\eta_i)\, \underset{\sim}{C}_2 \underset{\sim}{U}_2(\eta_i)\, E_2(\eta_i)\, A_2(\eta_i)$$

$$- \int_0^1 \frac{\eta}{\eta + \xi} \underset{\sim}{H}_2^{-1}(\eta) \underset{\sim}{C}_2 \; E_2(\eta) \; \underset{\sim}{A}_2(\eta) \, d\eta \; .$$

APPENDIX B

The $\underset{\sim}{Y}$ functionals that appear in Eqs. (82) and (83) are as follows:

$$\underset{\sim}{Y}_1(\xi, \nu_j) = \frac{\nu_j}{\xi + \nu_j} \underset{\sim}{H}_1^{-1}(\nu_j) \; \underset{\sim}{J}_1(\xi, \nu_j) \underset{\sim}{C}_1 \underset{\sim}{U}_1(\nu_j) - \frac{\nu_j}{\nu_j - \xi} \underset{\sim}{H}_1^{-1}(-\nu_j) \; \underset{\sim}{J}_2(\xi, \nu_j) \underset{\sim}{C}_1 \underset{\sim}{U}_1(\nu_j) \quad ,$$

$$\underset{\sim}{Y}_2(\xi, \nu') = \frac{\nu'}{\xi + \nu'} \underset{\sim}{H}_1^{-1}(\nu') \; \underset{\sim}{J}_1(\xi, \nu') \underset{\sim}{C}_1$$

$$- \frac{\nu'}{\nu' - \xi} \underset{\sim}{C}_1 \underset{\sim}{\widetilde{H}}_1(\nu') \left\{ \underset{\sim}{\lambda}_1(\nu') \underset{\sim}{C}_1^{-1} \; \underset{\sim}{J}_2(\xi, \nu') \underset{\sim}{C}_1 - \underset{\sim}{J}_2(\xi, \nu') \underset{\sim}{\Theta}_1(\nu') \underset{\sim}{\lambda}_1(\nu') \right\} ,$$

$$\underset{\sim}{Y}_3(\xi, \eta_j) = \left\{ \frac{\sigma_1 \eta_j}{\sigma_2 \eta_j + \sigma_1 \xi} \underset{\sim}{H}_1^{-1}(\sigma_2 \eta_j / \sigma_1) \underset{\sim}{J}_3(\xi, \eta_j) \underset{\sim}{k}_1 + \frac{\eta_j}{\eta_j + \xi} \underset{\sim}{H}_1^{-1}(\eta_j) \underset{\sim}{J}_3(\xi, \eta_j) \underset{\sim}{k}_2 \right\} \underset{\sim}{G}\underset{\sim}{C}_2 \underset{\sim}{U}_2(\eta_j)$$

$$- \left\{ \frac{\sigma_1 \eta_j}{\sigma_2 \eta_j - \sigma_1 \xi} \underset{\sim}{H}_1^{-1}(-\sigma_2 \eta_j / \sigma_1) \underset{\sim}{J}_4(\xi, \eta_j) \underset{\sim}{k}_1 + \frac{\eta_j}{\eta_j - \xi} \underset{\sim}{H}_1^{-1}(-\eta_j) \underset{\sim}{J}_4(\xi, \eta_j) \underset{\sim}{k}_2 \right\} \underset{\sim}{G}\underset{\sim}{C}_2 \underset{\sim}{U}_2(\eta_j) \, ,$$

$$\underset{\sim}{Y}_4(\xi, \eta') = \left\{ \frac{\sigma_1 \eta'}{\sigma_2 \eta' + \sigma_1 \xi} \underset{\sim}{H}_1^{-1}(\sigma_2 \eta' / \sigma_1) \underset{\sim}{J}_3(\xi, \eta') \underset{\sim}{k}_1 + \frac{\eta'}{\eta' + \xi} \underset{\sim}{H}_1^{-1}(\eta') \underset{\sim}{J}_3(\xi, \eta') \underset{\sim}{k}_2 \right\} \underset{\sim}{G}\underset{\sim}{C}_2$$

$$- \underset{\sim}{C}_1 \left\{ \frac{\sigma_1 \eta'}{\sigma_2 \eta' - \sigma_1 \xi} \underset{\sim}{\widetilde{H}}_1(\sigma_2 \eta' / \sigma_1) \underset{\sim}{\lambda}_1(\sigma_2 \eta' / \sigma_1) \underset{\sim}{C}_1^{-1} \underset{\sim}{J}_4(\xi, \eta') \underset{\sim}{k}_1 \right.$$

$$\left. + \frac{\eta'}{\eta' - \xi} \underset{\sim}{\widetilde{H}}_1(\eta') \underset{\sim}{\lambda}_1(\eta') \underset{\sim}{C}_1^{-1} \underset{\sim}{J}_4(\xi, \eta') \underset{\sim}{k}_2 \right\} \underset{\sim}{G}\underset{\sim}{C}_2$$

$$+ \underset{\sim}{C}_1 \left\{ \frac{\sigma_2 \eta'}{\sigma_2 \eta' - \sigma_1 \xi} \underset{\sim}{\widetilde{H}}_1(\sigma_2 \eta' / \sigma_1) \underset{\sim}{J}_4(\xi, \eta') \underset{\sim}{k}_1 + \frac{\eta'}{\eta' - \xi} \underset{\sim}{\widetilde{H}}_1(\eta') \underset{\sim}{J}_4(\xi, \eta') \underset{\sim}{k}_2 \right\} \underset{\sim}{\Theta}_2(\eta') \underset{\sim}{G} \underset{\sim}{\lambda}_2(\eta') \, ,$$

$$\underset{\sim}{Y}_5(\xi,\eta_j) = \frac{\eta_j}{\eta_j+\xi}\,\underset{\sim}{H}_2^{-1}(\eta_j)\,\underset{\sim}{J}_5(\xi,\eta_j)\,\underset{\sim}{C}_2\,\underset{\sim}{U}_2(\eta_j) - \frac{\eta_j}{\eta_j-\xi}\,\underset{\sim}{H}_2^{-1}(-\eta_j)\,\underset{\sim}{J}_6(\xi,\eta_j)\,\underset{\sim}{C}_2\,\underset{\sim}{U}_2(\eta_j) \quad ,$$

$$\underset{\sim}{Y}_6(\xi,\eta') = \frac{\eta'}{\eta'+\xi}\,\underset{\sim}{H}_2^{-1}(\eta')\,\underset{\sim}{J}_5(\xi,\eta')\,\underset{\sim}{C}_2$$

$$-\frac{\eta'}{\eta'-\xi}\,\underset{\sim}{C}_2\,\underset{\sim}{\widetilde{H}}_2(\eta')\,\{\underset{\sim}{\lambda}_2(\eta')\,\underset{\sim}{C}_2^{-1}\,\underset{\sim}{J}_6(\xi,\eta')\,\underset{\sim}{C}_2 - \underset{\sim}{J}_6(\xi,\eta')\,\underset{\sim}{\Theta}_2(\eta')\,\underset{\sim}{\lambda}_2(\eta')\}$$

$$\underset{\sim}{Y}_7(\xi,\nu_j) = \{\frac{\sigma_2\nu_j}{\sigma_1\nu_j+\sigma_2\xi}\,\underset{\sim}{H}_2^{-1}(\sigma_1\nu_j/\sigma_2)\,\underset{\sim}{J}_7(\xi,\nu_j)\,\underset{\sim}{k}_1 + \frac{\nu_j}{\nu_j+\xi}\,\underset{\sim}{H}_2^{-1}(\nu_j)\,\underset{\sim}{J}_7(\xi,\nu_j)\,\underset{\sim}{k}_2\}\underset{\sim}{G}^{-1}\,\underset{\sim}{C}_1\,\underset{\sim}{U}_1(\nu_j)$$

$$-\{\frac{\sigma_2\nu_j}{\sigma_1\nu_j-\sigma_2\xi}\,\underset{\sim}{H}_2^{-1}(-\sigma_1\nu_j/\sigma_2)\,\underset{\sim}{J}_8(\xi,\nu_j)\,\underset{\sim}{k}_1 + \frac{\nu_j}{\nu_j-\xi}\,\underset{\sim}{H}_2^{-1}(-\nu_j)\,\underset{\sim}{J}_8(\xi,\nu_j)\underset{\sim}{k}_2\}\underset{\sim}{G}^{-1}\,\underset{\sim}{C}_1\,\underset{\sim}{U}_1(\nu_j)\,,$$

$$\underset{\sim}{Y}_8(\xi,\nu') = \{\frac{\sigma_2\nu'}{\sigma_1\nu'+\sigma_2\xi}\,\underset{\sim}{H}_2^{-1}(\sigma_1\nu'/\sigma_2)\,\underset{\sim}{J}_7(\xi,\nu')\,\underset{\sim}{k}_1 + \frac{\nu'}{\nu'+\xi}\,\underset{\sim}{H}_2^{-1}(\nu')\,\underset{\sim}{J}_7(\xi,\nu')\,\underset{\sim}{k}_2\}\,\underset{\sim}{G}^{-1}\,\underset{\sim}{C}_1$$

$$-\underset{\sim}{C}_2\;\{\frac{\sigma_2\nu'}{\sigma_1\nu'-\sigma_2\xi}\,\underset{\sim}{\widetilde{H}}_2(\sigma_1\nu'/\sigma_2)\,\underset{\sim}{\lambda}_2(\sigma_1\nu'/\sigma_2)\,\underset{\sim}{C}_2^{-1}\,\underset{\sim}{J}_8(\xi,\nu')\,\underset{\sim}{k}_1$$

$$+\frac{\nu'}{\nu'-\xi}\,\underset{\sim}{\widetilde{H}}_2(\nu')\,\underset{\sim}{\lambda}_2(\nu')\,\underset{\sim}{C}_2^{-1}\,\underset{\sim}{J}_8(\xi,\nu')\,\underset{\sim}{k}_2\}\,\underset{\sim}{G}^{-1}\,\underset{\sim}{C}_1$$

$$+\underset{\sim}{C}_2\;\{\frac{\sigma_1\nu'}{\sigma_1\nu'-\sigma_2\xi}\,\underset{\sim}{\widetilde{H}}_2(\sigma_1\nu'/\sigma_2)\,\underset{\sim}{J}_8(\xi,\nu')\,\underset{\sim}{k}_1 + \frac{\nu'}{\nu'-\xi}\,\underset{\sim}{\widetilde{H}}_2(\nu')\,\underset{\sim}{J}_8(\xi,\nu')\,\underset{\sim}{k}_2\}\underset{\sim}{\Theta}_1(\nu')\underset{\sim}{G}^{-1}\,\underset{\sim}{\lambda}_1(\nu')\,,$$

where

$$\underset{\sim}{J}_1(x,y) = \begin{bmatrix} E_2(\sigma_1 x/\sigma_2) - E_1(y) & 0 \\ 0 & E_2(x) - E_1(y) \end{bmatrix} \underset{\sim}{J}^1(x),$$

$$\underset{\sim}{J}_2(x,y) = \begin{bmatrix} E_2(\sigma_1 x/\sigma_2) & 0 \\ 0 & E_2(x) \end{bmatrix} [E_1(x) - E_1(y)]\,\underset{\sim}{J}^1(x)\,,$$

$$\underset{\sim}{J_3}(x,y) = \begin{bmatrix} 1 - E_2(\sigma_1 x/\sigma_2)\, E_2(y) & 0 \\ 0 & 1 - E_2(x) E_2(y) \end{bmatrix} \underset{\sim}{J^1}(x),$$

$$\underset{\sim}{J_4}(x,y) = \begin{bmatrix} E_2(\sigma_1 x/\sigma_2) - E_2(y) & 0 \\ 0 & E_2(x) - E_2(y) \end{bmatrix} \underset{\sim}{J^1}(x),$$

$$\underset{\sim}{J_5}(x,y) = \begin{bmatrix} E_1(\sigma_2 x/\sigma_1) - E_2(y) & 0 \\ 0 & E_1(x) - E_2(y) \end{bmatrix} \underset{\sim}{J^2}(x),$$

$$\underset{\sim}{J_6}(x,y) = \begin{bmatrix} E_1(\sigma_2 x/\sigma_1) & 0 \\ 0 & E_1(x) \end{bmatrix} \begin{bmatrix} E_2(x) - E_2(y) \end{bmatrix} \underset{\sim}{J^2}(x),$$

$$\underset{\sim}{J_7}(x,y) = \begin{bmatrix} 1 - E_1(\sigma_2 x/\sigma_1)\, E_1(y) & 0 \\ 0 & 1 - E_1(x)\, E_1(y) \end{bmatrix} \underset{\sim}{J^2}(x),$$

$$\underset{\sim}{J_8}(x,y) = \begin{bmatrix} E_1(\sigma_2 x/\sigma_1) - E_1(y) & 0 \\ 0 & E_1(x) - E_1(y) \end{bmatrix} \underset{\sim}{J^2}(x),$$

with

$$\underset{\sim}{J^1}(x) = \begin{bmatrix} [1 - E_1(x)\, E_2(\sigma_1 x/\sigma_2)]^{-1} & 0 \\ 0 & [1 - E_1(x)\, E_2(x)]^{-1} \end{bmatrix},$$

$$\underset{\sim}{J^2}(x) = \begin{bmatrix} [1 - E_1(\sigma_2 x/\sigma_1)\, E_2(x)]^{-1} & 0 \\ 0 & [1 - E_1(x)\, E_2(x)]^{-1} \end{bmatrix}.$$

Table I

Definition of the cross section sets

Set	Material
1	H_2O
2	$H_2O + B$, $B/H = 3/2000$
3	$H_2O + U^{235}$, $U/H = 1/1000$
4	$H_2O + U^{235}$, $U/H = 1/500$
5	U^{235}

Table II

Macroscopic cross sections and the discrete eigenvalues

	Set 1	Set 2	Set 3	Set 4	Set 5
σ_1	2.9865	2.9664	2.9727	2.9628	25.826
σ_2	0.88798	0.88731	0.88721	0.88655	1.2782
σ_{11}	2.9676	2.8876	2.9183	2.8751	0.59234
σ_{12}	0.04749	0.04588	0.04635	0.04536	0.00001421
σ_{21}	0.000330	0.00106	0.000767	0.00116	0.000003357
σ_{22}	0.83975	0.83912	0.83892	0.83807	0.41677
$\nu_1\sigma_{f1}$	0.0	0.0	0.07391	0.14324	0.0
$\nu_2\sigma_{f2}$	0.0	0.0	0.00209	0.00412	0.0
χ_1	0.0	0.0	0.0	0.0	0.0
χ_2	0.0	0.0	1.0	1.0	0.0
Discrete eigenvalues					
	2.604020	2.551909	i4.721086	i3.437681	1.004466
	2.122979	1.070095	1.152128	–	–

Table IV

Critical half thickness of the core and percent errors of P_N approximation for the case of infinite reflector

Case	Exact	Percent errors	
	α	P_1	P_3
1	4.15767	1.0	< 0.1
2	2.1826	1.9	< 0.1

Table III

The group 1 scalar flux in two slabs with an incident flux

x	$\varphi_1(x)$ [a]	$\phi_1(x)$ [b]	$\phi_1(x)$ [c]	$\phi_1(x)$ [d]
0.0	0.16816	0.14545	0.16266	0.14054
0.2	0.36402	0.31225	0.35678	0.30612
0.4	0.46469	0.39937	0.46118	0.39702
0.6	0.51356	0.44804	0.51434	0.44991
0.8	0.52003	0.46878	0.52444	0.47433
1.0	0.48805	0.47039	0.49492	0.47875
1.2	0.43299	0.44800	0.44108	0.45801
1.4	0.36714	0.39637	0.37541	0.40663
1.6	0.28974	0.32156	0.29156	0.33078
1.8	0.20071	0.22618	0.20633	0.23313
2.0	0.08688	0.09781	0.08939	0.10089

a Case I, (Set 1, Set 2)
b Case I, (Set 2, Set 1)
c Case II, (Set 1, Set 2)
d Case II, (Set 2, Set 1)

Table V

Critical half-thickness for Case 1 with finite reflector

Reflector thickness γ	0	1	2	3	5
Core half-thickness a	6.85725	5.94147	5.22752	4.75065	4.31485

Table VI

Thermal disadvantage factor for two-slab cells and percent errors of S_N results

a_1	a_2	exact ξ	percent errors S_2	S_4	S_8	S_{16}
0.25	0.5	20.079	15.8	2.9	0.86	0.33
0.15	0.5	12.055	15.8	2.9	0.86	0.32
0.05	0.5	4.2910	18.6	3.1	0.91	0.36
0.15	0.2	9.7457	26.3	8.0	2.3	0.76
0.05	0.2	3.4757	27.9	8.4	2.3	0.78
0.025	0.2	2.1489	27.0	10.4	2.8	0.84

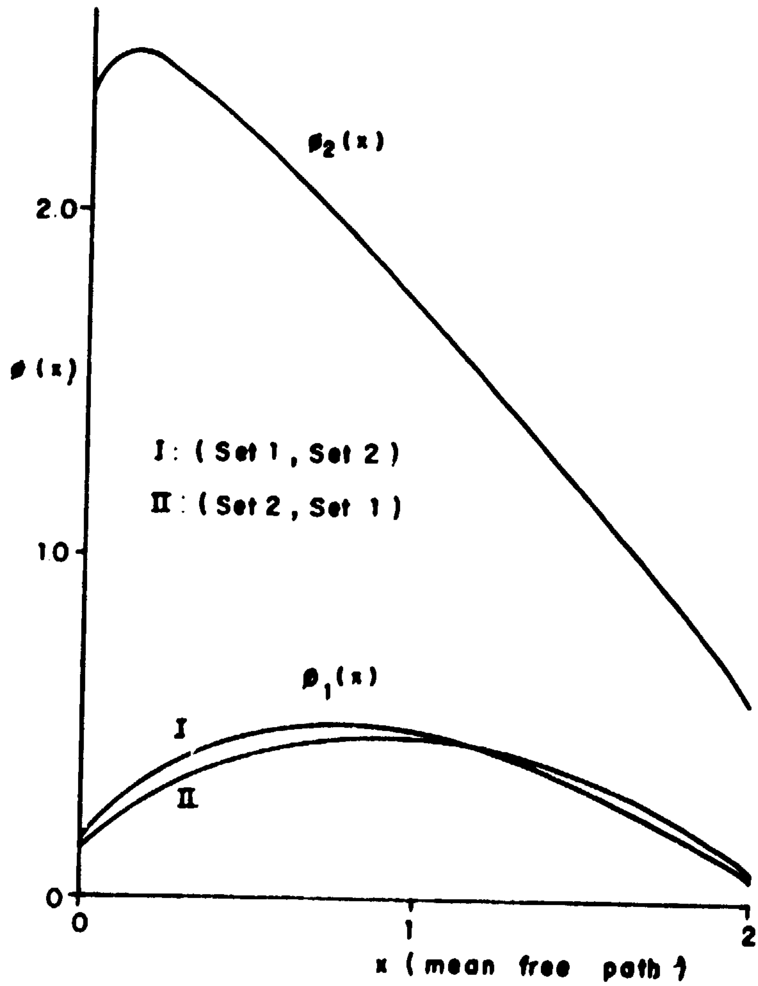


Figure 1 The scalar fluxes for Case 3 of the two slab problem with $\alpha_1 = \alpha_2 = 1$

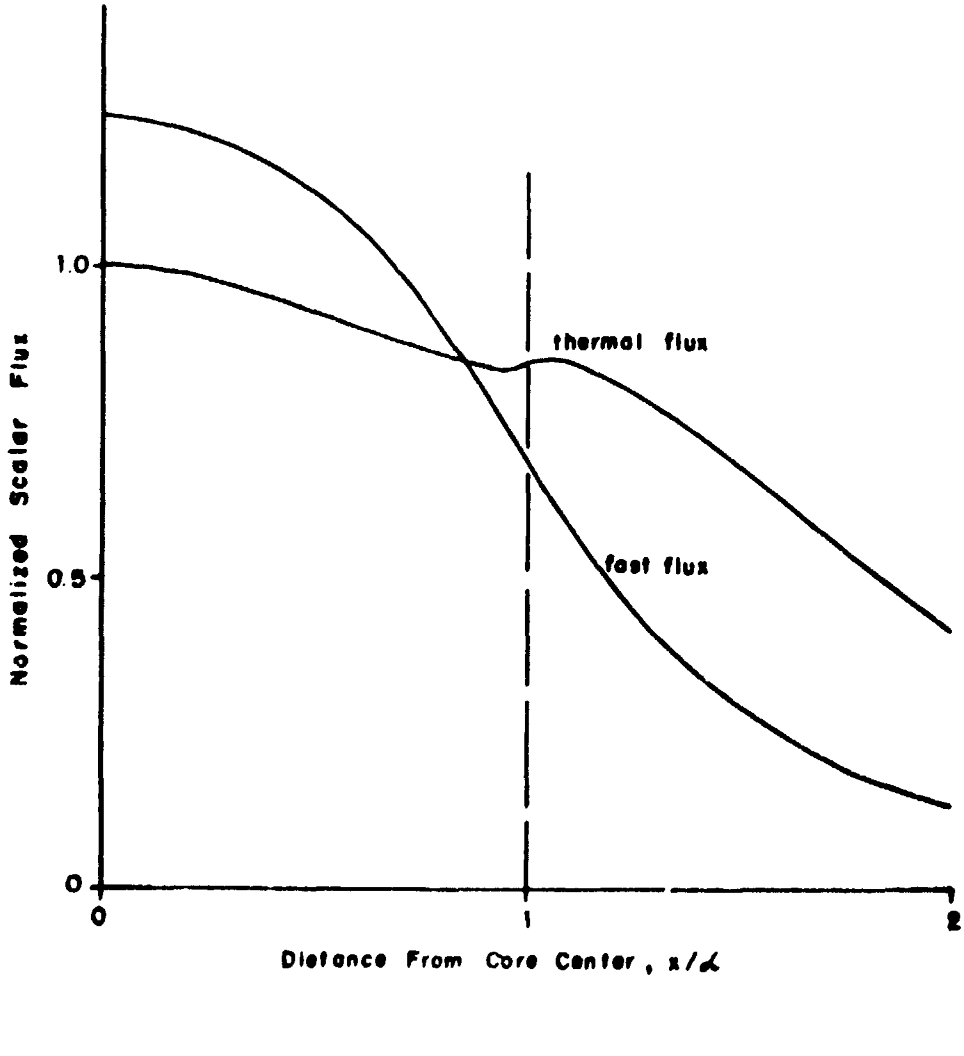


Figure 2 - The scalar fluxes for Case 1 of the critical problem with infinite reflector

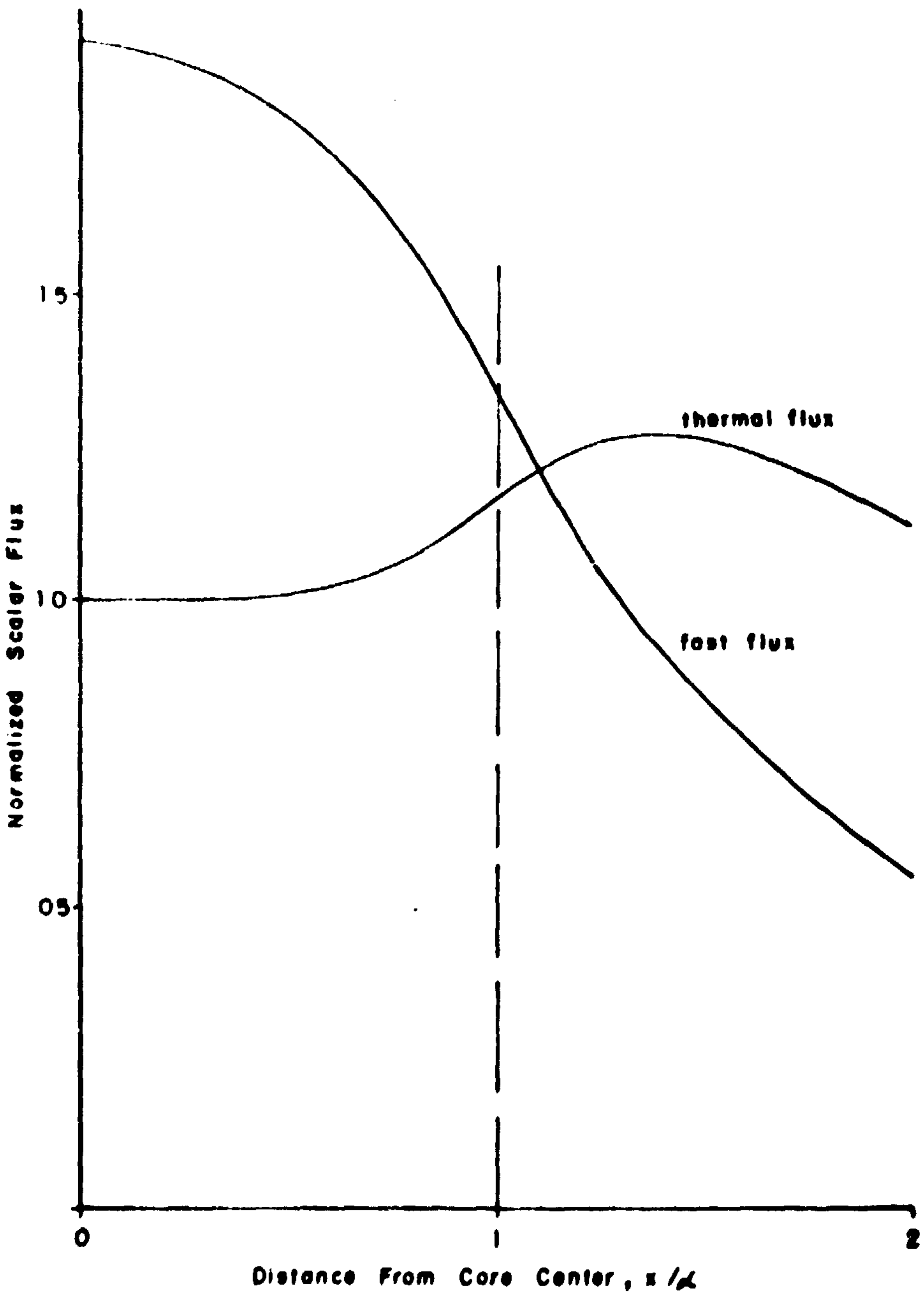


Figure 3 - The scalar fluxes for Case 2 of the critical problem with infinite reflector

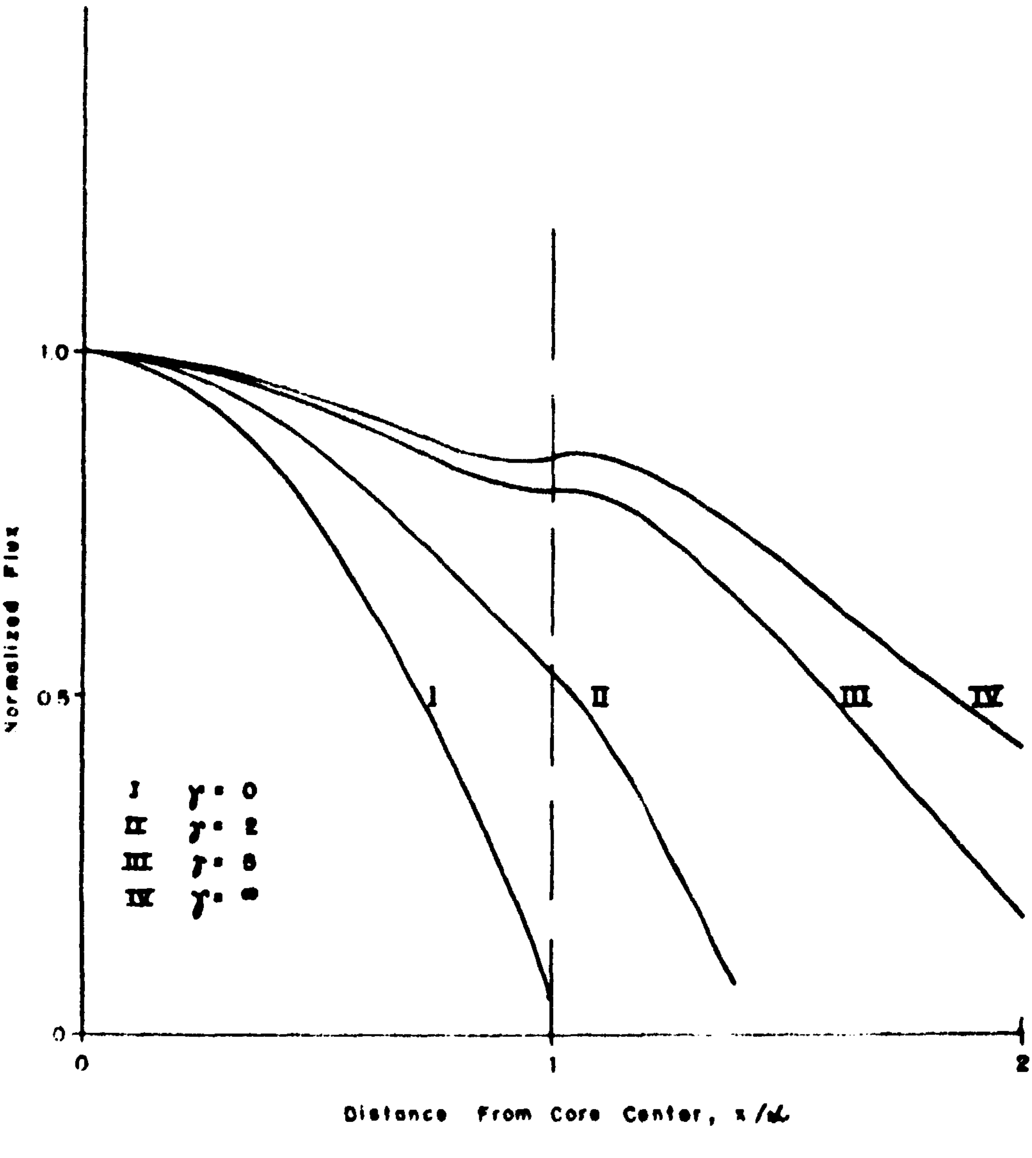


Figure 4 The thermal group scalar flux for Case 1 of the critical problem for various reflector thicknesses.

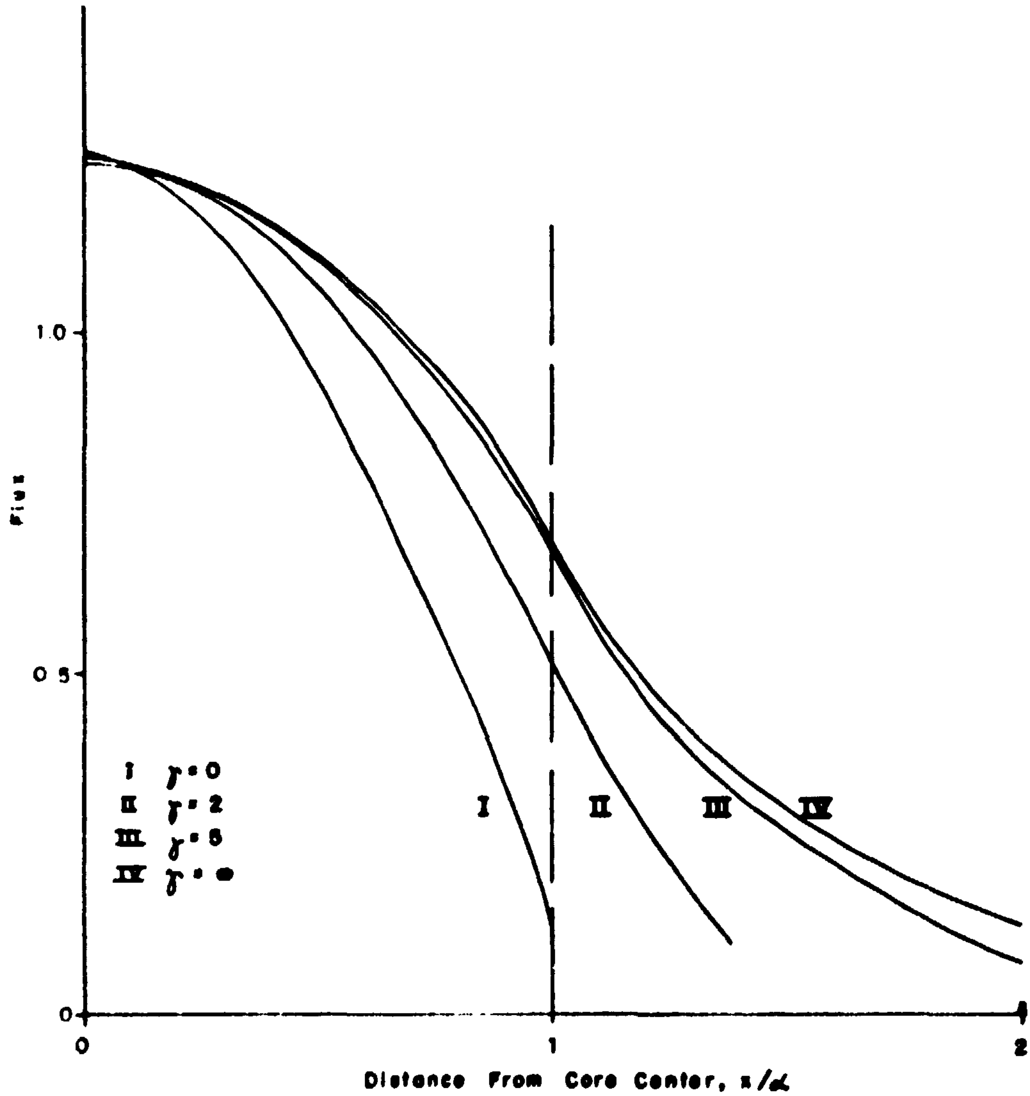


Figure 5 The fast group scalar fluxes for Case 1 of the critical problem for various reflector thicknesses

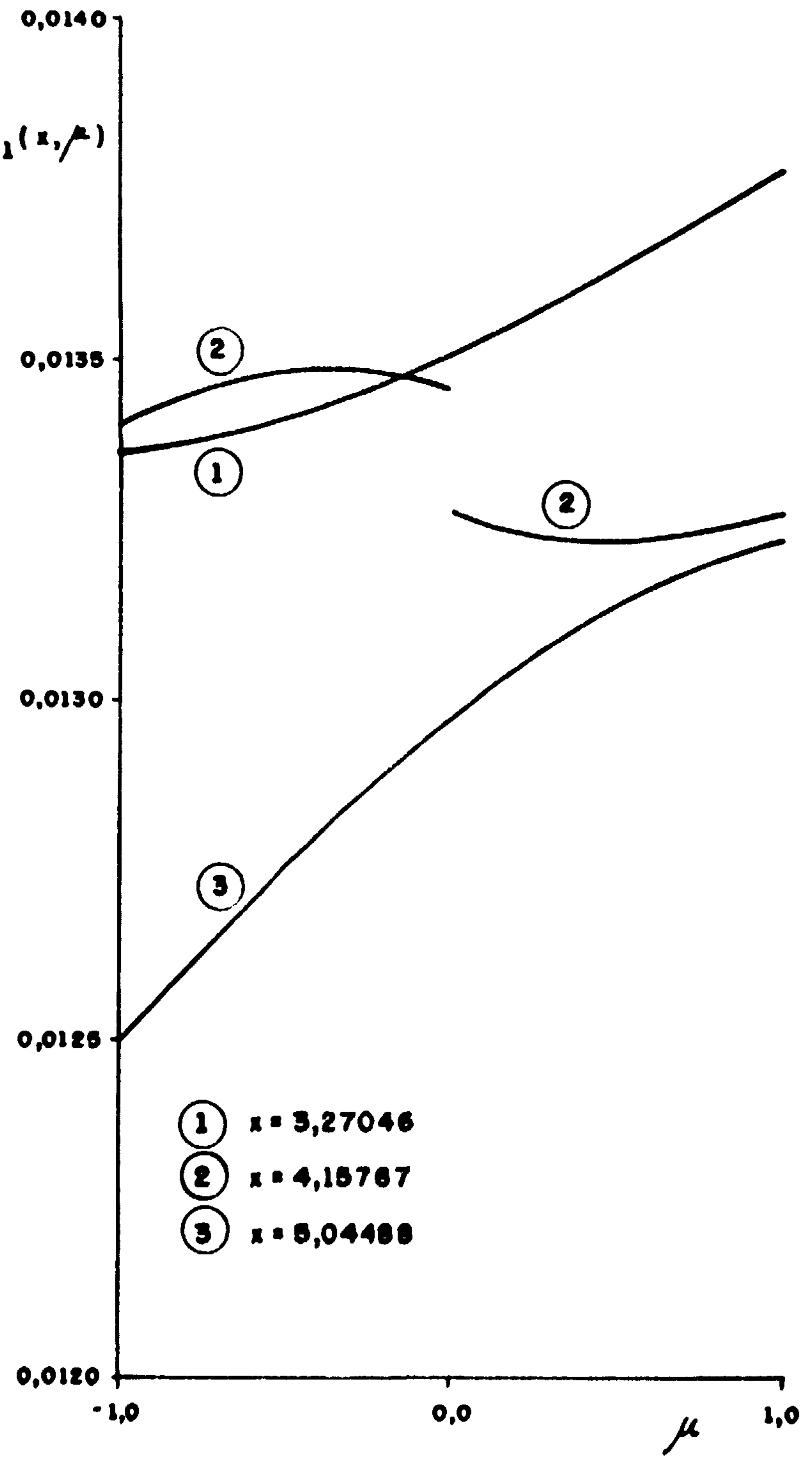


Figure 6 The thermal group angular flux for Case 1 of the critical problem with infinite reflector

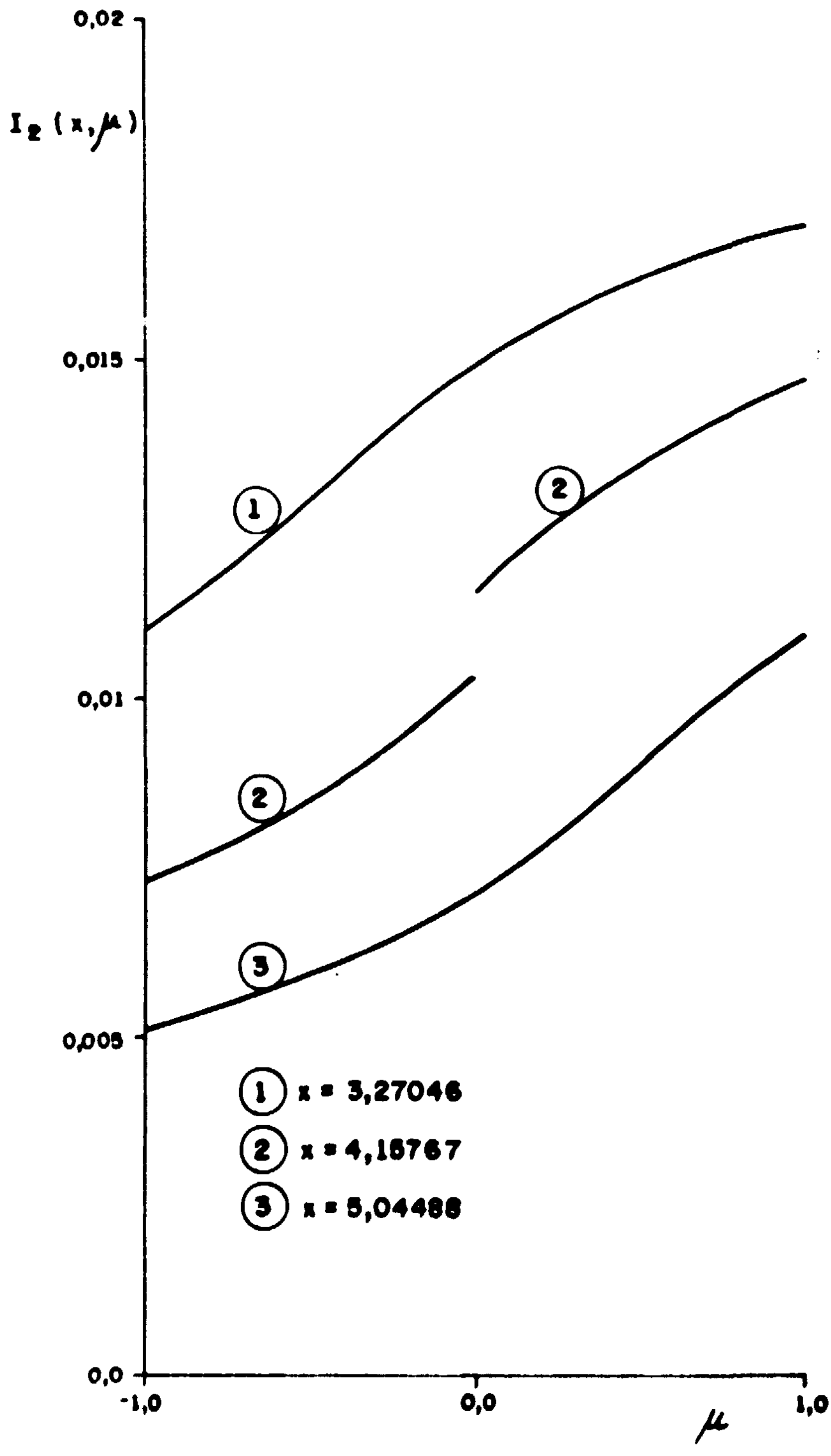


Figure 7 The fast group angular flux for Case 1 of the critical problem with infinite reflector

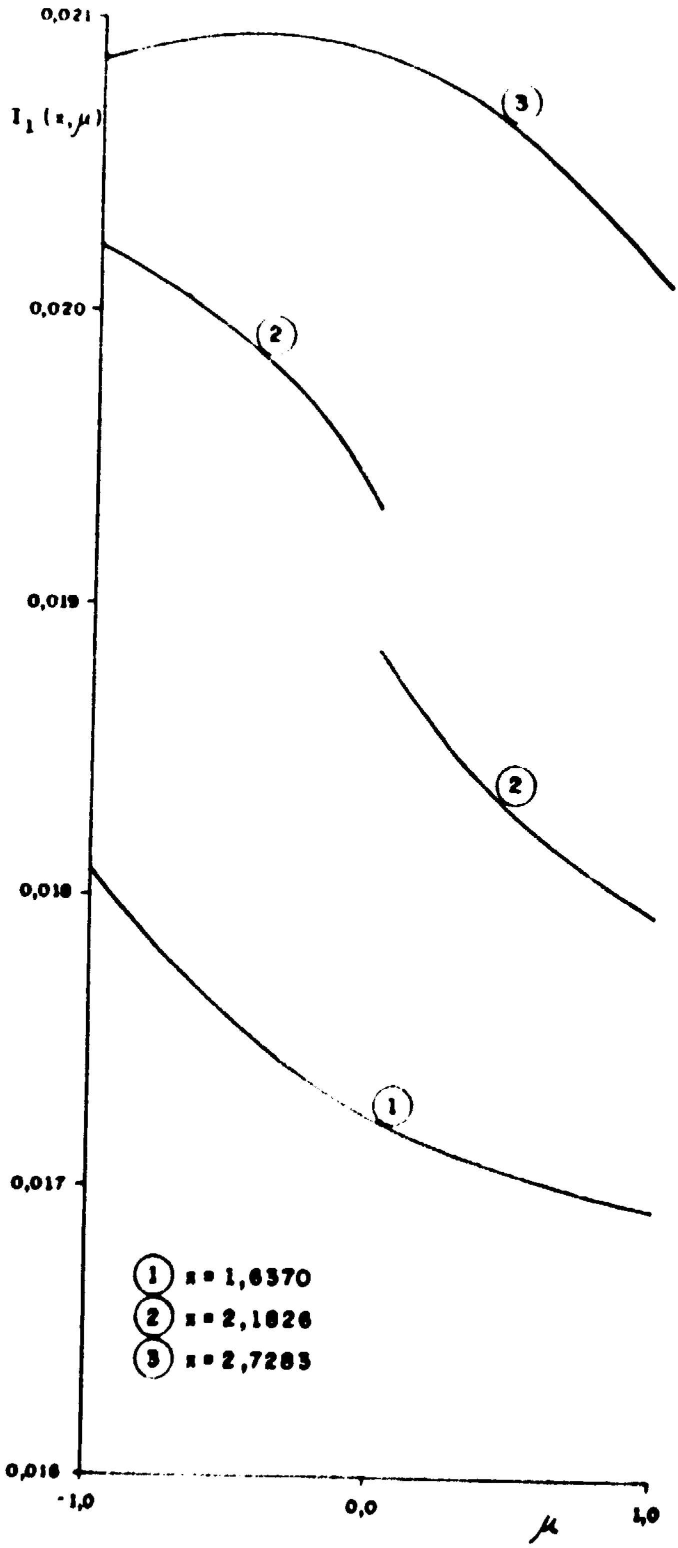


Figure 8 The thermal group angular flux for Case 2 of the critical problem with infinite reflector

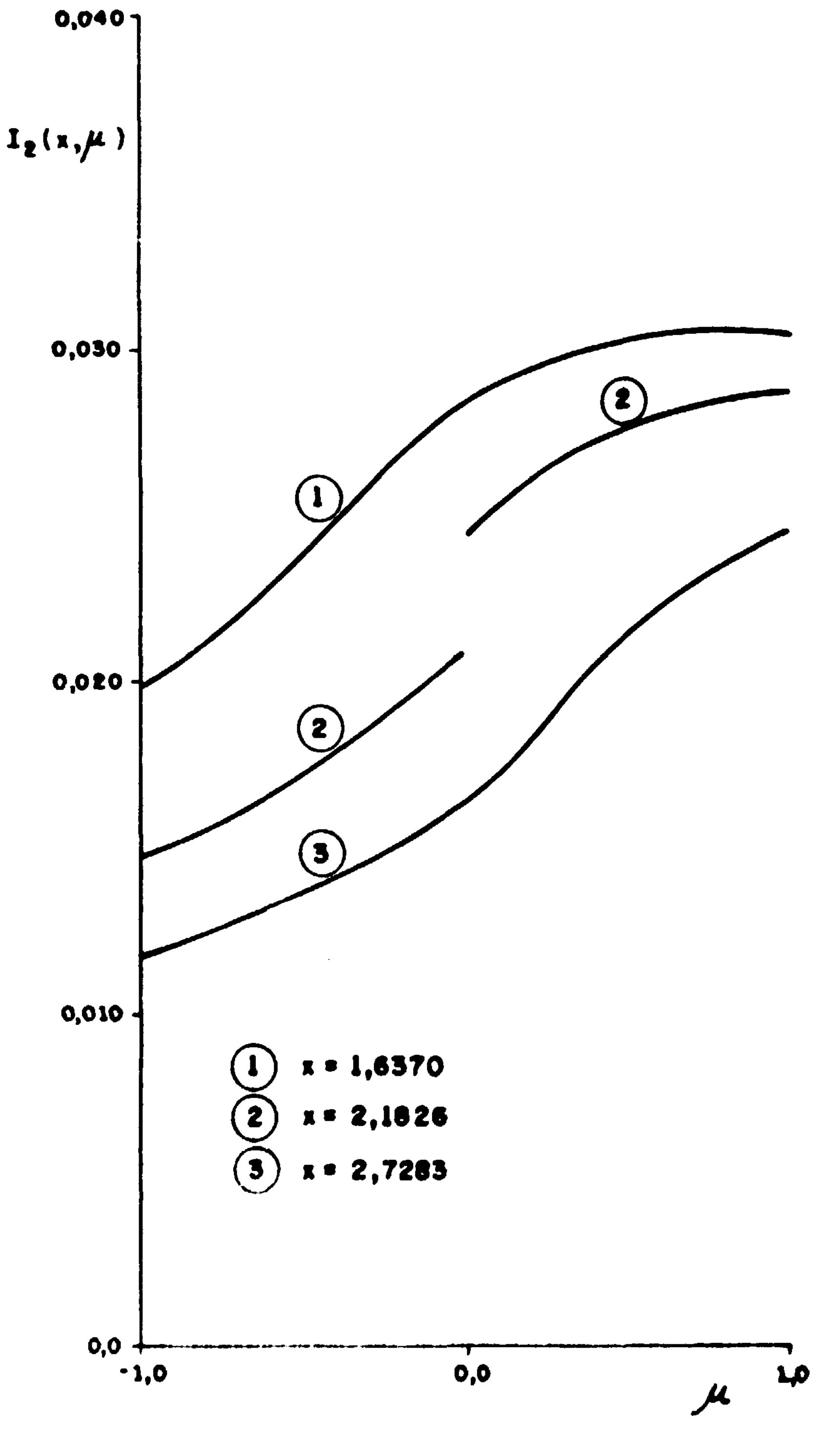


Figure 9 The fast group angular flux for Case 2 of the critical problem with infinite reflector

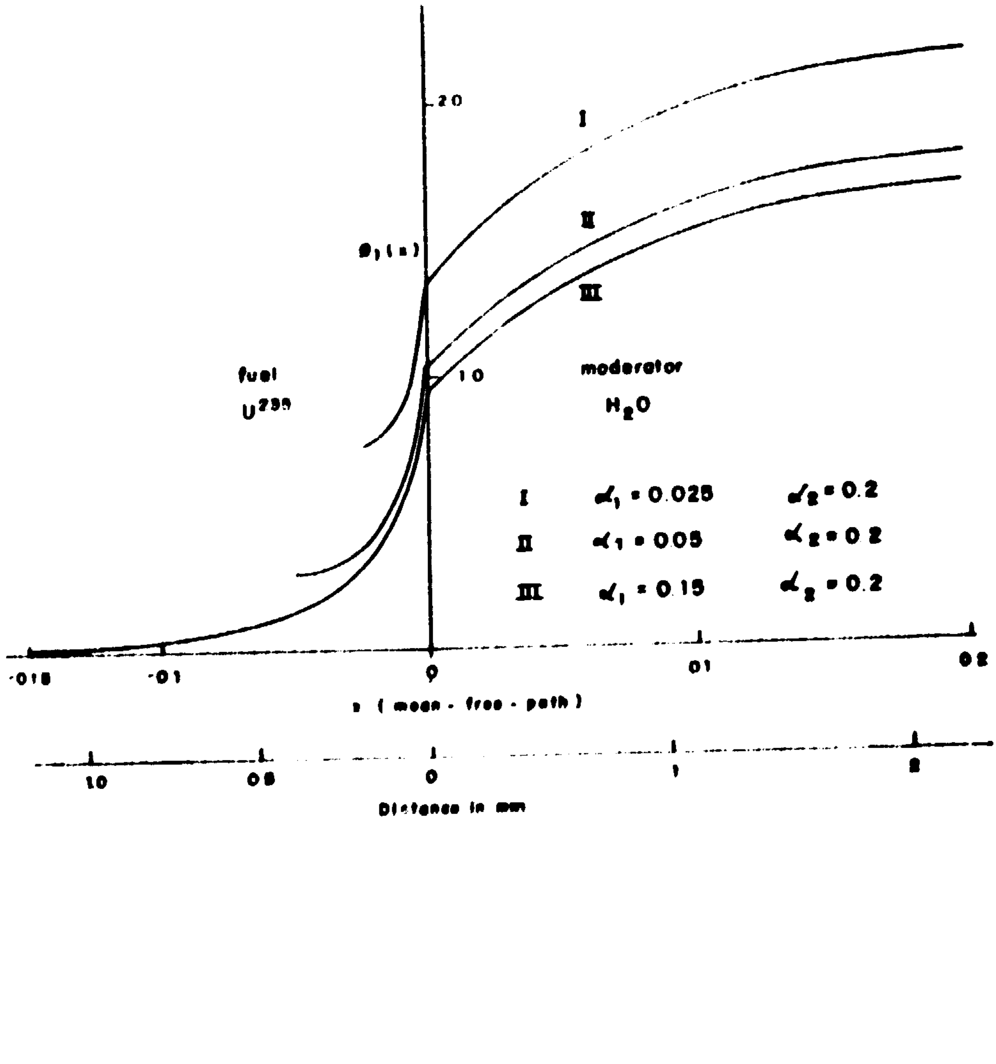
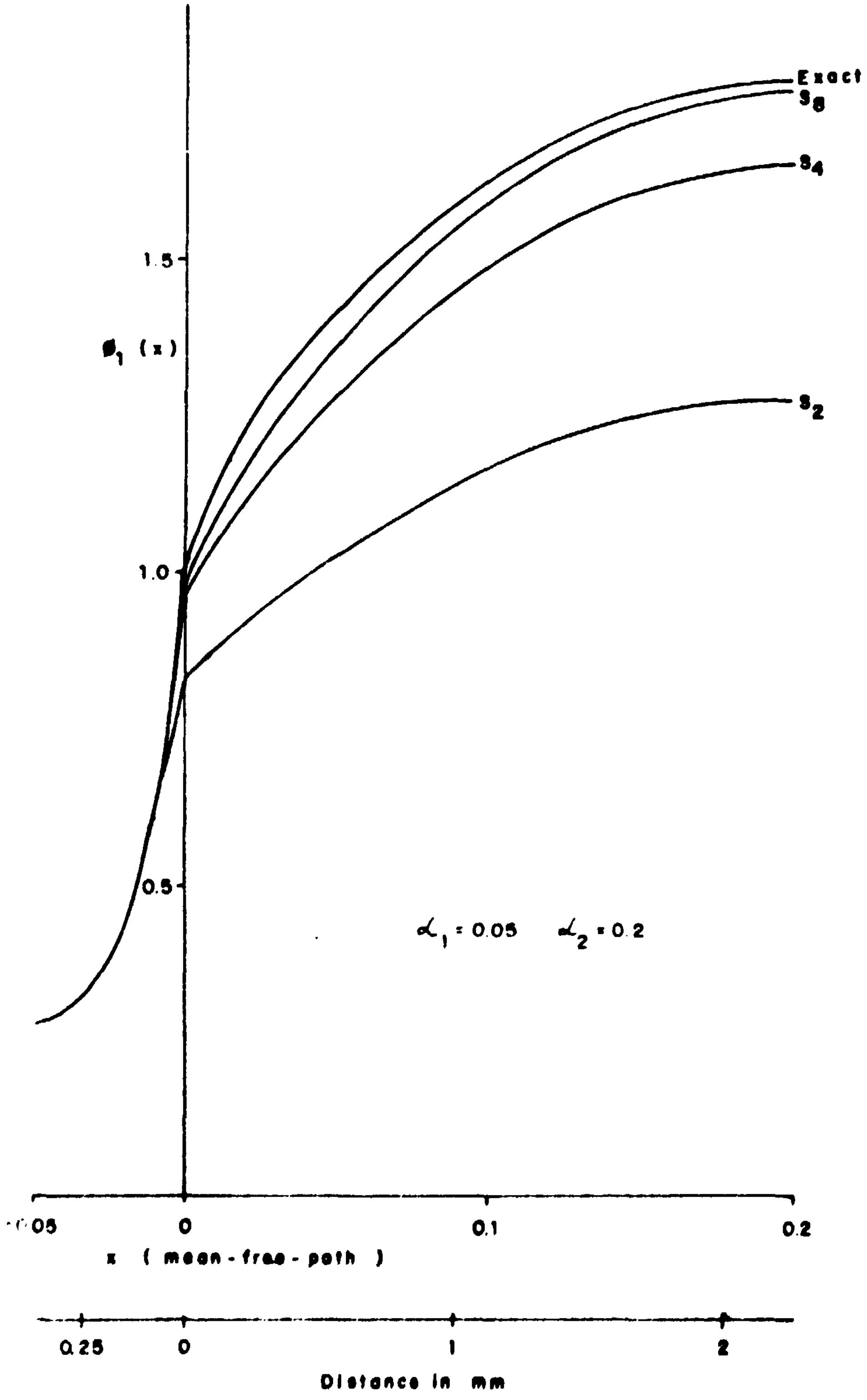


Figure 10 - The thermal group scalar flux for the cell problem



...re 11 A comparison of the exact and S-N results of the thermal group scalar flux for the cell problem.

RESUMO

Três problemas envolvendo dois meios em geometria plana são resolvidos na teoria de transporte de nêutrons no modelo de dois grupos e espalhamento isotrópico: duas placas com um fluxo incidente, o problema da criticalidade para reatores tipo placa refletida e o problema da célula. Cada problema é reduzido a um conjunto de equações integrais regulares para os coeficientes das expansões de Case, que é resolvido iterativamente. São publicados resultados numéricos para todos os problemas.

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