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A POINT SOURCE IN A FINITE SPHERE

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Abstract—Exact analysis and the F_N method are used to compute the radiation field due to a point source of radiation located at the center of a finite sphere.

1. INTRODUCTION

In a recent series of papers[1-4] the F_N method basic to radiative transfer and neutron-transport theory was introduced and used to solve concisely and accurately numerous basic problems. To date, however, the F_N method has been used primarily to compute surface quantities such as the albedo and the transmission factor. Here we wish to apply the method in order to establish the mean intensity J, as a function of the optical variable, interior to a finite sphere.

We consider the equation of transfer for isotropic scattering in the monochromatic form

$$\mu\frac{\partial}{\partial r}I(r,\mu)+\frac{(1-\mu^2)}{r}\frac{\partial}{\partial\mu}I(r,\mu)+I(r,\mu)=\frac{\omega}{2}\int_{-1}^{1}I(r,\mu)\,\mathrm{d}\mu+\frac{\delta(r)}{8\pi r^2}\cdot \tag{1}$$

Here, the isotropically emitting source term

$$S(r)=\delta(r)/[8\pi r^2] \tag{2}$$

is normalized so that

$$4\pi\int_{-1}^{1}\int_{0}^{\infty}r^2S(r)\,\mathrm{d}r\,\mathrm{d}\mu=1. \tag{3}$$

We thus seek a solution of Eq. (1) for $r\in(0,R]$, R is the radius of the sphere, subject to the condition of no entering radiation:

$$I(R,-\mu)=0,\ \mu>0. \tag{4}$$

The solution to this problem was formulated by Erdmann and Siewert[5] some years ago and, recently, the method of elementary solutions[6] was used to evaluate the solution numerically.[7] We thus have available accurate results with which to compare the solution obtained here by the F_N method.

As noted by Davison,[8] Eq. (1), along with the boundary condition given by Eq. (4), can be converted to the equivalent integral form

$$\rho(r)=\frac{1}{r}\int_{-R}^{R}r'E_1(|r-r'|)\left[\frac{\omega}{2}\rho(r')+S(r')\right]\mathrm{d}r',\ r\in[-R,R]. \tag{5}$$

Here

$$\rho(r)=2J(r)=\int_{-1}^{1}I(r,\mu)\,\mathrm{d}\mu \tag{6}$$

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and we have extended the range of r to $r \in [-R, R]$. We have also defined $\rho(-r) = \rho(r)$ and $S(-r) = S(r)$. The first exponential integral function is denoted by $E_1(x)$. Of course, once $\rho(r)$ is known, the complete radiation intensity $I(r,\mu)$ can readily be obtained from Eq. (1).

2. BASIC ANALYSIS

Following the paper of Wu and Siewert,[9] we find that we can express $\rho(r)$ as

$$\rho(r) = \frac{1}{r}\int_{-1}^{1} \Phi(r,\mu)\,\mathrm{d}\mu, \tag{7}$$

where $\Phi(r,\mu)$ is a solution of the pseudo-slab problem defined by

$$\mu\frac{\partial}{\partial r}\Phi(r,\mu) + \Phi(r,\mu) = \frac{\omega}{2}\int_{-1}^{1} \Phi(r,\mu)\,\mathrm{d}\mu, \qquad r \neq 0, \tag{8}$$

with the conditions

$$\Phi(-r,-\mu) = -\Phi(r,\mu), \tag{9a}$$

$$4\pi\mu^2[\Phi(0^+,\mu) - \Phi(0^-,\mu)] = 1,\ \mu \in (-1,1), \tag{9b}$$

and

$$\Phi(R-\mu) = 0,\ \mu > 0. \tag{9c}$$

To solve the pseudo-problem defined by Eqs. (8) and (9), we first write

$$\Phi(r,\mu) = A(\nu_0)\varphi(\nu_0,\mu)\,\mathrm{e}^{-r/\nu_0} + \int_0^1 A(\nu)\varphi(\nu,\mu)\,\mathrm{e}^{-r/\nu}\,\mathrm{d}\nu + \Phi_c(r,\mu),\ r > 0, \tag{10a}$$

and

$$\Phi(r,\mu) = -A(\nu_0)\varphi(-\nu_0,\mu)\,\mathrm{e}^{r/\nu_0} - \int_0^1 A(\nu)\varphi(-\nu,\mu)\,\mathrm{e}^{r/\nu}\,\mathrm{d}\nu + \Phi_c(r,\mu),\ r < 0, \tag{10b}$$

where

$$\Phi_c(r,\mu) = B(\nu_0)[\varphi(\nu_0,\mu)\,\mathrm{e}^{-r/\nu_0} - \varphi(-\nu_0,\mu)\,\mathrm{e}^{r/\nu_0}] + \int_0^1 B(\nu)[\varphi(\nu,\mu)\,\mathrm{e}^{-r/\nu} - \varphi(-\nu,\mu)\,\mathrm{e}^{r/\nu}]\,\mathrm{d}\nu \tag{11}$$

is a correction term to account for the fact that we are considering a finite sphere. Here, Case's[6] elementary solutions are

$$\varphi(\nu_0,\mu) = \omega\nu_0/[2(\nu_0 - \mu)] \tag{12}$$

where

$$\Lambda(\nu_0) = 1 + \frac{\omega\nu_0}{2}\int_{-1}^{1} \frac{\mathrm{d}\mu}{\mu - \nu_0} = 0 \tag{13}$$

and

$$\varphi(\nu,\mu) = \frac{\omega\nu}{2}\,Pv\left(\frac{1}{\nu - \mu}\right) + [1 - \omega\nu\tanh^{-1}\nu]\delta(\nu - \mu). \tag{14}$$

If we substitute Eqs. (10) into the jump condition, Eq. (9b), we can use Case's full-range theory[6] to find

$$A(\nu_0) = 1/[4\pi\nu_0 N(\nu_0)] \tag{15a}$$

and

$$A(\nu) = 1/[4\pi\nu N(\nu)], \tag{15b}$$

where

$$N(\nu_0) = \frac{\omega}{2}\nu_0^3\left(\frac{\omega}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right) \tag{16a}$$

and

$$N(\nu) = \nu\left([1 - \omega\nu \tanh^{-1}\nu]^2 + \frac{1}{4}\omega^2\nu^2\pi^2\right). \tag{16b}$$

Considering Eqs. (7) and (10), we see that we can now write

$$\rho(r) = \rho_\infty(r) + \rho_c(r),$$

where

$$\rho_\infty(r) = \frac{1}{4\pi r}\left[\frac{1}{\nu_0 N(\nu_0)} e^{-r/\nu_0} + \int_0^1 \frac{1}{\nu N(\nu)} e^{-r/\nu} \, d\nu\right] \tag{17}$$

and

$$\rho_c(r) = \frac{1}{r}\int_{-1}^{1} \Phi_c(r,\mu) \, d\mu. \tag{18}$$

Since $\Phi_c(r,\mu)$ satisfies Eq. (8) subject to

$$\Phi_c(-r, -\mu) = -\Phi_c(r,\mu) \tag{19a}$$

and

$$\Phi_c(-R,\mu) = K(\mu), \quad \mu > 0, \tag{19b}$$

with

$$K(\mu) = A(\nu_0)\varphi(-\nu_0,\mu) e^{-R/\nu_0} + \int_0^1 A(\nu)\varphi(-\nu,\mu) e^{-R/\nu} d\nu, \tag{20}$$

we can use the F_N method to compute this correction term and thus to complete the desired solution. Since the details of the F_N method are described in Refs. 1–4, we give here only a brief account of the analysis required for this application. We approximate the exit distribution by writing

$$\Phi_c(-R, -\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \qquad \mu > 0, \tag{21}$$

where the constants a_α, $\alpha = 0, 1, \ldots, N$, are found from the following system of linear algebraic equations:

$$\sum_{\alpha=0}^{N} a_\alpha\left[B_\alpha(\xi_\beta) - e^{-2R/\xi_\beta} A_\alpha(\xi_\beta)\right] = \Delta(\xi_\beta) \tag{22}$$

with

$$\Delta(\xi_\beta) = \frac{2}{\omega\xi_\beta}\left[\int_0^1 \varphi(-\xi_\beta,\mu)K(\mu)\mu\,\mathrm{d}\mu - \mathrm{e}^{-2R/\xi_\beta}\int_0^1 \varphi(\xi_\beta,\mu)K(\mu)\mu\,\mathrm{d}\mu\right]. \tag{23}$$

For the F_0-approximation, we use only $\xi_0 = \nu_0$ in Eq. (22); for the F_1-approximation, we use $\xi_0 = \nu_0$ and $\xi_1 = 0$; for the F_2-approximation, we use $\xi_0 = \nu_0$, $\xi_1 = 0$ and $\xi_2 = 1$ and, for higher-order approximations, we use additional values of ξ_β spaced equally in the interval [0, 1]. We note that the known r.h.s. of Eq. (22) can be expressed as

$$\Delta(\xi) = \Delta_1(\xi) - \mathrm{e}^{-2R/\xi}\Delta_2(\xi) \tag{24}$$

with

$$\Delta_1(\xi) = \frac{\omega}{8\pi}\left\{\frac{\mathrm{e}^{-R/\nu_0}}{N(\nu_0)}[A_0(\xi) - A_0(\nu_0)]\frac{1}{\nu_0 - \xi} + \int_0^1 \frac{\mathrm{e}^{-R/\nu}}{N(\nu)}[A_0(\xi) - A_0(\nu)]\frac{\mathrm{d}\nu}{\nu - \xi}\right\} \tag{25a}$$

and

$$\Delta_2(\xi) = \frac{\omega}{8\pi}\left\{\frac{\mathrm{e}^{-R/\nu_0}}{N(\nu_0)}[B_0(\xi) + A_0(\nu_0)]\frac{1}{\nu_0 + \xi} + \int_0^1 \frac{\mathrm{e}^{-R/\nu}}{N(\nu)}[B_0(\xi) + A_0(\nu)]\frac{\mathrm{d}\nu}{\nu + \xi}\right\}. \tag{25b}$$

The functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ appearing in Eqs. (22) and (25) are given by

$$A_0(\xi) = 1 - \xi\log\left(1 + \frac{1}{\xi}\right), \tag{26a}$$

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha+1},\ \alpha \geq 1, \tag{26b}$$

$$B_0(\xi) = \frac{2}{\omega} - 1 - \xi\log\left(1 + \frac{1}{\xi}\right), \tag{27a}$$

and

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \frac{1}{\alpha+1},\ \alpha \geq 1. \tag{27b}$$

Once we have solved the system of linear algebraic equations given by Eqs. (22) to obtain the constants a_α, $\alpha = 0, 1, 2, \ldots, N$, we can use Eq. (11) evaluated at $r = -R$, Eqs. (19b) and (21), and again Case's full-range theory[6] to find the expansion coefficients $B(\nu_0)$ and $B(\nu)$ required in Eq. (11) to complete the solution. We thus find that our final result can be expressed as

$$\rho(r) = \rho_\infty(r) - \frac{1}{4\pi r}\left[E(\nu_0)\,\mathrm{e}^{-R/\nu_0}\sinh(r/\nu_0) + \int_0^1 E(\nu)\,\mathrm{e}^{-R/\nu}\sinh(r/\nu)\,\mathrm{d}\nu\right], \tag{28}$$

where

$$E(\xi) = \frac{4\pi\omega\xi}{N(\xi)}\left[\Delta_2(\xi) - \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi)\right]. \tag{29}$$

3. NUMERICAL RESULTS

After finding ν_0 from Eq. (13), we approximated the integral in Eq. (17) by a Gaussian quadrature scheme in order to evaluate $\rho_\infty(r)$. We also evaluated Eqs. (25) in a similar manner and solved Eqs. (22) for the constants a_α required in Eq. (29), and thus we were able for various

Table 1. Numerical results for $\omega = 0.3$ and $R = 1$.

r	$4\pi r^2\rho_\infty(r)$ Exact	F_3	F_4	$4\pi r^2\rho(r)$ F_5	F_6	Exact
0.0	1	1	1	1	1	1
0.1	0.96440	0.96418	0.96418	0.96418	0.96418	0.96417
0.2	0.91965	0.91873	0.91872	0.91872	0.91872	0.91871
0.3	0.87101	0.86888	0.86887	0.86886	0.86886	0.86885
0.4	0.82091	0.81697	0.81695	0.81694	0.81693	0.81692
0.5	0.77076	0.76429	0.76426	0.76424	0.76423	0.76421
0.6	0.72147	0.71155	0.71151	0.71147	0.71146	0.71143
0.7	0.67364	0.65900	0.65894	0.65889	0.65888	0.65883
0.8	0.62765	0.60636	0.60628	0.60626	0.60624	0.60620
0.9	0.58374	0.55282	0.55270	0.55258	0.55256	0.55247
1.0	0.54204	0.49206	0.49192	0.49172	0.49169	0.49154

Table 2. Numerical results for $\omega = 0.9$ and $R = 1$

r	$4\pi r^2\rho_\infty(r)$ Exact	F_3	F_4	$4\pi r^2\rho(r)$ F_5	F_6	Exact
0.0	1	1	1	1	1	1
0.1	1.1208	1.1147	1.1147	1.1147	1.1147	1.1147
0.2	1.2342	1.2099	1.2099	1.2099	1.2099	1.2099
0.3	1.3383	1.2832	1.2832	1.2832	1.2832	1.2832
0.4	1.4326	1.3336	1.3336	1.3336	1.3336	1.3336
0.5	1.5169	1.3603	1.3602	1.3602	1.3602	1.3602
0.6	1.5914	1.3621	1.3620	1.3620	1.3620	1.3620
0.7	1.6567	1.3374	1.3373	1.3373	1.3372	1.3372
0.8	1.7129	1.2830	1.2829	1.2829	1.2828	1.2828
0.9	1.7607	1.1917	1.1916	1.1915	1.1915	1.1914
1.0	1.8006	1.0263	1.0260	1.0259	1.0259	1.0258

orders of the F_N approximation to evaluate Eq. (28) and establish $\rho(r)$. In Tables 1 and 2, we list our results for the F_N calculation of $\rho(r)$ along with $\rho_\infty(r)$ and the "exact" results taken from Ref. 7.

It is clear from Tables 1 and 2 that the F_N method yields better results for $\omega = 0.9$ than for $\omega = 0.3$. This result is obtained because the approximation given by Eq. (21) has greater validity when there is less absorption. We note also that the F_N results in Tables 1 and 2 improve as r is diminished from $r = R$. It thus appears that the errors introduced at the boundary, by the approximation given by Eq. (21), are reduced as the interior of the sphere is approached. In summary, we consider the F_N results given in Tables 1 and 2 to be remarkably good.

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