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# A SPHERICAL-HARMONICS SOLUTION FOR RADIATIVE-TRANSFER PROBLEMS WITH REFLECTING BOUNDARIES AND INTERNAL SOURCES

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**Abstract**—The spherical-harmonics method, including some recent improvements, is used to establish the complete solution for a general problem concerning radiative transfer in a plane-parallel medium. An $L$-th order Legendre expansion of the phase function is allowed, internal sources and reflecting boundaries are included in the model, and since a non-normally incident beam is impinging on one surface, all components in a Fourier decomposition of the intensity are required in the solution. Numerical results for two test problems are reported. © 1998 Elsevier Science Ltd. All rights reserved

## 1. INTRODUCTION

Some years ago, the $F_N$ method was used[1] to solve a general radiative-transfer problem that was based on a model that included internal sources, reflecting and emitting boundaries and a beam incident on one surface. Here we use some recent improvements in the spherical-harmonics method to solve this class of problems.

We let $I(\tau, \mu, \phi)$ denote the intensity (radiance) of the radiation field and utilize the equation of transfer[2] for a plane-parallel medium for our model. We write

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \phi) + I(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos\Theta) I(\tau, \mu', \phi') \, \mathrm{d}\phi' \, \mathrm{d}\mu' + S_0(\tau) \tag{1}$$

where $\tau \in (0, \tau_0)$ is the optical variable and $\varpi \in [0, 1]$ is the albedo for single scattering. In addition, $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$ are, respectively, the cosine of the polar angle (as measured from the *positive* $\tau$ axis) and the azimuthal angle which describe the direction of propagation of the radiation. Here we use $S_0(\tau)$ to represent an internal source of radiation. We note also that the phase function $p(\cos\Theta)$ is represented by a finite Legendre polynomial expansion in terms of the cosine of the scattering angle $\Theta$, *viz.*

$$p(\cos\Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos\Theta) \tag{2}$$

where the coefficients are such that $\beta_0 = 1$ and $|\beta_l| < 2l + 1$ for $0 < l \le L$. In regard to the physical parameters of this problem, we do not, for the moment, consider the special case of a conservative medium ($\varpi = 1$). The modifications to our general development for this special case are discussed in Section 6 of this paper.

We assume that the medium is illuminated uniformly at $\tau = 0$ by a beam with a direction specified by $(\mu_0, \phi_0)$ and that there is also an incident distribution, described by $T(\mu)$, of radiation on the surface located at $\tau = \tau_0$. We therefore seek a solution to Eq. (1) that satisfies the boundary conditions

$$I(0, \mu, \phi) = \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0) + \rho_1^s I(0, -\mu, \phi) + \frac{\rho_1^d}{\pi}\int_0^{2\pi}\int_0^1 I(0, -\mu', \phi')\mu' \,d\mu' \,d\phi' \tag{3a}$$

and

$$I(\tau_0, -\mu, \phi) = T(\mu) + \rho_2^s I(\tau_0, \mu, \phi) + \frac{\rho_2^d}{\pi}\int_0^{2\pi}\int_0^1 I(\tau_0, \mu', \phi')\mu' \,d\mu' \,d\phi' \tag{3b}$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. Here we use the symbols $\rho_\alpha^s$ and $\rho_\alpha^d$, for $\alpha = 1$ and 2, to characterize specular and diffuse reflection.

## 2. THE SINGULAR COMPONENT OF THE INTENSITY

Since the incident beam for the considered problem is represented by delta *functions*, the resulting intensity will also have a component containing generalized functions, and so, in order to remove the generalized functions from the computation, we use a decomposition of the form

$$I(\tau, \mu, \phi) = I_*(\tau, \mu, \phi) + \Delta(\tau, \mu, \phi) \tag{4}$$

where $\Delta(\tau, \mu, \phi)$ contains all of the generalized functions in the complete solution. Considering the homogeneous version of Eq. (1) for the case $\varpi = 0$, we write

$$\Delta(\tau, \mu, \phi) = F(\mu, \phi)e^{-\tau/\mu}. \tag{5}$$

We next substitute Eq. (5) into Eqs. (3) for the case without diffuse reflection and with $T(\mu) = 0$ to find

$$F(\mu, \phi) = \pi D^{-1}(\mu)\delta(\mu - \mu_0)\delta(\phi - \phi_0) \tag{6a}$$

and

$$F(-\mu, \phi) = \pi\rho_2^s D^{-1}(\mu)e^{-2\tau_0/\mu}\delta(\mu - \mu_0)\delta(\phi - \phi_0) \tag{6b}$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. Here we have used

$$D(\mu) = 1 - \rho_1^s\rho_2^s e^{-2\tau_0/\mu}. \tag{7}$$

## 3. THE NON-SINGULAR COMPONENT OF THE INTENSITY

We can now substitute Eq. (4) into Eqs. (1) and (3) and deduce that the non-singular component $I_*(\tau, \mu, \phi)$ is defined by

$$\mu\frac{\partial}{\partial\tau}I_*(\tau, \mu, \phi) + I_*(\tau, \mu, \phi) = \frac{\varpi}{4\pi}\int_{-1}^1\int_0^{2\pi} p(\cos\Theta)I_*(\tau, \mu', \phi')\,d\phi'\,d\mu' + S(\tau, \mu, \phi), \tag{8}$$

for $\tau \in (0, \tau_0)$, $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$, and the boundary conditions

$$I_*(0, \mu, \phi) = \rho_1^d K_1 + \rho_1^s I_*(0, -\mu, \phi) + \frac{\rho_1^d}{\pi}\int_0^{2\pi}\int_0^1 I_*(0, -\mu', \phi')\mu' \,d\mu' \,d\phi' \tag{9a}$$

and

$$I_*(\tau_0, -\mu, \phi) = \rho_2^d K_2 + T(\mu) + \rho_2^s I_*(\tau_0, \mu, \phi) + \frac{\rho_2^d}{\pi}\int_0^{2\pi}\int_0^1 I_*(\tau_0, \mu', \phi')\mu' \,d\mu' \,d\phi' \tag{9b}$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. Here, the known inhomogeneous term is

$$S(\tau, \mu, \phi) = S_0(\tau) + \frac{\varpi}{4\pi}\int_{-1}^1\int_0^{2\pi} p(\cos\Theta)\Delta(\tau, \mu', \phi')\,d\phi'\,d\mu'. \tag{10}$$

In addition,

$$K_1 = \mu_0\rho_2^s D^{-1}(\mu_0)e^{-2\tau_0/\mu_0} \tag{11a}$$

and

$$K_2 = \mu_0 D^{-1}(\mu_0)\mathrm{e}^{-\tau_0/\mu_0}. \tag{11b}$$

Before attempting a Fourier decomposition of the non-singular component of the intensity, we make use of the addition theorem[3] for the Legendre polynomials and express the scattering law as

$$p(\cos\Theta) = \sum_{m=0}^{L}(2-\delta_{0,m})\sum_{l=m}^{L}\beta_l P_l^m(\mu')P_l^m(\mu)\cos[m(\phi'-\phi)] \tag{12}$$

where

$$P_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}(1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu) \tag{13}$$

denotes a *normalized* associated Legendre function. Using Eq. (12), we rewrite Eq. (8) as

$$\mu\frac{\partial}{\partial\tau}I_*(\tau,\mu,\phi) + I_*(\tau,\mu,\phi) = \frac{\varpi}{4\pi}\sum_{m=0}^{L}(2-\delta_{0,m})\sum_{l=m}^{L}\beta_l P_l^m(\mu)$$
$$\times\int_{-1}^{1}\int_0^{2\pi}P_l^m(\mu')I_*(\tau,\mu',\phi')\cos[m(\phi'-\phi)]\,\mathrm{d}\phi'\,\mathrm{d}\mu' + S(\tau,\mu,\phi) \tag{14}$$

where now, after noting Eqs. (5), (6), (10) and (12), we write the inhomogeneous term as

$$S(\tau,\mu,\phi) = S_0(\tau) + S_1(\tau,\mu,\phi) \tag{15}$$

with

$$S_1(\tau,\mu,\phi) = \frac{\varpi}{4}D^{-1}(\mu_0)\sum_{m=0}^{L}(2-\delta_{0,m})\sum_{l=m}^{L}\beta_l P_l^m(\mu_0)P_l^m(\mu)$$
$$\times[\mathrm{e}^{-\tau/\mu_0} + (-1)^{l-m}\rho_2^s\mathrm{e}^{-(2\tau_0-\tau)/\mu_0}]\cos[m(\phi-\phi_0)]. \tag{16}$$

Introducing a finite Fourier decomposition, we substitute

$$I_*(\tau,\mu,\phi) = \frac{1}{2}\sum_{m=0}^{L}(2-\delta_{0,m})I_*^m(\tau,\mu)\cos[m(\phi-\phi_0)] \tag{17}$$

into Eq. (14) to find, for $m = 0, 1, \ldots, L$,

$$\mu\frac{\partial}{\partial\tau}I_*^m(\tau,\mu) + I_*^m(\tau,\mu) = \frac{\varpi}{2}\sum_{l=m}^{L}\beta_l P_l^m(\mu)\int_{-1}^{1}P_l^m(\mu')I_*^m(\tau,\mu')\,\mathrm{d}\mu' + Q^m(\tau,\mu) \tag{18}$$

where

$$Q^m(\tau,\mu) = 2S_0(\tau)\delta_{0,m} + \frac{\varpi}{2}D^{-1}(\mu_0)\sum_{l=m}^{L}\beta_l P_l^m(\mu_0)P_l^m(\mu)[\mathrm{e}^{-\tau/\mu_0} + (-1)^{l-m}\rho_2^s\mathrm{e}^{-(2\tau_0-\tau)/\mu_0}]. \tag{19}$$

To establish the boundary conditions subject to which we must solve Eq. (18), we substitute Eq. (17) into Eqs. (9) to find, for $m = 0, 1, \ldots, L$,

$$I_*^m(0,\mu) = 2\rho_1^d K_1\delta_{0,m} + \rho_1^s I_*^m(0,-\mu) + 2\rho_1^d\delta_{0,m}\int_0^1 I_*^m(0,-\mu')\mu'\,\mathrm{d}\mu' \tag{20a}$$

and

$$I_*^m(\tau_0,-\mu) = 2[\rho_2^d K_2 + T(\mu)]\delta_{0,m} + \rho_2^s I_*^m(\tau_0,\mu) + 2\rho_2^d\delta_{0,m}\int_0^1 I_*^m(\tau_0,\mu')\mu'\,\mathrm{d}\mu' \tag{20b}$$

for $\mu \in (0, 1]$. At this point, we are ready to develop our spherical-harmonics solution to the set of problems defined by Eqs. (18)–(20).

## 4. A SPHERICAL-HARMONICS SOLUTION

In order to solve the collection of Fourier-component problems defined by Eqs. (18)–(20) we use a form of the solution to the moments of the homogeneous version of Eq. (18) that was reported in Ref. 4 and a new particular solution that was worked out by Siewert and McCormick in a work[5] on polarization that contains the scalar case considered here as the first component in a Stokes-vector formulation. In view of Refs. 4 and 5, our presentation here is brief. First of all, we note that, for $N$ odd,

$$I_*^m(\tau,\mu) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu) \sum_{j=1}^{J} [A_j \mathrm{e}^{-\tau/\xi_j} + (-1)^{l-m} B_j \mathrm{e}^{-(\tau_0-\tau)/\xi_j}]\, g_l^m(\xi_j) + I_p^m(\tau,\mu), \tag{21}$$

where

$$I_p^m(\tau,\mu) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j} [\mathcal{A}_j(\tau) + (-1)^{l-m} \mathcal{B}_j(\tau)]\, g_l^m(\xi_j) \tag{22}$$

is a particular solution, satisfies the first $N + 1$ moments of Eq. (18), i.e.

$$\int_{-1}^{1} P_{m+\alpha}^m(\mu) \left[ \mu \frac{\partial}{\partial \tau} I_*^m(\tau,\mu) + I_*^m(\tau,\mu) - \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I_*^m(\tau,\mu')\, \mathrm{d}\mu' - Q^m(\tau,\mu) \right] \mathrm{d}\mu = 0 \tag{23}$$

for $\alpha = 0, 1, \ldots, N$. Here $M = m + N$, $J = (N + 1)/2$, and we use the *normalized* Chandrasekhar polynomials,[2,6] with the starting value

$$g_m^m(\xi) = (2m-1)!![(2m)!]^{-1/2}, \tag{24}$$

that satisfy, for $l \geqslant m$,

$$h_l \xi g_l^m(\xi) = [(l+1)^2 - m^2]^{1/2} g_{l+1}^m(\xi) + (l^2 - m^2)^{1/2} g_{l-1}^m(\xi) \tag{25}$$

where

$$h_l = 2l + 1 - \varpi \beta_l, \quad \text{for} \quad 0 \le l \le L, \tag{26a}$$

and

$$h_l = 2l + 1, \quad \text{for} \quad l > L. \tag{26b}$$

In addition, the eigenvalues $\{\xi_j\}$ are the positive zeros of $g_{M+1}^m(\xi)$, and the constants $\{A_j\}$ and $\{B_j\}$ are to be determined from the boundary conditions. In regard to the particular solution given by Eq. (22), we note that the constants $\{C_j\}$ are given by

$$C_j = 2 \left( \sum_{l=m}^{M} h_l\, [g_l^m(\xi_j)]^2 \right)^{-1} \tag{27}$$

and that

$$\mathcal{A}_j(\tau) = \int_0^{\tau} \left\{ \int_{-1}^{1} X_j(\mu) Q^m(x,\mu)\, \mathrm{d}\mu \right\} \mathrm{e}^{-(\tau-x)/\xi_j}\, \mathrm{d}x \tag{28a}$$

and

$$\mathcal{B}_j(\tau) = \int_{\tau}^{\tau_0} \left\{ \int_{-1}^{1} Y_j(\mu) Q^m(x,\mu)\, \mathrm{d}\mu \right\} \mathrm{e}^{-(x-\tau)/\xi_j}\, \mathrm{d}x \tag{28b}$$

where

$$X_j(\mu) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu) g_l^m(\xi_j) \tag{29a}$$

and

$$Y_j(\mu) = \sum_{l=m}^{M} \frac{2l+1}{2} (-1)^{l-m} P_l^m(\mu) g_l^m(\xi_j). \tag{29b}$$

As we wish to use the Mark conditions[7] to find the unknown constants $\{A_j\}$ and $\{B_j\}$ in Eq. (21), we first let $\{\mu_i\}$ denote the $J$ positive zeros of $P_{M+1}^m(\mu)$. Then we substitute Eq. (21) into Eqs. (20) evaluated at $\mu = \mu_i$, for $i = 1, 2, \ldots, J$, to find

$$\sum_{j=1}^{J} [X_j(\mu_i) - \rho_1^s Y_j(\mu_i) - 2\rho_1^d \delta_{0,m} \bar{Y}_j] A_j + \sum_{j=1}^{J} \mathrm{e}^{-\tau_0/\xi_j} [Y_j(\mu_i) - \rho_1^s X_j(\mu_i) - 2\rho_1^d \delta_{0,m} \bar{X}_j] B_j = R_1(\mu_i) \tag{30a}$$

and

$$\sum_{j=1}^{J} [X_j(\mu_i) - \rho_2^s Y_j(\mu_i) - 2\rho_2^d \delta_{0,m} \bar{Y}_j] B_j + \sum_{j=1}^{J} \mathrm{e}^{-\tau_0/\xi_j} [Y_j(\mu_i) - \rho_2^s X_j(\mu_i) - 2\rho_2^d \delta_{0,m} \bar{X}_j] A_j = R_2(\mu_i) \tag{30b}$$

where

$$\bar{X}_j = \int_0^1 X_j(\mu)\mu \, \mathrm{d}\mu \qquad \text{and} \qquad \bar{Y}_j = \int_0^1 Y_j(\mu)\mu \, \mathrm{d}\mu. \tag{31a and b}$$

In addition, the known right-hand sides of Eqs. (30) are defined as

$$R_1(\mu) = 2\rho_1^d K_1 \delta_{0,m} - I_p^m(0, \mu) + \rho_1^s I_p^m(0, -\mu) + 2\rho_1^d \delta_{0,m} \int_0^1 I_p^m(0, -\mu')\mu' \, \mathrm{d}\mu' \tag{32a}$$

and

$$R_2(\mu) = 2[\rho_2^d K_2 + T(\mu)]\delta_{0,m} - I_p^m(\tau_0, -\mu) + \rho_2^s I_p^m(\tau_0, \mu) + 2\rho_2^d \delta_{0,m} \int_0^1 I_p^m(\tau_0, \mu')\mu' \, \mathrm{d}\mu' \tag{32b}$$

or, more explicitly,

$$R_1(\mu_i) = 2\rho_1^d K_1 \delta_{0,m} + \sum_{j=1}^{J} \frac{C_j}{\xi_j} \mathcal{B}_j(0)[-Y_j(\mu_i) + \rho_1^s X_j(\mu_i) + 2\rho_1^d \delta_{0,m} \bar{X}_j] \tag{33a}$$

and

$$R_2(\mu_i) = 2[\rho_2^d K_2 + T(\mu_i)]\delta_{0,m} + \sum_{j=1}^{J} \frac{C_j}{\xi_j} \mathcal{A}_j(\tau_0)[-Y_j(\mu_i) + \rho_2^s X_j(\mu_i) + 2\rho_2^d \delta_{0,m} \bar{X}_j]. \tag{33b}$$

It is clear that to complete the expression given by Eq. (21) we have to solve the system of linear algebraic equations defined by Eqs. (30). Considering that we solved that system of equations to find the constants $\{A_j\}$ and $\{B_j\}$ that are required in Eq. (21), we now go on to establish our final representation for the $I_*^m(\tau,\mu)$ that are required in Eq. (17). As mentioned previously, we use Eq. (21) only to compute the moments of $I_*^m(\tau,\mu)$, and so, following and generalizing other works,[8–11] we go back to Eq. (18) and solve for $I_*^m(\tau,\mu)$ with the assumption that we know the right-hand side of that equation. We thus write

$$I_*^m(\tau, \mu) = I_*^m(0, \mu)\mathrm{e}^{-\tau/\mu} + M(\tau, \mu) \tag{34a}$$

and

$$I_*^m(\tau, -\mu) = I_*^m(\tau_0, -\mu)\mathrm{e}^{-(\tau_0-\tau)/\mu} + M(\tau, -\mu) \tag{34b}$$

for $\mu \in (0, 1]$. Here

$$M(\tau, \mu) = \frac{1}{\mu}\int_0^\tau B(x, \mu)\mathrm{e}^{-(\tau-x)/\mu}\,\mathrm{d}x \tag{35a}$$

and

$$M(\tau, -\mu) = \frac{1}{\mu}\int_\tau^{\tau_0} B(x, -\mu)\mathrm{e}^{-(x-\tau)/\mu}\,\mathrm{d}x \tag{35b}$$

where, for $\mu \in (0, 1]$,

$$B(x, \pm\mu) = \frac{\varpi}{2}\sum_{l=m}^{L}\beta_l P_l^m(\pm\mu)\int_{-1}^{1} P_l^m(\mu')I_*^m(x, \mu')\,\mathrm{d}\mu' + Q^m(x, \pm\mu). \tag{36}$$

In order to complete Eqs. (34) we must find the surface quantities $I_*^m(0, \mu)$ and $I_*^m(\tau_0, -\mu)$ for $\mu \in (0,1]$. Thus we consider the system of equations obtained from Eq. (34a) evaluated at $\tau = \tau_0$, Eq. (34b) evaluated at $\tau = 0$ and Eqs. (20), *viz*.

$$I_*^m(\tau_0, \mu) = I_*^m(0, \mu)\mathrm{e}^{-\tau_0/\mu} + M(\tau_0, \mu), \tag{37a}$$

$$I_*^m(0, -\mu) = I_*^m(\tau_0, -\mu)\mathrm{e}^{-\tau_0/\mu} + M(0, -\mu), \tag{37b}$$

$$I_*^m(0, \mu) = \rho_1^s I_*^m(0, -\mu) + 2\rho_1^d(K_1 + J_1)\delta_{0,m} \tag{37c}$$

and

$$I_*^m(\tau_0, -\mu) = \rho_2^s I_*^m(\tau_0, \mu) + 2[\rho_2^d(K_2 + J_2) + T(\mu)]\delta_{0,m} \tag{37d}$$

for $\mu \in (0, 1]$. Here we use

$$J_1 = \int_0^1 I_*^0(0, -\mu)\mu\,\mathrm{d}\mu \tag{38a}$$

and

$$J_2 = \int_0^1 I_*^0(\tau_0, \mu)\mu\,\mathrm{d}\mu. \tag{38b}$$

We can solve Eqs. (37) to find, for $\mu \in (0, 1]$,

$$I_*^m(0, \mu) = D^{-1}(\mu)\big\{2\rho_1^d L_1\delta_{0,m} + \rho_1^s M(0, -\mu) + \rho_1^s \mathrm{e}^{-\tau_0/\mu}\big[2[\rho_2^d L_2 + T(\mu)]\delta_{0,m} + \rho_2^s M(\tau_0, \mu)\big]\big\} \tag{39a}$$

and

$$I_*^m(\tau_0, -\mu) = D^{-1}(\mu)\big\{2[\rho_2^d L_2 + T(\mu)]\delta_{0,m} + \rho_2^s M(\tau_0, \mu) + \rho_2^s \mathrm{e}^{-\tau_0/\mu}[2\rho_1^d L_1\delta_{0,m} + \rho_1^s M(0, -\mu)]\big\} \tag{39b}$$

where $L_1 = J_1 + K_1$ and $L_2 = J_2 + K_2$. We note that $J_1$ and $J_2$ are still unknown, but we can integrate Eqs. (21) and (22) to obtain

$$J_1 = \sum_{l=0}^{N}\frac{2l+1}{2}\Delta_l\sum_{j=1}^{J}[(-1)^l A_j + B_j\mathrm{e}^{-\tau_0/\xi_j} + \frac{C_j}{\xi_j}\mathcal{B}_j(0)]\,g_l^0(\xi_j) \tag{40a}$$

and

$$J_2 = \sum_{l=0}^{N}\frac{2l+1}{2}\Delta_l\sum_{j=1}^{J}[(-1)^l B_j + A_j\mathrm{e}^{-\tau_0/\xi_j} + \frac{C_j}{\xi_j}\mathcal{A}_j(\tau_0)]\,g_l^0(\xi_j) \tag{40b}$$

where

$$\Delta_l = \int_0^1 P_l(\mu)\mu \, d\mu. \tag{41}$$

At this point, we note that once we find explicit expressions for the quantities $M(\tau, \mu)$ and $M(\tau, -\mu)$ that appear in Eqs. (34), including the specific values $M(\tau_0, \mu)$ and $M(0, -\mu)$ required in Eqs. (39), our final results for $I_*^m(\tau, \mu)$ and $I_*^m(\tau, -\mu)$ are completely determined. To this end, we can substitute Eqs. (21) and (22) into Eq. (36) and perform the required integrations, to find that Eqs. (35) yield, for $\tau \in [0, \tau_0]$ and $\mu \in (0, 1]$,

$$M(\tau, \mu) = \Gamma^m(\tau, \mu) + \frac{\varpi}{2}[\Upsilon^m(\tau, \mu) + \Xi^m(\tau, \mu)] \tag{42a}$$

and

$$M(\tau, -\mu) = \Gamma^m(\tau, -\mu) + \frac{\varpi}{2}[\Upsilon^m(\tau, -\mu) + \Xi^m(\tau, -\mu)]. \tag{42b}$$

Here we let $K = \min\{M, L\}$ so that we can write

$$\Upsilon^m(\tau, \mu) = \sum_{l=m}^{K} \beta_l P_l^m(\mu) \sum_{j=1}^{J} \xi_j [A_j C(\tau : \mu, \xi_j) + (-1)^{l-m} B_j e^{-(\tau_0-\tau)/\xi_j} S(\tau : \mu, \xi_j)] g_l^m(\xi_j) \tag{43a}$$

and

$$\Upsilon^m(\tau, -\mu) = \sum_{l=m}^{K} \beta_l P_l^m(\mu) \sum_{j=1}^{J} \xi_j [(-1)^{l-m} A_j e^{-\tau/\xi_j} S(\tau_0 - \tau : \mu, \xi_j) + B_j C(\tau_0 - \tau : \mu, \xi_j)] g_l^m(\xi_j) \tag{43b}$$

where

$$C(\tau : \mu, \xi) = \frac{e^{-\tau/\mu} - e^{-\tau/\xi}}{\mu - \xi} \tag{44a}$$

and

$$S(\tau : \mu, \xi) = \frac{1 - e^{-\tau/\mu} e^{-\tau/\xi}}{\mu + \xi}. \tag{44b}$$

In addition, if we let

$$U_j^m(z) = \int_{-1}^{1} X_j(\mu) Q^m(z, \mu) \, d\mu \tag{45a}$$

and

$$V_j^m(z) = \int_{-1}^{1} Y_j(\mu) Q^m(z, \mu) \, d\mu, \tag{45b}$$

then we can write

$$\Xi^m(\tau, \mu) = \sum_{l=m}^{K} \beta_l P_l^m(\mu) \sum_{j=1}^{J} C_j \Big\{ (-1)^{l-m} S(\tau : \mu, \xi_j) \mathcal{B}_j(\tau) + \int_0^{\tau} [U_j^m(z) C(\tau - z : \mu, \xi_j) + (-1)^{l-m} V_j^m(z) e^{-(\tau-z)/\mu} S(z : \mu, \xi_j)] \, dz \Big\} g_l^m(\xi_j) \tag{46a}$$

and

$$\Xi^m(\tau, -\mu) = \sum_{l=m}^{K} \beta_l P_l^m(\mu) \sum_{j=1}^{J} C_j \Big\{ (-1)^{l-m} S(\tau_0 - \tau : \mu, \xi_j) \mathcal{A}_j(\tau) + \int_{\tau}^{\tau_0} [V_j^m(z) C(z - \tau : \mu, \xi_j) + (-1)^{l-m} U_j^m(z) e^{-(z-\tau)/\mu} S(\tau_0 - z : \mu, \xi_j)] \, dz \Big\} g_l^m(\xi_j). \tag{46b}$$

To complete Eqs. (42), we first use Eq. (19) in

$$\Gamma^m(\tau, \mu) = \frac{1}{\mu}\int_0^{\tau} Q^m(x, \mu)\mathrm{e}^{-(\tau-x)/\mu}\,\mathrm{d}x \tag{47a}$$

and

$$\Gamma^m(\tau, -\mu) = \frac{1}{\mu}\int_{\tau}^{\tau_0} Q^m(x, -\mu)\mathrm{e}^{-(x-\tau)/\mu}\,\mathrm{d}x \tag{47b}$$

to find

$$\Gamma^m(\tau, \mu) = \delta_{0,m}\frac{2}{\mu}\int_0^{\tau} S_0(x)\mathrm{e}^{-(\tau-x)/\mu}\,\mathrm{d}x + \Lambda^m(\tau, \mu) \tag{48a}$$

and

$$\Gamma^m(\tau, -\mu) = \delta_{0,m}\frac{2}{\mu}\int_{\tau}^{\tau_0} S_0(x)\mathrm{e}^{-(x-\tau)/\mu}\,\mathrm{d}x + \Lambda^m(\tau, -\mu) \tag{48b}$$

where

$$\Lambda^m(\tau, \mu) = \frac{\varpi}{2}\mu_0 D^{-1}(\mu_0)\sum_{l=m}^{L}\beta_l P_l^m(\mu)P_l^m(\mu_0)[C(\tau : \mu, \mu_0) + (-1)^{l-m}\rho_2^s\mathrm{e}^{-(2\tau_0-\tau)/\mu_0}S(\tau : \mu, \mu_0)] \tag{49a}$$

and

$$\begin{aligned}\Lambda^m(\tau, -\mu) = {}&\frac{\varpi}{2}\mu_0 D^{-1}(\mu_0)\sum_{l=m}^{L}\beta_l P_l^m(\mu)P_l^m(\mu_0)\\ &\times[\rho_2^s\mathrm{e}^{-\tau_0/\mu_0}C(\tau_0-\tau : \mu, \mu_0) + (-1)^{l-m}\mathrm{e}^{-\tau/\mu_0}S(\tau_0-\tau : \mu, \mu_0)].\end{aligned} \tag{49b}$$

Continuing, we substitute Eq. (19) into Eqs. (28) to find

$$\mathcal{A}_j(\tau) = 2\delta_{0,m}\int_0^{\tau} S_0(x)\mathrm{e}^{-(\tau-x)/\xi_j}\,\mathrm{d}x + a_j^m(\tau) \tag{50a}$$

and

$$\mathcal{B}_j(\tau) = 2\delta_{0,m}\int_{\tau}^{\tau_0} S_0(x)\mathrm{e}^{-(x-\tau)/\xi_j}\,\mathrm{d}x + b_j^m(\tau) \tag{50b}$$

where

$$a_j^m(\tau) = \frac{\varpi}{2}D^{-1}(\mu_0)\mu_0\xi_j[E_j(\mu_0)C(\tau : \mu_0, \xi_j) + \rho_2^s F_j(\mu_0)\mathrm{e}^{-(2\tau_0-\tau)/\mu_0}S(\tau : \mu_0, \xi_j)] \tag{51a}$$

and

$$b_j^m(\tau) = \frac{\varpi}{2}D^{-1}(\mu_0)\mu_0\xi_j[F_j(\mu_0)\mathrm{e}^{-\tau/\mu_0}S(\tau_0-\tau : \mu_0, \xi_j) + \rho_2^s E_j(\mu_0)\mathrm{e}^{-\tau_0/\mu_0}C(\tau_0-\tau : \mu_0, \xi_j)]. \tag{51b}$$

Here

$$E_j(\mu) = \sum_{l=m}^{K}\beta_l P_l^m(\mu)g_l^m(\xi_j) \tag{52a}$$

and

$$F_j(\mu) = \sum_{l=m}^{K}\beta_l(-1)^{l-m}P_l^m(\mu)g_l^m(\xi_j). \tag{52b}$$

Finally in order to complete Eqs. (46), we can use Eq. (19) in Eqs. (45), to obtain, after noting

Eqs. (52),

$$U_j^m(z) = 2S_0(z)\delta_{0,m} + \frac{\varpi}{2} D^{-1}(\mu_0)[E_j(\mu_0)e^{-z/\mu_0} + \rho_2^s F_j(\mu_0)e^{-(2\tau_0-z)/\mu_0}] \tag{53a}$$

and

$$V_j^m(z) = 2S_0(z)\delta_{0,m} + \frac{\varpi}{2} D^{-1}(\mu_0)[F_j(\mu_0)e^{-z/\mu_0} + \rho_2^s E_j(\mu_0)e^{-(2\tau_0-z)/\mu_0}]. \tag{53b}$$

As our spherical-harmonics solutions for the Fourier-component problems are completely defined, we now move on to consider some numerical calculations.

## 5. NUMERICAL RESULTS

Since our preferred method of computing the required eigenvalues $\{\xi_j\}$ is discussed, for example, in Ref. 4, and since we have reported in Ref. 6 an accurate and efficient way of computing the Chandrasekhar polynomials $\{g_l^m(\xi_j)\}$, we have here no particularly difficult computations to do. Having solved numerically the (well-conditioned) system of linear algebraic equations given by Eqs. (30), we have simply to evaluate the final expressions to find the non-singular component of the intensity. We note that, in general, we can use a Gauss-quadrature scheme to evaluate integrals involving the assumed given inhomogeneous source term $S_0(\tau)$; however, for the specific test cases considered here, we were able to perform all of the required integrations analytically.

For our first calculation we consider the problem solved in Ref. 1 with the $F_N$ method. Here the scattering law is given by $L = 8$, with $\beta_0=1$, $\beta_1=2.00916$, $\beta_2=1.56339$, $\beta_3=0.67407$, $\beta_4=0.22215$, $\beta_5=0.04725$, $\beta_6=0.00671$, $\beta_7=0.00068$ and $\beta_8=0.00005$. Other parameters used to define this calculation are: $\varpi = 0.9$, $\tau_0=4.0$, $\mu_0=0.5$, and $\rho_1^s=\rho_1^d=\rho_2^s=\rho_2^d=0.25$. In addition, the problem here has $T(\mu) = 0.4$ and

$$S_0(\tau) = S_0 + S_1(\tau/\tau_0) + S_2(\tau/\tau_0)^2 + S_3(\tau/\tau_0)^3 \tag{54}$$

with $S_0=0.01$, $S_1=0.008$, $S_2=0.0064$ and $S_3= -0.0064$. We show in Tables 1–3 the results,

Table 1. First problem: the non-singular component $I_*(\tau, \mu, \phi)$ for $\phi - \phi_0=0$

| $\mu$ | $\tau/\tau_0 = 0.0$ | $\tau/\tau_0 = 0.25$ | $\tau/\tau_0 = 0.5$ | $\tau/\tau_0 = 0.75$ | $\tau/\tau_0 = 1.0$ |
|---|---|---|---|---|---|
| −1.0 | 3.9191(−1) | 4.0746(−1) | 4.5654(−1) | 5.2409(−1) | 5.8833(−1) |
| −0.9 | 4.4386(−1) | 4.1415(−1) | 4.4966(−1) | 5.1776(−1) | 5.9433(−1) |
| −0.8 | 4.8707(−1) | 4.1804(−1) | 4.4055(−1) | 5.0770(−1) | 5.9577(−1) |
| −0.7 | 5.3908(−1) | 4.2469(−1) | 4.3159(−1) | 4.9576(−1) | 5.9620(−1) |
| −0.6 | 6.0318(−1) | 4.3531(−1) | 4.2349(−1) | 4.8220(−1) | 5.9640(−1) |
| −0.5 | 6.8200(−1) | 4.5092(−1) | 4.1695(−1) | 4.6716(−1) | 5.9690(−1) |
| −0.4 | 7.7816(−1) | 4.7253(−1) | 4.1279(−1) | 4.5098(−1) | 5.9811(−1) |
| −0.3 | 8.9429(−1) | 5.0121(−1) | 4.1185(−1) | 4.3460(−1) | 6.0022(−1) |
| −0.2 | 1.0326 | 5.3817(−1) | 4.1476(−1) | 4.2016(−1) | 6.0333(−1) |
| −0.1 | 1.1925 | 5.8508(−1) | 4.2176(−1) | 4.1024(−1) | 6.0763(−1) |
| −0.0 | 1.3423 | 6.4441(−1) | 4.3328(−1) | 4.0430(−1) | 6.1460(−1) |
| 0.0 | 4.4240(−1) | 6.4441(−1) | 4.3328(−1) | 4.0430(−1) | 4.7496(−1) |
| 0.1 | 4.0494(−1) | 7.1969(−1) | 4.5028(−1) | 4.0152(−1) | 4.4708(−1) |
| 0.2 | 3.6497(−1) | 8.0384(−1) | 4.7436(−1) | 4.0211(−1) | 4.2988(−1) |
| 0.3 | 3.3039(−1) | 8.6207(−1) | 5.0579(−1) | 4.0670(−1) | 4.1744(−1) |
| 0.4 | 3.0136(−1) | 8.8120(−1) | 5.3889(−1) | 4.1553(−1) | 4.0899(−1) |
| 0.5 | 2.7732(−1) | 8.6659(−1) | 5.6552(−1) | 4.2714(−1) | 4.0417(−1) |
| 0.6 | 2.5761(−1) | 8.2536(−1) | 5.7968(−1) | 4.3848(−1) | 4.0215(−1) |
| 0.7 | 2.4159(−1) | 7.6238(−1) | 5.7776(−1) | 4.4597(−1) | 4.0137(−1) |
| 0.8 | 2.2858(−1) | 6.7970(−1) | 5.5720(−1) | 4.4595(−1) | 3.9964(−1) |
| 0.9 | 2.1778(−1) | 5.7465(−1) | 5.1383(−1) | 4.3391(−1) | 3.9387(−1) |
| 1.0 | 2.0480(−1) | 3.8405(−1) | 4.0193(−1) | 3.8312(−1) | 3.6985(−1) |

Table 2. First problem: the non-singular component $I_*(\tau, \mu, \phi)$ for $\phi - \phi_0 = \pi/2$

| $\mu$ | $\tau/\tau_0 = 0.0$ | $\tau/\tau_0 = 0.25$ | $\tau/\tau_0 = 0.5$ | $\tau/\tau_0 = 0.75$ | $\tau/\tau_0 = 1.0$ |
|---|---|---|---|---|---|
| −1.0 | 3.9191(−1) | 4.0746(−1) | 4.5654(−1) | 5.2409(−1) | 5.8833(−1) |
| −0.9 | 3.8994(−1) | 3.9943(−1) | 4.4482(−1) | 5.1436(−1) | 5.8850(−1) |
| −0.8 | 3.8911(−1) | 3.9223(−1) | 4.3281(−1) | 5.0333(−1) | 5.8894(−1) |
| −0.7 | 3.8951(−1) | 3.8617(−1) | 4.2074(−1) | 4.9079(−1) | 5.8970(−1) |
| −0.6 | 3.9111(−1) | 3.8158(−1) | 4.0894(−1) | 4.7658(−1) | 5.9084(−1) |
| −0.5 | 3.9375(−1) | 3.7879(−1) | 3.9792(−1) | 4.6062(−1) | 5.9244(−1) |
| −0.4 | 3.9703(−1) | 3.7805(−1) | 3.8830(−1) | 4.4312(−1) | 5.9460(−1) |
| −0.3 | 4.0017(−1) | 3.7949(−1) | 3.8075(−1) | 4.2497(−1) | 5.9743(−1) |
| −0.2 | 4.0161(−1) | 3.8308(−1) | 3.7559(−1) | 4.0832(−1) | 6.0106(−1) |
| −0.1 | 3.9788(−1) | 3.8875(−1) | 3.7271(−1) | 3.9570(−1) | 6.0576(−1) |
| −0.0 | 3.7348(−1) | 3.9633(−1) | 3.7193(−1) | 3.8645(−1) | 6.1308(−1) |
| 0.0 | 2.0019(−1) | 3.9633(−1) | 3.7193(−1) | 3.8645(−1) | 4.6886(−1) |
| 0.1 | 2.0629(−1) | 4.0525(−1) | 3.7326(−1) | 3.7951(−1) | 4.3958(−1) |
| 0.2 | 2.0722(−1) | 4.1243(−1) | 3.7663(−1) | 3.7473(−1) | 4.2079(−1) |
| 0.3 | 2.0686(−1) | 4.1353(−1) | 3.8145(−1) | 3.7203(−1) | 4.0627(−1) |
| 0.4 | 2.0608(−1) | 4.0987(−1) | 3.8643(−1) | 3.7121(−1) | 3.9495(−1) |
| 0.5 | 2.0525(−1) | 4.0415(−1) | 3.9064(−1) | 3.7180(−1) | 3.8630(−1) |
| 0.6 | 2.0459(−1) | 3.9812(−1) | 3.9387(−1) | 3.7329(−1) | 3.7990(−1) |
| 0.7 | 2.0419(−1) | 3.9271(−1) | 3.9628(−1) | 3.7532(−1) | 3.7535(−1) |
| 0.8 | 2.0410(−1) | 3.8842(−1) | 3.9822(−1) | 3.7767(−1) | 3.7233(−1) |
| 0.9 | 2.0430(−1) | 3.8549(−1) | 4.0000(−1) | 3.8027(−1) | 3.7056(−1) |
| 1.0 | 2.0480(−1) | 3.8405(−1) | 4.0193(−1) | 3.8312(−1) | 3.6985(−1) |

Table 3. First problem: the non-singular component $I_*(\tau, \mu, \phi)$ for $\phi - \phi_0 = \pi$

| $\mu$ | $\tau/\tau_0 = 0.0$ | $\tau/\tau_0 = 0.25$ | $\tau/\tau_0 = 0.5$ | $\tau/\tau_0 = 0.75$ | $\tau/\tau_0 = 1.0$ |
|---|---|---|---|---|---|
| −1.0 | 3.9191(−1) | 4.0746(−1) | 4.5654(−1) | 5.2409(−1) | 5.8833(−1) |
| −0.9 | 3.6094(−1) | 3.8929(−1) | 4.4097(−1) | 5.1169(−1) | 5.8437(−1) |
| −0.8 | 3.4765(−1) | 3.7676(−1) | 4.2708(−1) | 4.9999(−1) | 5.8447(−1) |
| −0.7 | 3.3639(−1) | 3.6524(−1) | 4.1309(−1) | 4.8697(−1) | 5.8549(−1) |
| −0.6 | 3.2593(−1) | 3.5460(−1) | 3.9912(−1) | 4.7221(−1) | 5.8712(−1) |
| −0.5 | 3.1570(−1) | 3.4491(−1) | 3.8555(−1) | 4.5552(−1) | 5.8928(−1) |
| −0.4 | 3.0526(−1) | 3.3626(−1) | 3.7295(−1) | 4.3705(−1) | 5.9197(−1) |
| −0.3 | 2.9406(−1) | 3.2858(−1) | 3.6188(−1) | 4.1765(−1) | 5.9524(−1) |
| −0.2 | 2.8141(−1) | 3.2163(−1) | 3.5259(−1) | 3.9946(−1) | 5.9924(−1) |
| −0.1 | 2.6621(−1) | 3.1512(−1) | 3.4484(−1) | 3.8503(−1) | 6.0423(−1) |
| −0.0 | 2.4425(−1) | 3.0873(−1) | 3.3830(−1) | 3.7363(−1) | 6.1184(−1) |
| 0.0 | 1.6788(−1) | 3.0873(−1) | 3.3830(−1) | 3.7363(−1) | 4.6390(−1) |
| 0.1 | 1.7337(−1) | 3.0201(−1) | 3.3277(−1) | 3.6410(−1) | 4.3349(−1) |
| 0.2 | 1.7717(−1) | 2.9408(−1) | 3.2805(−1) | 3.5617(−1) | 4.1350(−1) |
| 0.3 | 1.8033(−1) | 2.8489(−1) | 3.2382(−1) | 3.4966(−1) | 3.9752(−1) |
| 0.4 | 1.8313(−1) | 2.7632(−1) | 3.1993(−1) | 3.4440(−1) | 3.8442(−1) |
| 0.5 | 1.8574(−1) | 2.6987(−1) | 3.1671(−1) | 3.4031(−1) | 3.7367(−1) |
| 0.6 | 1.8830(−1) | 2.6647(−1) | 3.1488(−1) | 3.3753(−1) | 3.6503(−1) |
| 0.7 | 1.9092(−1) | 2.6701(−1) | 3.1549(−1) | 3.3650(−1) | 3.5850(−1) |
| 0.8 | 1.9373(−1) | 2.7329(−1) | 3.2031(−1) | 3.3824(−1) | 3.5442(−1) |
| 0.9 | 1.9705(−1) | 2.9058(−1) | 3.3367(−1) | 3.4531(−1) | 3.5402(−1) |
| 1.0 | 2.0480(−1) | 3.8405(−1) | 4.0193(−1) | 3.8312(−1) | 3.6985(−1) |

obtained by use of the spherical-harmonics method with $N$ typically between 99 and 499, we found for the non-singular component of the intensity. We note that Tables 1–3 are in essentially perfect agreement with the calculations reported by Devaux, Siewert and Yuan in Ref. 1.

In regard to radiative-transfer problems formulated for use in the field of atmospheric sciences, it is common to have scattering processes that require many terms in a Legendre expansion of the phase function. And so, as a second computational problem, we consider a more challenging set of basic data. To have a specific scattering law for testing our solution technique, and to avoid having to provide a table of the coefficients $\{\beta_l\}$, we use $L = 299$ with the binomial scattering law[12]

$$p(\cos\Theta) = \frac{L+1}{2^L}(1+\cos\Theta)^L \tag{55}$$

which can be represented exactly with $L + 1$ Legendre coefficients that can be computed with $\beta_0 = 1$ and[13]

$$\beta_l = \left(\frac{2l+1}{2l-1}\right)\left(\frac{L+1-l}{L+1+l}\right)\beta_{l-1}. \tag{56}$$

Continuing to define our problem, we use $\varpi = 0.99$, $\tau_0 = 2.0$, $\mu_0 = 0.3$, $\rho_1^s = 0.1$, $\rho_1^d = 0.2$, $\rho_2^s = 0.3$ and $\rho_2^d = 0.4$. In addition, we use here

$$T(\mu) = 1 + \mu + \mu^2 \tag{57}$$

and

$$S_0(\tau) = \sin(\pi\tau/\tau_0). \tag{58}$$

In order to have some confidence that the results listed in Tables 4–6 for the non-singular component of the intensity are correct to within plus or minus one in the last place given, we have typically used values of $N$ between 299 and 499 in our spherical-harmonics solution.

Table 4. Second problem: the non-singular component $I_*(\tau, \mu, \phi)$ for $\phi - \phi_0 = 0$

| $\mu$ | $\tau/\tau_0 = 0.0$ | $\tau/\tau_0 = 0.25$ | $\tau/\tau_0 = 0.5$ | $\tau/\tau_0 = 0.75$ | $\tau/\tau_0 = 1.0$ |
|---|---|---|---|---|---|
| −1.0 | 7.1939 | 7.0478 | 6.6376 | 6.2269 | 6.0796 |
| −0.9 | 7.0906 | 6.9214 | 6.4595 | 5.9977 | 5.8290 |
| −0.8 | 7.0949 | 6.8877 | 6.3529 | 5.8203 | 5.6197 |
| −0.7 | 7.3391 | 7.0691 | 6.4232 | 5.7790 | 5.5140 |
| −0.6 | 8.0223 | 7.6875 | 6.9108 | 6.1254 | 5.7602 |
| −0.5 | 9.2426 | 8.8741 | 8.0086 | 7.1741 | 6.8506 |
| −0.4 | 1.0930(1) | 1.0508(1) | 9.4791 | 8.6719 | 8.7946 |
| −0.3 | 1.2959(1) | 1.2628(1) | 1.1064(1) | 9.7043 | 9.9909 |
| −0.2 | 1.4524(1) | 1.5472(1) | 1.3519(1) | 1.0654(1) | 9.6407 |
| −0.1 | 1.3276(1) | 1.8048(1) | 1.7356(1) | 1.3489(1) | 8.4224 |
| −0.0 | 6.7985 | 1.8636(1) | 2.0995(1) | 1.8233(1) | 6.4043 |
| 0.0 | 2.2540 | 1.8636(1) | 2.0995(1) | 1.8233(1) | 1.1264(1) |
| 0.1 | 2.9017 | 1.9404(1) | 2.3652(1) | 2.2353(1) | 1.7625(1) |
| 0.2 | 3.0265 | 3.1125(1) | 2.9064(1) | 2.5455(1) | 2.1252(1) |
| 0.3 | 2.8700 | 4.2233(1) | 3.4249(1) | 2.6850(1) | 2.1920(1) |
| 0.4 | 2.6672 | 2.0198(1) | 2.1011(1) | 1.9361(1) | 1.7365(1) |
| 0.5 | 2.4984 | 5.2549 | 7.8824 | 9.6144 | 1.0252(1) |
| 0.6 | 2.3763 | 2.8730 | 4.0275 | 5.2192 | 5.9175 |
| 0.7 | 2.3080 | 2.5734 | 3.2501 | 3.9647 | 4.3301 |
| 0.8 | 2.2836 | 2.5047 | 3.0576 | 3.6150 | 3.8492 |
| 0.9 | 2.2832 | 2.4778 | 2.9652 | 3.4524 | 3.6466 |
| 1.0 | 2.2935 | 2.4680 | 2.9056 | 3.3425 | 3.5152 |

Table 5. Second problem: the non-singular component $I_*(\tau, \mu, \phi)$ for $\phi - \phi_0 = \pi/2$

| $\mu$ | $\tau/\tau_0 = 0.0$ | $\tau/\tau_0 = 0.25$ | $\tau/\tau_0 = 0.5$ | $\tau/\tau_0 = 0.75$ | $\tau/\tau_0 = 1.0$ |
|---|---|---|---|---|---|
| −1.0 | 7.1939 | 7.0478 | 6.6376 | 6.2269 | 6.0796 |
| −0.9 | 7.0872 | 6.9192 | 6.4581 | 5.9968 | 5.8285 |
| −0.8 | 7.0551 | 6.8575 | 6.3314 | 5.8062 | 5.6112 |
| −0.7 | 7.1326 | 6.8912 | 6.2770 | 5.6667 | 5.4358 |
| −0.6 | 7.3894 | 7.0763 | 6.3345 | 5.6037 | 5.3195 |
| −0.5 | 7.9462 | 7.5168 | 6.5769 | 5.6674 | 5.3019 |
| −0.4 | 8.9453 | 8.3817 | 7.1210 | 5.9263 | 5.4544 |
| −0.3 | 1.0315(1) | 9.8635 | 8.1631 | 6.4295 | 5.8140 |
| −0.2 | 1.1189(1) | 1.1781(1) | 9.9905 | 7.3320 | 6.1724 |
| −0.1 | 9.8304 | 1.2762(1) | 1.2287(1) | 9.2077 | 6.0316 |
| −0.0 | 5.2706 | 1.1010(1) | 1.3086(1) | 1.1563(1) | 5.1791 |
| 0.0 | 2.1012 | 1.1010(1) | 1.3086(1) | 1.1563(1) | 7.1802 |
| 0.1 | 2.5572 | 7.2552 | 1.0986(1) | 1.1956(1) | 9.6555 |
| 0.2 | 2.6930 | 4.4501 | 7.6562 | 9.8771 | 9.6914 |
| 0.3 | 2.6056 | 3.3492 | 5.3233 | 7.2847 | 7.9968 |
| 0.4 | 2.4686 | 2.9412 | 4.1900 | 5.5462 | 6.2314 |
| 0.5 | 2.3687 | 2.7342 | 3.6537 | 4.6190 | 5.0898 |
| 0.6 | 2.3131 | 2.6123 | 3.3575 | 4.1155 | 4.4484 |
| 0.7 | 2.2874 | 2.5406 | 3.1723 | 3.8073 | 4.0694 |
| 0.8 | 2.2796 | 2.4994 | 3.0492 | 3.5995 | 3.8208 |
| 0.9 | 2.2828 | 2.4773 | 2.9645 | 3.4513 | 3.6449 |
| 1.0 | 2.2935 | 2.4680 | 2.9056 | 3.3425 | 3.5152 |

Table 6. Second problem: the non-singular component $I_*(\tau, \mu, \phi)$ for $\phi - \phi_0 = \pi$

| $\mu$ | $\tau/\tau_0 = 0.0$ | $\tau/\tau_0 = 0.25$ | $\tau/\tau_0 = 0.5$ | $\tau/\tau_0 = 0.75$ | $\tau/\tau_0 = 1.0$ |
|---|---|---|---|---|---|
| −1.0 | 7.1939 | 7.0478 | 6.6376 | 6.2269 | 6.0796 |
| −0.9 | 7.0872 | 6.9192 | 6.4581 | 5.9968 | 5.8285 |
| −0.8 | 7.0550 | 6.8575 | 6.3314 | 5.8062 | 5.6112 |
| −0.7 | 7.1325 | 6.8911 | 6.2770 | 5.6667 | 5.4358 |
| −0.6 | 7.3892 | 7.0762 | 6.3344 | 5.6036 | 5.3195 |
| −0.5 | 7.9458 | 7.5165 | 6.5767 | 5.6673 | 5.3019 |
| −0.4 | 8.9443 | 8.3810 | 7.1206 | 5.9260 | 5.4542 |
| −0.3 | 1.0313(1) | 9.8620 | 8.1620 | 6.4288 | 5.8136 |
| −0.2 | 1.1186(1) | 1.1779(1) | 9.9886 | 7.3307 | 6.1717 |
| −0.1 | 9.8274 | 1.2759(1) | 1.2284(1) | 9.2054 | 6.0308 |
| −0.0 | 5.2691 | 1.1007(1) | 1.3082(1) | 1.1560(1) | 5.1785 |
| 0.0 | 2.1010 | 1.1007(1) | 1.3082(1) | 1.1560(1) | 7.1784 |
| 0.1 | 2.5569 | 7.2533 | 1.0983(1) | 1.1953(1) | 9.6526 |
| 0.2 | 2.6927 | 4.4493 | 7.6547 | 9.8750 | 9.6889 |
| 0.3 | 2.6054 | 3.3488 | 5.3226 | 7.2836 | 7.9953 |
| 0.4 | 2.4685 | 2.9410 | 4.1897 | 5.5458 | 6.2307 |
| 0.5 | 2.3687 | 2.7342 | 3.6536 | 4.6188 | 5.0895 |
| 0.6 | 2.3130 | 2.6123 | 3.3575 | 4.1155 | 4.4483 |
| 0.7 | 2.2874 | 2.5406 | 3.1723 | 3.8073 | 4.0694 |
| 0.8 | 2.2796 | 2.4994 | 3.0492 | 3.5995 | 3.8208 |
| 0.9 | 2.2828 | 2.4773 | 2.9645 | 3.4513 | 3.6449 |
| 1.0 | 2.2935 | 2.4680 | 2.9056 | 3.3425 | 3.5152 |

## 6. THE SPECIAL CASE OF $\varpi = 1$ AND $m = 0$

In order to complete our analysis, we make a couple of observations about modifications that are required in our spherical-harmonics solution for the special case $\varpi = 1$. First of all, we note that no modifications are required for the case $m > 0$, and so our discussion here is relevant only to the case $\varpi = 1$ and $m = 0$. Since we discuss here only this special case, we suppress the superscript that we used in previous sections to denote the Fourier index.

As was noted in Ref. 4, the Chandrasekhar polynomial $g_{N+1}(\xi)$ is a polynomial only of degree $N-1$ when $\varpi = 1$, and so we find only $J-1$ positive eigenvalues that are finite (see Ref. 4 to find our method of computing these eigenvalues). As the missing (positive) eigenvalue has become infinite for $\varpi = 1$, we follow Ref. 4 and express the solution to the moments of Eq. (18) as

$$I_*(\tau, \mu) = \frac{1}{2}A(\tau_0 - \tau + 3\mu/h_1) + \frac{1}{2}B(\tau - 3\mu/h_1) + \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=2}^{J} [A_j \mathrm{e}^{-\tau/\xi_j} + (-1)^l B_j \mathrm{e}^{-(\tau_0-\tau)/\xi_j}]\, g_l(\xi_j) + I_p(\tau, \mu), \tag{59}$$

where $I_p(\tau, \mu)$ is a particular solution. Of course, the constants $A$ and $B$ appearing in Eq. (59) are arbitrary, as are the constants $A_j$ and $B_j$ for $j = 2, 3, \ldots, J$.

It is clear from Eq. (22) that the particular solution we have used for all other cases must also be modified here. Considering Eqs. (22) and (59), we propose that the desired particular solution, for the case $\varpi = 1$ with $m = 0$, be written in the form

$$I_p(\tau, \mu) = \frac{1}{2}\mathcal{A}(\tau)(\tau_0 - \tau + 3\mu/h_1) + \frac{1}{2}\mathcal{B}(\tau)(\tau - 3\mu/h_1) + \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=2}^{J} \frac{C_j}{\xi_j} [\mathcal{A}_j(\tau) + (-1)^l \mathcal{B}_j(\tau)] g_l(\xi_j) \tag{60}$$

where $\mathcal{A}(\tau)$ and $\mathcal{B}(\tau)$ are functions to be determined. Here we continue to use $\mathcal{A}_j(\tau)$ and $\mathcal{B}_j(\tau)$ as defined by Eqs. (28), and

$$C_j = 2\left(\sum_{l=2}^{N} h_l [g_l(\xi_j)]^2\right)^{-1}. \tag{61}$$

If we substitute Eq. (60) into Eq. (18), take the appropriate moments and use the identity (valid only for $\varpi = 1$)

$$[1 + (-1)^{\alpha+l}] h_l \sum_{j=2}^{J} C_j g_\alpha(\xi_j) g_l(\xi_j) = 2\delta_{\alpha,l}, \quad \text{for} \quad \alpha, l = 2, 3, \ldots, N, \tag{62}$$

that was reported in Ref. 14, then we find that Eq. (60) will be the desired particular solution if the functions $\mathcal{A}(\tau)$ and $\mathcal{B}(\tau)$ are defined by

$$\mathcal{A}(\tau) = \frac{1}{\tau_0} \int_0^\tau [h_1 x Q_0(x) + Q_1(x)]\, \mathrm{d}x \tag{63a}$$

and

$$\mathcal{B}(\tau) = \frac{1}{\tau_0} \int_\tau^{\tau_0} [h_1(\tau_0 - x) Q_0(x) - Q_1(x)]\, \mathrm{d}x \tag{63b}$$

where

$$Q_l(\tau) = (2l+1) \int_{-1}^{1} P_l(\mu) Q(\tau, \mu)\, \mathrm{d}\mu. \tag{64}$$

## 7. CONCLUDING REMARKS

To conclude this work, we would like to make note of some of the improvements we have made in regard to the use of the spherical-harmonics method for solving a general class of radiative-transfer problems. First of all, we note that the particular solution[5] we have used here for the case $\varpi \in (0, 1)$ is considered especially useful. In fact, even for the case of no reflections and no internal source of radiation, the particular solution used here is not singular, as is the one used by Chandrasekhar[2] when $\mu_0$ happens to be exactly one of the eigenvalues basic to the spherical-harmonics method. We note that the singularity in Chandrasekhar's particular solution was resolved in Ref. 4, but the particular solution used here does not have the singularity at all. We note also that a general particular solution is worked out here in Section 6 for the special case of $\varpi = 1$ and $m = 0$.

Before this work, we were of the opinion that there was very little difference in the computational accuracy obtained (in, say, high order) with the use of the Mark or the Marshak boundary conditions. However, having implemented the Mark boundary conditions into our general code, we have seen, for various values ($m > 0$) of the Fourier index $m$, that the use of the Mark boundary conditions yielded considerably better results for the intensities than using Marshak boundary conditions for the same order of approximation. We may report on this observation in more detail at a later date.

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