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# A HIGH-ORDER SPHERICAL HARMONICS SOLUTION TO THE STANDARD PROBLEM IN RADIATIVE TRANSFER

M. BENASSI,[1] R. D. M. GARCIA,[2,3] A. H. KARP,[2] AND C. E. SIEWERT[1,2]
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## ABSTRACT

The classical spherical harmonics method is used, in high order, to compute an accurate solution to Chandrasekhar's standard problem in radiative transfer: the illumination of a plane-parallel layer by a solar beam. While the analytical formulation used follows Chandrasekhar's basic approach, several additions to the theory are reported: (i) the solution for the specific case of pure scattering ($\omega = 1$) is developed in detail; (ii) a particular solution (required when the uncollided component of the radiation intensity is split off from the complete solution) that is valid even for the case when the direction cosine of the incident beam is equal to one of the eigenvalues is reported; and (iii) a final iteration of the equation of transfer is used to obtain an improved analytical expression for the intensity that is, in addition, especially accurate for the case of strong absorption. An explicit presentation of the numerical methods used is given, and the algorithm is shown to be computationally stable even for very high order ($N = 499$) and for many-term ($L = 299$) scattering laws.

For all five of the test problems recently posed by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics, the method is shown to yield, with modest computational effort, results for the complete radiation field that are continuous in all variables and accurate, in general, to five significant figures.

*Subject heading:* radiative transfer

## I. INTRODUCTION

The technique commonly known as the spherical harmonics method was, as pointed out by Dave (1975), suggested as a solution technique in the field of radiative transfer by Jeans as early as 1917. Later, as noted by Davison (1957), the method was introduced into the literature relevant to neutron-transport theory by Wick (1943) and Marshak (1947) and extensively developed by Mark (1944, 1945, 1947). The technique has subsequently become one of the popular standard methods for solving problems in radiative transfer (Dave 1975; Canosa and Penafiel 1973; Devaux *et al.* 1973; Karp, Greenstadt, and Fillmore 1980) as well as neutron-transport theory (Kofink 1959; Pomraning 1965; Gelbard 1968). Here we use the method in high order to obtain accurate numerical results for the complete radiation field for a set of challenging test problems. Since the basic aspects of the method are so well known, our presentation here is brief.

## II. FORMULATION OF THE PROBLEM

We consider the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \phi) + I(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta) I(\tau, \mu', \phi') d\phi' d\mu' \tag{1}$$

where $\omega \in [0, 1]$ is the single-scattering albedo, $\mu$ is the direction cosine, as measured from the *positive* $\tau$ axis, of the propagating radiation, and $\phi$ is the azimuthal angle. In addition $\Theta$ is the scattering angle, and we consider phase functions that have a Legendre expansion of the form

$$p(\cos \Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos \Theta), \qquad \beta_0 = 1 . \tag{2}$$

We consider Chandrasekhar's (1950) standard problem and therefore seek, for all $\mu \in [-1, 1]$, $\phi \in [0, 2\pi]$, and $\tau \in [0, \tau_0]$, a solution of equation (1) subject to the boundary conditions

$$I(0, \mu, \phi) = \pi \delta(\mu - \mu_0) \delta(\phi - \phi_0) \tag{3a}$$

and

$$I(\tau_0, -\mu, \phi) = 0 \tag{3b}$$

[1] Mathematics Department, North Carolina State University.
[2] IBM Scientific Center, Palo Alto, CA 94304.
[3] Divisão de Física de Reatores, Instituto de Pesquisas Energéticas e Nucleares, Cidade Universitária, São Paulo, Brasil.

for $\mu > 0$ and all $\phi$. We can use the addition theorem for the Legendre polynomials (Abramowitz and Stegun 1964) and a Fourier expansion (Devaux and Siewert 1980) to express the desired solution as

$$I(\tau, \mu, \phi) = \sum_{m=0}^{L} [I^m(\tau, \mu) - I^m(0, \mu)e^{-\tau/\mu}] \cos [m(\phi - \phi_0)] + \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0)e^{-\tau/\mu} \tag{4a}$$

and

$$I(\tau, -\mu, \phi) = \sum_{m=0}^{L} I^m(\tau, -\mu) \cos [m(\phi - \phi_0)] \tag{4b}$$

for $\mu > 0$, $\tau \in [0, \tau_0]$ and all $\phi$. Here the components $I^m(\tau, \mu)$ are (Chandrasekhar 1950) solutions to

$$\mu \frac{\partial}{\partial \tau} I^m(\tau, \mu) + I^m(\tau, \mu) = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu')I^m(\tau, \mu')d\mu' \tag{5}$$

subject to the boundary conditions

$$I^m(0, \mu) = \tfrac{1}{2}(2 - \delta_{0,m})\delta(\mu - \mu_0) \tag{6a}$$

and

$$I^m(\tau_0, -\mu) = 0 \tag{6b}$$

for $\mu > 0$. Here we use $P_l^m(\mu)$ to denote the associated Legendre functions, i.e.,

$$P_l^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu), \tag{7}$$

and

$$\beta_l^m = \frac{(l - m)!}{(l + m)!} \beta_l . \tag{8}$$

We proceed to develop an approximate solution, for each $m$, to equations (5) and (6).

## III. ANALYSIS

We start by writing

$$I^m(\tau, \mu) = \tfrac{1}{2}(2 - \delta_{0,m})[\delta(\mu - \mu_0)e^{-\tau/\mu} + I_*^m(\tau, \mu)], \tag{9}$$

where $I_*^m(\tau, \mu)$ satisfies

$$\mu \frac{\partial}{\partial \tau} I_*^m(\tau, \mu) + I_*^m(\tau, \mu) = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu')I_*^m(\tau, \mu')d\mu' + S^m(\tau, \mu) \tag{10}$$

and, for $\mu > 0$,

$$I_*^m(0, \mu) = I_*^m(\tau_0, -\mu) = 0 . \tag{11}$$

Here the known inhomogeneous term is

$$S^m(\tau, \mu) = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu)P_l^m(\mu_0)e^{-\tau/\mu_0} . \tag{12}$$

We consider first that $\omega \neq 1$ and note that, for $N$ odd,

$$H^m(\tau, \mu) = \sum_{l=m}^{M} \left(\frac{2l + 1}{2}\right) \frac{(l - m)!}{(l + m)!} P_l^m(\mu) \sum_{j=1}^{J} [A_j e^{-\tau/\xi_j} + (-1)^{l-m} B_j e^{-(\tau_0 - \tau)/\xi_j}] g_l^m(\xi_j) \tag{13}$$

is a solution of the first $N + 1$ moments of the homogeneous version of equation (10), i.e.,

$$\int_{-1}^{1} P_{m+\alpha}^m(\mu) \left[\mu \frac{\partial}{\partial \tau} H^m(\tau, \mu) + H^m(\tau, \mu) - \frac{\omega}{2} \sum_{l=m}^{M} \beta_l^m P_l^m(\mu) H_l^m(\tau)\right] d\mu = 0 \tag{14}$$

for $\alpha = 0, 1, 2, \ldots, N$. Here $M = m + N$, $J = (N + 1)/2$, and

$$H_l^m(\tau) = \int_{-1}^{1} P_l^m(\mu) H^m(\tau, \mu) d\mu . \tag{15}$$

The polynomials $g_l^m(\xi)$ appearing in equation (13) are those introduced by Chandrasekhar (1950), i.e., $g_m^m(\xi) = (2m-1)!!$ and, for $l \geq m$,

$$(l-m+1)g_{l+1}^m(\xi) = h_l \xi g_l^m(\xi) - (l+m)(1-\delta_{m,l})g_{l-1}^m(\xi) \tag{16}$$

with

$$h_l = 2l + 1 - \omega\beta_l\,. \tag{17}$$

In addition, the eigenvalues $\xi_j$ are the *positive* zeros of $g_{M+1}^m(\xi)$ and the constants $A_j$ and $B_j$ are to be determined from the boundary conditions. Continuing, we let

$$N^m(\tau, \mu) = \sum_{l=m}^{M} \left(\frac{2l+1}{2}\right) \frac{(l-m)!}{(l+m)!} P_l^m(\mu) N_l^m(\tau) \tag{18}$$

denote a particular solution of the first $N+1$ moments of equation (10) and substitute

$$I_*^m(\tau, \mu) = H^m(\tau, \mu) + N^m(\tau, \mu) \tag{19}$$

into equation (11) to obtain, for $\mu > 0$,

$$\sum_{l=m}^{M} \left(\frac{2l+1}{2}\right) \frac{(l-m)!}{(l+m)!} P_l^m(\mu) \sum_{j=1}^{J} [A_j + (-1)^{l-m} B_j e^{-\tau_0/\xi_j}] g_l^m(\xi_j) = -N^m(0, \mu) \tag{20a}$$

and

$$\sum_{l=m}^{M} \left(\frac{2l+1}{2}\right) \frac{(l-m)!}{(l+m)!} P_l^m(\mu) \sum_{j=1}^{J} [(-1)^{l-m} A_j e^{-\tau_0/\xi_j} + B_j] g_l^m(\xi_j) = -N^m(\tau_0, -\mu)\,. \tag{20b}$$

To use a generalization (Dave 1975) of the Marshak projection (Davison 1957) we multiply equations (20) by $P_{m+2\alpha+1}^m(\mu)$, $\alpha = 0, 1, \ldots, (N-1)/2$, and integrate over $\mu$ from 0 to 1 to obtain the system of linear algebraic equations

$$\sum_{l=m}^{M} \sum_{j=1}^{J} (2l+1) \frac{(l-m)!}{(l+m)!} S_{\alpha,l}^m [A_j + (-1)^{l-m} B_j e^{-\tau_0/\xi_j}] g_l^m(\xi_j) = R_{1,\alpha}^m \tag{21a}$$

and

$$\sum_{l=m}^{M} \sum_{j=1}^{J} (2l+1) \frac{(l-m)!}{(l+m)!} S_{\alpha,l}^m [(-1)^{l-m} A_j e^{-\tau_0/\xi_j} + B_j] g_l^m(\xi_j) = R_{2,\alpha}^m \tag{21b}$$

where

$$R_{1,\alpha}^m = -2 \int_0^1 P_{m+2\alpha+1}^m(\mu) N^m(0, \mu) d\mu\,, \tag{22a}$$

$$R_{2,\alpha}^m = -2 \int_0^1 P_{m+2\alpha+1}^m(\mu) N^m(\tau_0, -\mu) d\mu \tag{22b}$$

and

$$S_{\alpha,l}^m = \int_0^1 P_{m+2\alpha+1}^m(\mu) P_l^m(\mu) d\mu\,. \tag{23}$$

With the convention that $\beta_l = 0$ for $l > L$, we write equation (10) as

$$\frac{\partial}{\partial\tau}[I_*^m(\tau, \mu)e^{\tau/\mu}] = \frac{\omega}{2\mu} e^{\tau/\mu} \left\{ \sum_{l=m}^{M} \beta_l^m P_l^m(\mu)[H_l^m(\tau) + N_l^m(\tau)] + \sum_{l=m}^{L} \beta_l^m P_l^m(\mu) P_l^m(\mu_0) e^{-\tau/\mu_0} \right\}. \tag{24}$$

Thus, following other works (Dave 1975; Devaux *et al.* 1973; Karp and Petrack 1983), we consider our spherical harmonics approximation to be a way to compute the moments $H_l^m(\tau)$ and $N_l^m(\tau)$ so that the intensity $I_*^m(\tau, \mu)$ can be found by integrating equation (24).

At this point we express our particular solution (Chandrasekhar 1950) as

$$N^m(\tau, \mu) = \sum_{l=m}^{M} \left(\frac{2l+1}{2}\right) \frac{(l-m)!}{(l+m)!} P_l^m(\mu)[\gamma g_l^m(\mu_0) - P_l^m(\mu_0)] e^{-\tau/\mu_0} \tag{25}$$

and note that $N^m(\tau, \mu)$ will satisfy the first $N+1$ moments of equation (10) if

$$\gamma = P_{M+1}^m(\mu_0)/g_{M+1}^m(\mu_0)\,. \tag{26}$$

The solution given by equation (25) clearly is not valid if $\mu_0$ is a zero of $g_{M+1}^m(\xi)$. As we can always add a solution of the homogeneous version of the moment equations to our particular solution, we can add a variant of equation (13), with $A_j = -\gamma\delta_{i,j}$ and $B_j = 0$, to equation (25) and take the limit as $\mu_0 \to \xi_i$ to find, for the case $g_{M+1}^m(\mu_0) = 0$,

$$N^m(\tau, \mu) = \sum_{l=m}^{M} \left(\frac{2l+1}{2}\right) \frac{(l-m)!}{(l+m)!} P_l^m(\mu) \left\{ \left[ \dot{g}_l^m(\mu_0) + \frac{\tau}{\mu_0{}^2} g_l^m(\mu_0) \right] \frac{P_{M+1}^m(\mu_0)}{\dot{g}_{M+1}^m(\mu_0)} - P_l^m(\mu_0) \right\} e^{-\tau/\mu_0} , \tag{27}$$

where

$$\dot{g}_l^m(\xi) = \frac{d}{d\xi} g_l^m(\xi) . \tag{28}$$

Using equation (13) and equation (25) or equation (27) to compute $H_l^m(\tau)$ and $N_l^m(\tau)$, we now integrate equation (24) to obtain, for $\tau \in [0, \tau_0]$,

$$I_*^m(\tau, -\mu) = \frac{\omega}{2} \left[ \mu_0 e^{-\tau/\mu_0} S(\tau_0 - \tau : \mu, \mu_0) \sum_{l=M+1}^{L} (-1)^{l-m} \beta_l^m P_l^m(\mu) P_l^m(\mu_0) + \Xi^m(\tau, -\mu) + \Upsilon^m(\tau, -\mu) \right] \tag{29a}$$

and

$$I_*^m(\tau, \mu) = \frac{\omega}{2} \left[ \mu_0 C(\tau : \mu, \mu_0) \sum_{l=M+1}^{L} \beta_l^m P_l^m(\mu) P_l^m(\mu_0) + \Xi^m(\tau, \mu) + \Upsilon^m(\tau, \mu) \right] , \tag{29b}$$

where, for $\mu \in [0, 1]$,

$$\Upsilon^m(\tau, -\mu) = \sum_{l=m}^{M} \beta_l^m P_l^m(\mu) \sum_{j=1}^{J} \xi_j [(-1)^{l-m} A_j e^{-\tau/\xi_j} S(\tau_0 - \tau : \mu, \xi_j) + B_j C(\tau_0 - \tau : \mu, \xi_j)] g_l^m(\xi_j) , \tag{30a}$$

$$\Upsilon^m(\tau, \mu) = \sum_{l=m}^{M} \beta_l^m P_l^m(\mu) \sum_{j=1}^{J} \xi_j [A_j C(\tau : \mu, \xi_j) + (-1)^{l-m} B_j e^{-(\tau_0 - \tau)/\xi_j} S(\tau : \mu, \xi_j)] g_l^m(\xi_j) , \tag{30b}$$

$$\Xi^m(\tau, -\mu) = \mu_0 \sum_{l=m}^{M} (-1)^{l-m} \beta_l^m P_l^m(\mu) [S(\tau_0 - \tau : \mu, \mu_0) X_l^m(\mu_0) + E(\tau_0 - \tau : \mu, \mu_0) Y_l^m(\mu_0)] e^{-\tau/\mu_0} , \tag{31a}$$

and

$$\Xi^m(\tau, \mu) = \mu_0 \sum_{l=m}^{M} \beta_l^m P_l^m(\mu) [C(\tau : \mu, \mu_0) X_l^m(\mu_0) + D(\tau : \mu, \mu_0) Y_l^m(\mu_0)] . \tag{31b}$$

Here we use

$$C(\tau : \mu, \xi) = \frac{e^{-\tau/\mu} - e^{-\tau/\xi}}{\mu - \xi} , \tag{32a}$$

$$S(\tau : \mu, \xi) = \frac{1 - e^{-\tau/\mu} e^{-\tau/\xi}}{\mu + \xi} , \tag{32b}$$

$$D(\tau : \mu, \mu_0) = \frac{\tau e^{-\tau/\mu_0} - \mu\mu_0 C(\tau : \mu, \mu_0)}{\mu_0 - \mu} , \tag{33a}$$

and

$$E(\tau : \mu, \mu_0) = \frac{\tau_0(1 - e^{-\tau/\mu} e^{-\tau/\mu_0}) - \tau + \mu\mu_0 S(\tau : \mu, \mu_0)}{\mu + \mu_0} . \tag{33b}$$

In addition

$$X_l^m(\mu_0) = \gamma g_l^m(\mu_0) \tag{34a}$$

and

$$Y_l^m(\mu_0) = 0 \tag{34b}$$

for the case $g_{M+1}^m(\mu_0) \neq 0$, and

$$X_l^m(\mu_0) = \dot{g}_l^m(\mu_0) P_{M+1}^m(\mu_0) / \dot{g}_{M+1}^m(\mu_0) \tag{35a}$$

and

$$Y_l^m(\mu_0) = \frac{1}{\mu_0^2} g_l^m(\mu_0) P_{M+1}^m(\mu_0) / \dot{g}_{M+1}^m(\mu_0) \tag{35b}$$

for the special case $g^m_{M+1}(\mu_0) = 0$. At this point we can use equation (25) or equation (27) in equations (22) to obtain

$$R^m_{1,\alpha} = -\sum_{l=m}^{M} (2l+1)\frac{(l-m)!}{(l+m)!} S^m_{\alpha,l}[X^m_l(\mu_0) - P^m_l(\mu_0)] \tag{36a}$$

and

$$R^m_{2,\alpha} = -\sum_{l=m}^{M} (-1)^{l-m}(2l+1)\frac{(l-m)!}{(l+m)!} S^m_{\alpha,l}[X^m_l(\mu_0) + \tau_0 Y^m_l(\mu_0) - P^m_l(\mu_0)]e^{-\tau_0/\mu_0} \tag{36b}$$

so that now we can solve equations (21) to find the constants $\{A_j\}$ and $\{B_j\}$ required to complete the desired solution.

We now discuss the modifications to the foregoing analysis that are required for the special case $m = 0$ and $\omega = 1$. For this special case $g_{N+1}(\xi)$ is a polynomial of degree $N - 1$, and so we can obtain only $J - 1$ finite eigenvalues $\{\xi_j\}$. We therefore rewrite equation (13) as

$$H(\tau, \mu) = \frac{1}{2}A\left(\tau_0 - \tau + \frac{3}{h_1}\mu\right) + \frac{1}{2}B\left(\tau - \frac{3}{h_1}\mu\right) + \sum_{l=0}^{N}\left(\frac{2l+1}{2}\right)P_l(\mu)\sum_{j=1}^{J-1}[A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0-\tau)/\xi_j}]g_l(\xi_j)\,, \tag{37}$$

where the first two terms are the result of the missing eigenvalue having become infinite. Continuing, we rewrite equations (21) as

$$\sum_{l=0}^{N}\sum_{j=1}^{J-1}(2l+1)S_{\alpha,l}[A_j + (-1)^l B_j e^{-\tau_0/\xi_j}]g_l(\xi_j) + \left(S_{\alpha,0}\tau_0 + \frac{3}{h_1}S_{\alpha,1}\right)A - \frac{3}{h_1}S_{\alpha,1}B = R_{1,\alpha} \tag{38a}$$

and

$$\sum_{l=0}^{N}\sum_{j=1}^{J-1}(2l+1)S_{\alpha,l}[(-1)^l A_j e^{-\tau_0/\xi_j} + B_j]g_l(\xi_j) + \left(S_{\alpha,0}\tau_0 + \frac{3}{h_1}S_{\alpha,1}\right)B - \frac{3}{h_1}S_{\alpha,1}A = R_{2,\alpha}\,, \tag{38b}$$

and equations (30) as

$$\begin{aligned}\Upsilon(\tau, -\mu) = \sum_{l=0}^{N}\beta_l P_l(\mu)\sum_{j=1}^{J-1}\xi_j[(-1)^l A_j e^{-\tau/\xi_j}S(\tau_0 - \tau{:}\,\mu, \xi_j) + B_j C(\tau_0 - \tau{:}\,\mu, \xi_j)]g_l(\xi_j)\\ + A\{\tau_0 - \tau - (3\mu/h_1)[1 - e^{-(\tau_0-\tau)/\mu}]\} + B\{\tau - \tau_0 + [\tau_0 + (3\mu/h_1)][1 - e^{-(\tau_0-\tau)/\mu}]\}\end{aligned} \tag{39a}$$

and

$$\begin{aligned}\Upsilon(\tau, \mu) = \sum_{l=0}^{N}\beta_l P_l(\mu)\sum_{j=1}^{J-1}\xi_j[A_j C(\tau{:}\,\mu, \xi_j) + (-1)^l B_j e^{-(\tau_0-\tau)/\xi_j}S(\tau{:}\,\mu, \xi_j)]g_l(\xi_j)\\ + A\left[\left(\tau_0 + \frac{3\mu}{h_1}\right)(1 - e^{-\tau/\mu}) - \tau\right] + B\left[\tau - \frac{3\mu}{h_1}(1 - e^{-\tau/\mu})\right].\end{aligned} \tag{39b}$$

To complete this section we write the partial fluxes

$$q_\pm(\tau) = \int_0^1\int_0^{2\pi}\mu I(\tau, \pm\mu, \phi)d\phi\, d\mu \tag{40}$$

and the net flux

$$q(\tau) = \int_{-1}^1\int_0^{2\pi}\mu I(\tau, \mu, \phi)d\phi\, d\mu \tag{41}$$

as

$$q_+(\tau) = \pi[\mu_0 e^{-\tau/\mu_0} + H_+(\tau) + N_+(\tau)]\,, \tag{42a}$$

$$q_-(\tau) = \pi[H_-(\tau) + N_-(\tau)] \tag{42b}$$

and

$$q(\tau) = \pi\left\{(1-\omega)\sum_{j=1}^{J}\xi_j[A_j e^{-\tau/\xi_j} - B_j e^{-(\tau_0-\tau)/\xi_j}] + [X_1(\mu_0) + \tau Y_1(\mu_0)]e^{-\tau/\mu_0}\right\}, \tag{43}$$

where

$$H_\pm(\tau) = \sum_{l=0}^{N}\left(\frac{2l+1}{2}\right)(\pm 1)^l S_{0,l}\sum_{j=1}^{J}[A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0-\tau)/\xi_j}]g_l(\xi_j) \tag{44}$$

and

$$N_{\pm}(\tau) = \sum_{l=0}^{N} \left(\frac{2l+1}{2}\right)(\pm 1)^l S_{0,l}[X_l(\mu_0) + \tau Y_l(\mu_0) - P_l(\mu_0)]e^{-\tau/\mu_0} . \tag{45}$$

For the special case $\omega = 1$, we find

$$q_+(\tau) = \pi\{\mu_0 e^{-\tau/\mu_0} + H_+(\tau) + N_+(\tau) + \tfrac{1}{4}A[\tau_0 - \tau + (2/h_1)] + \tfrac{1}{4}B[\tau - (2/h_1)]\} , \tag{46a}$$

$$q_-(\tau) = \pi\{H_-(\tau) + N_-(\tau) + \tfrac{1}{4}A[\tau_0 - \tau - (2/h_1)] + \tfrac{1}{4}B[\tau + (2/h_1)]\} , \tag{46b}$$

and

$$q(\tau) = \frac{\pi}{h_1}(A - B) , \tag{47}$$

where now we must use

$$H_{\pm}(\tau) = \sum_{l=0}^{N} \left(\frac{2l+1}{2}\right)(\pm 1)^l S_{0,l} \sum_{j=1}^{J-1} [A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0 - \tau)/\xi_j}] g_l(\xi_j) . \tag{48}$$

## IV. NUMERICAL RESULTS AND CONCLUSIONS

To begin our numerical calculations we note that Dave (1975) has reported recursion relations that establish a very convenient way to evaluate the constants defined by equation (23); we express our version of the formulas required to compute all non-zero values of these constants as

$$S^m_{\alpha, m+2\alpha+1} = \left(\frac{1}{4\alpha + 3 + 2m}\right) \frac{(2\alpha + 1 + 2m)!}{(2\alpha + 1)!} , \tag{49a}$$

$$S^m_{\alpha, l+2} = \left(\frac{1 - l + 2\alpha + m}{4 + l + 2\alpha + m}\right)\left(\frac{l + 1 + m}{l + 1 - 2\alpha - m}\right)\left(\frac{l + 2 + 2\alpha + m}{l + 2 - m}\right) S^m_{\alpha, l} , \tag{49b}$$

and

$$S^m_{\alpha+1, m} = -\frac{1}{2}\left(\frac{2\alpha + 1}{\alpha + 2 + m}\right)\left(\frac{2\alpha + 3 + 2m}{2\alpha + 3}\right)\left(\frac{\alpha + 1 + m}{\alpha + 1}\right) S^m_{\alpha, m} , \tag{49c}$$

with

$$S^m_{0,m} = \frac{1}{2}\left(\frac{2m + 1}{m + 1}\right)[(2m - 1)!!]^2 . \tag{49d}$$

Considering next the calculation of the eigenvalues $\{\xi_j\}$ that are the zeros of the polynomial $g^m_{M+1}(\xi)$ as defined by equation (16), we first deduce from equation (16) a recursion formula involving only even polynomials. Thus for $l = m, m + 2, \ldots$, we find

$$(1 - \delta_{l,m})\left[\frac{(l+m)(l+m-1)}{h_{l-1}}\right] g^m_{l-2}(\xi) + \left\{\left[\frac{(l+1)^2 - m^2}{h_{l+1}}\right] + \left[\frac{l^2 - m^2}{h_{l-1}}\right] - h_l \xi^2\right\} g^m_l(\xi) + \left[\frac{(l-m+2)(l-m+1)}{h_{l+1}}\right] g^m_{l+2}(\xi) = 0 . \tag{50}$$

By using equation (50) for $l = m, m + 2, \ldots, M - 1$, we see that the problem of finding the positive zeros of $g^m_{M+1}(\xi)$ can be restated as one of finding the eigenvalues $\{\xi_j^2\}$ of the tridiagonal matrix $E$ of order $(N + 1)/2$ with elements

$$E_{\alpha, \alpha+1} = \frac{2\alpha(2\alpha - 1)}{h_{m+2\alpha-2}\, h_{m+2\alpha-1}} , \tag{51a}$$

$$E_{\alpha, \alpha} = \frac{1}{h_{m+2\alpha-2}}\left[\frac{4(\alpha - 1)(m + \alpha - 1)}{h_{m+2\alpha-3}} + \frac{(2\alpha - 1)(2m + 2\alpha - 1)}{h_{m+2\alpha-1}}\right] , \tag{51b}$$

and

$$E_{\alpha+1, \alpha} = \frac{2(m + \alpha)(2m + 2\alpha - 1)}{h_{m+2\alpha}\, h_{m+2\alpha-1}} , \tag{51c}$$

where $\alpha \geq 1$. In the event that $\omega = 1$ and $m = 0$ there is a pair of unbounded eigenvalues and the foregoing scheme requires a modification. Since $h_0 = 0$ in this case, we note from equation (50) that both $g_0(\xi)$ and $g_2(\xi)$ are constants. After using

TABLE 1

BASIC DATA

| Case | Model | $\tau_0$ | $\omega$ | $\mu_0$ |
|---|---|---|---|---|
| 1 | Haze L | 1 | 1 | 1 |
| 2 | Haze L | 1 | 0.9 | 1 |
| 3 | Haze L | 1 | 0.9 | 0.5 |
| 4 | Cloud $C_1$ | 64 | 1 | 1 |
| 5 | Cloud $C_1$ | 64 | 0.9 | 1 |

equation (50) for $l = 2, 4, \ldots, N-1$, and expressing $g_0(\xi)$ in terms of $g_2(\xi)$ we conclude that the square of the bounded zeros of $g_{N+1}(\xi)$ for $\omega = 1$ are the eigenvalues of the tridiagonal matrix $\boldsymbol{F}$ of order $(N-1)/2$ given, for $\alpha \geq 1$, by

$$F_{\alpha,\alpha+1} = E_{\alpha+1,\alpha+2} , \tag{52a}$$

$$F_{\alpha,\alpha} = E_{\alpha+1,\alpha+1} - \left(\frac{4}{h_1 h_2}\right)\delta_{\alpha,1} , \tag{52b}$$

and

$$F_{\alpha+1,\alpha} = E_{\alpha+2,\alpha+1} . \tag{52c}$$

In this work we have used a FORTRAN program in the EISPACK program package (Smith *et al.* 1976) to compute the required eigenvalues, and we have used equation (16), in either a forward ($\xi \leq 1$) or backward ($\xi > 1$) manner to compute the polynomials $g_l^m(\xi)$ for $\xi = \mu_0$ or $\xi \in \{\xi_j\}$. In contrast to the opinion of Stamnes and Swanson (1981), we do not believe that standard subroutine packages that can yield the eigenvectors (as well as the eigenvalues) have rendered obsolete the use of Chandrasekhar's (1950) recursion formula. In fact, we believe that the recursion formula, when used correctly (Gautschi 1967), constitutes a fast and particularly accurate procedure for computing the polynomials $g_l^m(\xi)$, $\xi = \mu_0$ or $\xi \in \{\xi_j\}$.

For our numerical examples we consider the test problems recently posed (Lenoble 1977) by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics. The five test cases are listed in Table 1, and in Tables 2 and 3 we list the Legendre coefficients computed by de Haan (1982) and Karp (1982). It is worthwhile to note that it is a particularly challenging task to compute accurately the Mie scattering law averaged over a distribution of particle sizes. Thus both de Haan who used his own code and Karp who modified a code of Dave (1970) invested considerable computer time to obtain the Legendre coefficients given in Tables 2 and 3. The calculations of de Haan and Karp agreed with one another to within $\pm 1$ in the last digits given in the tables, and both de Haan (1982) and Karp (1982) are of the opinion that Tables 2 and 3 are correct to within $\pm 1$ in the last digits given.

We now write

$$I(\tau, \mu, \phi) = I_*(\tau, \mu, \phi) + \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0)e^{-\tau/\mu} \tag{53}$$

TABLE 2

THE LEGENDRE COEFFICIENTS FOR THE HAZE L PHASE FUNCTION

| $l$ | $\beta_l$ | $\beta_{l+16}$ | $\beta_{l+32}$ | $\beta_{l+48}$ | $\beta_{l+64}$ | $\beta_{l+80}$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0.34688 | 0.01711 | 0.00107 | 0.00008 | 0.00001 |
| 1 | 2.41260 | 0.28351 | 0.01298 | 0.00082 | 0.00006 | 0.00001 |
| 2 | 3.23047 | 0.23317 | 0.01198 | 0.00077 | 0.00006 | 0.00001 |
| 3 | 3.37296 | 0.18963 | 0.00904 | 0.00059 | 0.00005 | |
| 4 | 3.23150 | 0.15788 | 0.00841 | 0.00055 | 0.00004 | |
| 5 | 2.89350 | 0.12739 | 0.00634 | 0.00043 | 0.00004 | |
| 6 | 2.49594 | 0.10762 | 0.00592 | 0.00040 | 0.00003 | |
| 7 | 2.11361 | 0.08597 | 0.00446 | 0.00031 | 0.00003 | |
| 8 | 1.74812 | 0.07381 | 0.00418 | 0.00029 | 0.00002 | |
| 9 | 1.44692 | 0.05828 | 0.00316 | 0.00023 | 0.00002 | |
| 10 | 1.17714 | 0.05089 | 0.00296 | 0.00021 | 0.00002 | |
| 11 | 0.96643 | 0.03971 | 0.00225 | 0.00017 | 0.00001 | |
| 12 | 0.78237 | 0.03524 | 0.00210 | 0.00015 | 0.00001 | |
| 13 | 0.64114 | 0.02720 | 0.00160 | 0.00012 | 0.00001 | |
| 14 | 0.51966 | 0.02451 | 0.00150 | 0.00011 | 0.00001 | |
| 15 | 0.42563 | 0.01874 | 0.00115 | 0.00009 | 0.00001 | |

TABLE 3

THE LEGENDRE COEFFICIENTS FOR THE CLOUD $C_1$ PHASE FUNCTION

| $l$ | $\beta_l$ | $\beta_{l+35}$ | $\beta_{l+70}$ | $\beta_{l+105}$ | $\beta_{l+140}$ | $\beta_{l+175}$ | $\beta_{l+210}$ | $\beta_{l+245}$ | $\beta_{l+280}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 19.884 | 16.144 | 6.990 | 2.025 | 0.440 | 0.079 | 0.012 | 0.002 |
| 1 | 2.544 | 20.024 | 15.883 | 6.785 | 1.940 | 0.422 | 0.074 | 0.011 | 0.002 |
| 2 | 3.883 | 20.145 | 15.606 | 6.573 | 1.869 | 0.401 | 0.071 | 0.011 | 0.001 |
| 3 | 4.568 | 20.251 | 15.338 | 6.377 | 1.790 | 0.384 | 0.067 | 0.010 | 0.001 |
| 4 | 5.235 | 20.330 | 15.058 | 6.173 | 1.723 | 0.364 | 0.064 | 0.009 | 0.001 |
| 5 | 5.887 | 20.401 | 14.784 | 5.986 | 1.649 | 0.349 | 0.060 | 0.009 | 0.001 |
| 6 | 6.457 | 20.444 | 14.501 | 5.790 | 1.588 | 0.331 | 0.057 | 0.008 | 0.001 |
| 7 | 7.177 | 20.477 | 14.225 | 5.612 | 1.518 | 0.317 | 0.054 | 0.008 | 0.001 |
| 8 | 7.859 | 20.489 | 13.941 | 5.424 | 1.461 | 0.301 | 0.052 | 0.008 | 0.001 |
| 9 | 8.494 | 20.483 | 13.662 | 5.255 | 1.397 | 0.288 | 0.049 | 0.007 | 0.001 |
| 10 | 9.286 | 20.467 | 13.378 | 5.075 | 1.344 | 0.273 | 0.047 | 0.007 | 0.001 |
| 11 | 9.856 | 20.427 | 13.098 | 4.915 | 1.284 | 0.262 | 0.044 | 0.006 | 0.001 |
| 12 | 10.615 | 20.382 | 12.816 | 4.744 | 1.235 | 0.248 | 0.042 | 0.006 | 0.001 |
| 13 | 11.229 | 20.310 | 12.536 | 4.592 | 1.179 | 0.238 | 0.039 | 0.006 | 0.001 |
| 14 | 11.851 | 20.236 | 12.257 | 4.429 | 1.134 | 0.225 | 0.038 | 0.005 | 0.001 |
| 15 | 12.503 | 20.136 | 11.978 | 4.285 | 1.082 | 0.215 | 0.035 | 0.005 | 0.001 |
| 16 | 13.058 | 20.036 | 11.703 | 4.130 | 1.040 | 0.204 | 0.034 | 0.005 | 0.001 |
| 17 | 13.626 | 19.909 | 11.427 | 3.994 | 0.992 | 0.195 | 0.032 | 0.005 | 0.001 |
| 18 | 14.209 | 19.785 | 11.156 | 3.847 | 0.954 | 0.185 | 0.030 | 0.004 | 0.001 |
| 19 | 14.660 | 19.632 | 10.884 | 3.719 | 0.909 | 0.177 | 0.029 | 0.004 | 0.001 |
| 20 | 15.231 | 19.486 | 10.618 | 3.580 | 0.873 | 0.167 | 0.027 | 0.004 | |
| 21 | 15.641 | 19.311 | 10.350 | 3.459 | 0.832 | 0.160 | 0.026 | 0.004 | |
| 22 | 16.126 | 19.145 | 10.090 | 3.327 | 0.799 | 0.151 | 0.024 | 0.003 | |
| 23 | 16.539 | 18.949 | 9.827 | 3.214 | 0.762 | 0.145 | 0.023 | 0.003 | |
| 24 | 16.934 | 18.764 | 9.574 | 3.090 | 0.731 | 0.137 | 0.022 | 0.003 | |
| 25 | 17.325 | 18.551 | 9.318 | 2.983 | 0.696 | 0.131 | 0.021 | 0.003 | |
| 26 | 17.673 | 18.348 | 9.072 | 2.866 | 0.668 | 0.124 | 0.020 | 0.003 | |
| 27 | 17.999 | 18.119 | 8.822 | 2.766 | 0.636 | 0.118 | 0.018 | 0.003 | |
| 28 | 18.329 | 17.901 | 8.584 | 2.656 | 0.610 | 0.112 | 0.018 | 0.002 | |
| 29 | 18.588 | 17.659 | 8.340 | 2.562 | 0.581 | 0.107 | 0.017 | 0.002 | |
| 30 | 18.885 | 17.428 | 8.110 | 2.459 | 0.557 | 0.101 | 0.016 | 0.002 | |
| 31 | 19.103 | 17.174 | 7.874 | 2.372 | 0.530 | 0.097 | 0.015 | 0.002 | |
| 32 | 19.345 | 16.931 | 7.652 | 2.274 | 0.508 | 0.091 | 0.014 | 0.002 | |
| 33 | 19.537 | 16.668 | 7.424 | 2.193 | 0.483 | 0.087 | 0.013 | 0.002 | |
| 34 | 19.721 | 16.415 | 7.211 | 2.102 | 0.463 | 0.082 | 0.013 | 0.002 | |

TABLE 4

INTENSITY $I_*(\tau, \mu, \phi)$ FOR THE HAZE L PHASE FUNCTION WITH $\tau_0 = 1$, $\omega = 1$, AND $\mu_0 = 1$

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 3.6145(−2) | 3.4339(−2) | 3.2511(−2) | 2.8812(−2) | 1.7629(−2) | 8.5259(−3) | |
| −0.9 | 3.9782(−2) | 3.7872(−2) | 3.5921(−2) | 3.1930(−2) | 1.9620(−2) | 9.4573(−3) | |
| −0.8 | 4.2731(−2) | 4.0841(−2) | 3.8873(−2) | 3.4768(−2) | 2.1602(−2) | 1.0396(−2) | |
| −0.7 | 4.8005(−2) | 4.6132(−2) | 4.4131(−2) | 3.9829(−2) | 2.5248(−2) | 1.2217(−2) | |
| −0.6 | 5.5821(−2) | 5.4043(−2) | 5.2059(−2) | 4.7599(−2) | 3.1184(−2) | 1.5362(−2) | |
| −0.5 | 6.6094(−2) | 6.4630(−2) | 6.2845(−2) | 5.8497(−2) | 4.0274(−2) | 2.0562(−2) | |
| −0.4 | 7.8148(−2) | 7.7440(−2) | 7.6251(−2) | 7.2705(−2) | 5.3730(−2) | 2.9128(−2) | |
| −0.3 | 8.9968(−2) | 9.0771(−2) | 9.0878(−2) | 8.9471(−2) | 7.2964(−2) | 4.3469(−2) | |
| −0.2 | 9.7081(−2) | 1.0042(−1) | 1.0279(−1) | 1.0551(−1) | 9.8378(−2) | 6.7995(−2) | |
| −0.1 | 9.2932(−2) | 9.9819(−2) | 1.0519(−1) | 1.1350(−1) | 1.2404(−1) | 1.0840(−1) | |
| −0.0 | 6.99 (−2) | 8.4667(−2) | 9.4166(−2) | 1.0873(−1) | 1.3576(−1) | 1.4278(−1) | |
| 0.0 | | 8.4667(−2) | 9.4166(−2) | 1.0873(−1) | 1.3576(−1) | 1.4278(−1) | 1.15 (−1) |
| 0.1 | | 2.9541(−2) | 5.2434(−2) | 8.4565(−2) | 1.3510(−1) | 1.5611(−1) | 1.5697(−1) |
| 0.2 | | 1.6490(−2) | 3.2281(−2) | 6.0753(−2) | 1.2435(−1) | 1.5893(−1) | 1.7682(−1) |
| 0.3 | | 1.2342(−2) | 2.4849(−2) | 4.9397(−2) | 1.1481(−1) | 1.5794(−1) | 1.8830(−1) |
| 0.4 | | 1.1188(−2) | 2.2645(−2) | 4.5755(−2) | 1.1227(−1) | 1.6086(−1) | 2.0002(−1) |
| 0.5 | | 1.1796(−2) | 2.3791(−2) | 4.8000(−2) | 1.1908(−1) | 1.7319(−1) | 2.1963(−1) |
| 0.6 | | 1.4205(−2) | 2.8458(−2) | 5.6873(−2) | 1.3905(−1) | 2.0144(−1) | 2.5598(−1) |
| 0.7 | | 1.9583(−2) | 3.8925(−2) | 7.6745(−2) | 1.8200(−1) | 2.5899(−1) | 3.2512(−1) |
| 0.8 | | 3.1953(−2) | 6.2943(−2) | 1.2204(−1) | 2.7718(−1) | 3.8277(−1) | 4.6866(−1) |
| 0.9 | | 6.8727(−2) | 1.3392(−1) | 2.5426(−1) | 5.4460(−1) | 7.1945(−1) | 8.4608(−1) |
| 1 | | 3.6494(−1) | 7.0027(−1) | 1.2895 | 2.5226 | 3.0932 | 3.3809 |

TABLE 5

INTENSITY $I_*(\tau, \mu, \phi)$ FOR THE HAZE L PHASE FUNCTION WITH $\tau_0 = 1$, $\omega = 0.9$, AND $\mu_0 = 1$

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 2.7972(−2) | 2.6583(−2) | 2.5179(−2) | 2.2342(−2) | 1.3752(−2) | 6.7044(−3) | |
| −0.9 | 3.0180(−2) | 2.8743(−2) | 2.7276(−2) | 2.4282(−2) | 1.5037(−2) | 7.3279(−3) | |
| −0.8 | 3.1448(−2) | 3.0071(−2) | 2.8641(−2) | 2.5662(−2) | 1.6096(−2) | 7.8550(−3) | |
| −0.7 | 3.4384(−2) | 3.3056(−2) | 3.1641(−2) | 2.8606(−2) | 1.8314(−2) | 9.0026(−3) | |
| −0.6 | 3.9131(−2) | 3.7891(−2) | 3.6513(−2) | 3.3428(−2) | 2.2098(−2) | 1.1062(−2) | |
| −0.5 | 4.5638(−2) | 4.4617(−2) | 4.3384(−2) | 4.0403(−2) | 2.8009(−2) | 1.4514(−2) | |
| −0.4 | 5.3511(−2) | 5.2985(−2) | 5.2141(−2) | 4.9686(−2) | 3.6854(−2) | 2.0231(−2) | |
| −0.3 | 6.1542(−2) | 6.1991(−2) | 6.1979(−2) | 6.0886(−2) | 4.9616(−2) | 2.9828(−2) | |
| −0.2 | 6.6956(−2) | 6.9056(−2) | 7.0500(−2) | 7.2037(−2) | 6.6697(−2) | 4.6300(−2) | |
| −0.1 | 6.5529(−2) | 7.0004(−2) | 7.3432(−2) | 7.8590(−2) | 8.4463(−2) | 7.3611(−2) | |
| −0.0 | 5.18 (−2) | 6.1710(−2) | 6.8016(−2) | 7.7466(−2) | 9.3998(−2) | 9.7484(−2) | |
| 0.0 | | 6.1710(−2) | 6.8016(−2) | 7.7466(−2) | 9.3998(−2) | 9.7484(−2) | 7.94 (−2) |
| 0.1 | | 2.2495(−2) | 3.9518(−2) | 6.2653(−2) | 9.6254(−2) | 1.0872(−1) | 1.0819(−1) |
| 0.2 | | 1.3101(−2) | 2.5340(−2) | 4.6773(−2) | 9.1592(−2) | 1.1381(−1) | 1.2421(−1) |
| 0.3 | | 1.0194(−2) | 2.0270(−2) | 3.9472(−2) | 8.7467(−2) | 1.1662(−1) | 1.3571(−1) |
| 0.4 | | 9.5290(−3) | 1.9068(−2) | 3.7770(−2) | 8.8332(−2) | 1.2250(−1) | 1.4827(−1) |
| 0.5 | | 1.0264(−2) | 2.0502(−2) | 4.0649(−2) | 9.6418(−2) | 1.3582(−1) | 1.6752(−1) |
| 0.6 | | 1.2529(−2) | 2.4909(−2) | 4.9066(−2) | 1.1534(−1) | 1.6223(−1) | 2.0070(−1) |
| 0.7 | | 1.7417(−2) | 3.4415(−2) | 6.7081(−2) | 1.5398(−1) | 2.1356(−1) | 2.6167(−1) |
| 0.8 | | 2.8562(−2) | 5.6020(−2) | 1.0770(−1) | 2.3849(−1) | 3.2255(−1) | 3.8692(−1) |
| 0.9 | | 6.1633(−2) | 1.1976(−1) | 2.2612(−1) | 4.7610(−1) | 6.1970(−1) | 7.1775(−1) |
| 1 | | 3.2812(−1) | 6.2907(−1) | 1.1563 | 2.2484 | 2.7415 | 2.9777 |

where

$$I_*(\tau, \mu, \phi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) I_*^m(\tau, \mu) \cos [m(\phi - \phi_0)] , \tag{54}$$

and in Tables 4–10 we list our final results which we believe to be correct to within ±1 in the last digits given. We note that Tables 4, 5, 9, and 10 are based on $N = 499$ calculations; Tables 6, 7, and 8 are based on $N = 499$ for the Fourier components $m = 0$–5 and reduced orders for $m = 6$–82. In Table 11 we list the fluxes $q_+(\tau)$, $q_-(\tau)$, and $q(\tau)$ obtained from $N = 299$ calculations and correct, we believe, to within ±1 in the last digits given.

To assess the accuracy of our results given in Tables 4–11 we have noted apparent convergence as $N \to 499$. In addition we found that all our results relevant to the haze problems agreed to within ±1 in the last digit of all figures given in our tables with $F_N$ results of Garcia and Siewert (1983) and for $\tau = 0$, $\tau = \tau_0/2$, and $\tau = \tau_0$ with doubling results of de Haan

TABLE 6

INTENSITY $I_*(\tau, \mu, \phi)$ FOR THE HAZE L PHASE FUNCTION WITH $\tau_0 = 1$, $\omega = 0.9$, $\mu_0 = 0.5$, AND $\phi - \phi_0 = 0$

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 2.2819(−2) | 2.1417(−2) | 1.9992(−2) | 1.7157(−2) | 9.3472(−3) | 4.0251(−3) | |
| −0.9 | 4.1112(−2) | 3.8628(−2) | 3.6056(−2) | 3.0870(−2) | 1.6454(−2) | 6.8105(−3) | |
| −0.8 | 6.4998(−2) | 6.1297(−2) | 5.7396(−2) | 4.9400(−2) | 2.6608(−2) | 1.1021(−2) | |
| −0.7 | 9.9944(−2) | 9.4742(−2) | 8.9121(−2) | 7.7351(−2) | 4.2613(−2) | 1.7933(−2) | |
| −0.6 | 1.5099(−1) | 1.4408(−1) | 1.3634(−1) | 1.1966(−1) | 6.8117(−2) | 2.9506(−2) | |
| −0.5 | 2.2477(−1) | 2.1626(−1) | 2.0617(−1) | 1.8348(−1) | 1.0907(−1) | 4.9312(−2) | |
| −0.4 | 3.2933(−1) | 3.2018(−1) | 3.0804(−1) | 2.7889(−1) | 1.7521(−1) | 8.4105(−2) | |
| −0.3 | 4.7253(−1) | 4.6552(−1) | 4.5295(−1) | 4.1872(−1) | 2.8208(−1) | 1.4724(−1) | |
| −0.2 | 6.5683(−1) | 6.5839(−1) | 6.4949(−1) | 6.1521(−1) | 4.5147(−1) | 2.6618(−1) | |
| −0.1 | 8.7032(−1) | 8.9452(−1) | 8.9745(−1) | 8.7217(−1) | 6.9751(−1) | 4.9155(−1) | |
| −0.0 | 1.032 | 1.1478 | 1.1880 | 1.1944 | 1.0087 | 7.9575(−1) | |
| 0.0 | | 1.1478 | 1.1880 | 1.1944 | 1.0087 | 7.9575(−1) | 5.24 (−1) |
| 0.1 | | 6.0754(−1) | 9.9859(−1) | 1.3847 | 1.4118 | 1.1581 | 8.7647(−1) |
| 0.2 | | 5.0888(−1) | 9.0722(−1) | 1.4379 | 1.8263 | 1.6127 | 1.2914 |
| 0.3 | | 5.4999(−1) | 9.9990(−1) | 1.6483 | 2.3079 | 2.1559 | 1.8055 |
| 0.4 | | 6.3861(−1) | 1.1693 | 1.9578 | 2.8730 | 2.7782 | 2.4010 |
| 0.5 | | 6.3427(−1) | 1.1687 | 1.9832 | 3.0333 | 3.0343 | 2.7105 |
| 0.6 | | 4.1826(−1) | 7.7809(−1) | 1.3462 | 2.1840 | 2.2951 | 2.1537 |
| 0.7 | | 2.0488(−1) | 3.8581(−1) | 6.8393(−1) | 1.1933 | 1.3320 | 1.3268 |
| 0.8 | | 8.6475(−2) | 1.6515(−1) | 3.0101(−1) | 5.6958(−1) | 6.7861(−1) | 7.1993(−1) |
| 0.9 | | 3.1366(−2) | 6.0901(−2) | 1.1461(−1) | 2.3734(−1) | 3.0321(−1) | 3.4348(−1) |
| 1 | | 5.0711(−3) | 1.0119(−2) | 2.0044(−2) | 4.7608(−2) | 6.7349(−2) | 8.3758(−2) |

TABLE 7

INTENSITY $I_*(\tau, \mu, \phi)$ FOR THE HAZE L PHASE FUNCTION WITH $\tau_0 = 1$, $\omega = 0.9$, $\mu_0 = 0.5$, AND $\phi - \phi_0 = \pi/2$

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 2.2819(−2) | 2.1417(−2) | 1.9992(−2) | 1.7157(−2) | 9.3472(−3) | 4.0251(−3) | |
| −0.9 | 2.6986(−2) | 2.5400(−2) | 2.3770(−2) | 2.0489(−2) | 1.1251(−2) | 4.8400(−3) | |
| −0.8 | 3.2325(−2) | 3.0543(−2) | 2.8684(−2) | 2.4882(−2) | 1.3858(−2) | 5.9869(−3) | |
| −0.7 | 3.9091(−2) | 3.7129(−2) | 3.5036(−2) | 3.0662(−2) | 1.7462(−2) | 7.6343(−3) | |
| −0.6 | 4.7519(−2) | 4.5445(−2) | 4.3159(−2) | 3.8227(−2) | 2.2493(−2) | 1.0059(−2) | |
| −0.5 | 5.7696(−2) | 5.5680(−2) | 5.3327(−2) | 4.7996(−2) | 2.9570(−2) | 1.3724(−2) | |
| −0.4 | 6.9292(−2) | 6.7684(−2) | 6.5550(−2) | 6.0259(−2) | 3.9548(−2) | 1.9442(−2) | |
| −0.3 | 8.0972(−2) | 8.0408(−2) | 7.9037(−2) | 7.4715(−2) | 5.3455(−2) | 2.8676(−2) | |
| −0.2 | 8.9408(−2) | 9.0899(−2) | 9.1160(−2) | 8.9386(−2) | 7.1822(−2) | 4.4111(−2) | |
| −0.1 | 8.8632(−2) | 9.3608(−2) | 9.6567(−2) | 9.9164(−2) | 9.1541(−2) | 6.9449(−2) | |
| −0.0 | 6.76 (−2) | 8.1602(−2) | 8.9222(−2) | 9.8376(−2) | 1.0348(−1) | 9.3237(−2) | |
| 0.0 | | 8.1602(−2) | 8.9222(−2) | 9.8376(−2) | 1.0348(−1) | 9.3237(−2) | 6.30 (−2) |
| 0.1 | | 2.7447(−2) | 4.8362(−2) | 7.5757(−2) | 1.0462(−1) | 1.0439(−1) | 8.9590(−2) |
| 0.2 | | 1.4133(−2) | 2.7594(−2) | 5.0906(−2) | 9.2887(−2) | 1.0468(−1) | 1.0118(−1) |
| 0.3 | | 9.2687(−3) | 1.8729(−2) | 3.6874(−2) | 7.8547(−2) | 9.7400(−2) | 1.0299(−1) |
| 0.4 | | 6.9663(−3) | 1.4241(−2) | 2.8824(−2) | 6.6803(−2) | 8.8295(−2) | 9.9518(−2) |
| 0.5 | | 5.7571(−3) | 1.1785(−2) | 2.4072(−2) | 5.8234(−2) | 8.0115(−2) | 9.4319(−2) |
| 0.6 | | 5.1103(−3) | 1.0425(−2) | 2.1282(−2) | 5.2383(−2) | 7.3754(−2) | 8.9330(−2) |
| 0.7 | | 4.7970(−3) | 9.7344(−3) | 1.9763(−2) | 4.8720(−2) | 6.9368(−2) | 8.5463(−2) |
| 0.8 | | 4.7111(−3) | 9.5050(−3) | 1.9150(−2) | 4.6840(−2) | 6.6882(−2) | 8.3120(−2) |
| 0.9 | | 4.8064(−3) | 9.6424(−3) | 1.9264(−2) | 4.6500(−2) | 6.6213(−2) | 8.2499(−2) |
| 1 | | 5.0711(−3) | 1.0119(−2) | 2.0044(−2) | 4.7608(−2) | 6.7349(−2) | 8.3758(−2) |

(1982); for the cloud problems we found again for $\tau = 0$, $\tau = \tau_0/2$ and $\tau = \tau_0$ the same agreement with doubling results of de Haan (1982). In studying the convergence of our spherical harmonics solution as $N$ increases, we have noted, not surprisingly, that the boundary results for $\mu \to 0$ are the slowest to converge.

Although the special case $\omega = 1$ has been considered separately here, we wish to remark that we have found the given formulation of the spherical harmonics method to be computationally sound for $\omega$ near to one; e.g., we encountered no loss of accuracy in our calculations for either the haze problem or the cloud problem for the case $\omega = 1\text{–}10^{-12}$. We note also that the accuracy of the present formulation of the classical spherical harmonics method does not deteriorate with increasing $\tau_0$; this clearly represents an advantage over an alternative formulation (Cuzzi, Ackerman, and Helmle 1982), which, as reported, is not computationally viable for sufficiently large $\tau_0$.

In conclusion we wish to point out that, in this study, the classical spherical harmonics method has proved, for the considered beam problems, to be an efficient solution technique that can yield, in high order and with modest computational

TABLE 8

INTENSITY $I_*(\tau, \mu, \phi)$ FOR THE HAZE L PHASE FUNCTION WITH $\tau_0 = 1$, $\omega = 0.9$, $\mu_0 = 0.5$, AND $\phi - \phi_0 = \pi$

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 2.2819(−2) | 2.1417(−2) | 1.9992(−2) | 1.7157(−2) | 9.3472(−3) | 4.0251(−3) | |
| −0.9 | 2.6185(−2) | 2.4622(−2) | 2.3041(−2) | 1.9904(−2) | 1.1159(−2) | 4.9719(−3) | |
| −0.8 | 3.2037(−2) | 3.0173(−2) | 2.8283(−2) | 2.4515(−2) | 1.3893(−2) | 6.2445(−3) | |
| −0.7 | 3.7482(−2) | 3.5436(−2) | 3.3333(−2) | 2.9081(−2) | 1.6759(−2) | 7.6101(−3) | |
| −0.6 | 4.1214(−2) | 3.9270(−2) | 3.7201(−2) | 3.2868(−2) | 1.9528(−2) | 9.0230(−3) | |
| −0.5 | 4.9946(−2) | 4.7780(−2) | 4.5434(−2) | 4.0452(−2) | 2.4660(−2) | 1.1664(−2) | |
| −0.4 | 5.5013(−2) | 5.3239(−2) | 5.1157(−2) | 4.6434(−2) | 2.9881(−2) | 1.4774(−2) | |
| −0.3 | 6.4493(−2) | 6.3013(−2) | 6.1080(−2) | 5.6413(−2) | 3.8598(−2) | 2.0438(−2) | |
| −0.2 | 7.0886(−2) | 7.0367(−2) | 6.9145(−2) | 6.5532(−2) | 4.9114(−2) | 2.9204(−2) | |
| −0.1 | 6.9559(−2) | 7.1089(−2) | 7.1378(−2) | 7.0121(−2) | 5.9030(−2) | 4.2527(−2) | |
| −0.0 | 5.48 (−2) | 6.2332(−2) | 6.5528(−2) | 6.8120(−2) | 6.3940(−2) | 5.3941(−2) | |
| 0.0 | | 6.2332(−2) | 6.5528(−2) | 6.8120(−2) | 6.3940(−2) | 5.3941(−2) | 3.42 (−2) |
| 0.1 | | 1.9992(−2) | 3.4255(−2) | 5.1006(−2) | 6.2501(−2) | 5.8017(−2) | 4.6680(−2) |
| 0.2 | | 9.4112(−3) | 1.7986(−2) | 3.1893(−2) | 5.2723(−2) | 5.5654(−2) | 5.0696(−2) |
| 0.3 | | 5.5298(−3) | 1.1012(−2) | 2.1054(−2) | 4.1504(−2) | 4.8798(−2) | 4.9126(−2) |
| 0.4 | | 3.6730(−3) | 7.4517(−3) | 1.4805(−2) | 3.2470(−2) | 4.1253(−2) | 4.4801(−2) |
| 0.5 | | 2.6731(−3) | 5.4640(−3) | 1.1069(−2) | 2.5926(−2) | 3.4795(−2) | 3.9991(−2) |
| 0.6 | | 2.1101(−3) | 4.3193(−3) | 8.8224(−3) | 2.1493(−2) | 2.9972(−2) | 3.5933(−2) |
| 0.7 | | 1.8110(−3) | 3.6999(−3) | 7.5687(−3) | 1.8839(−2) | 2.6960(−2) | 3.3333(−2) |
| 0.8 | | 1.7224(−3) | 3.5058(−3) | 7.1536(−3) | 1.7961(−2) | 2.6113(−2) | 3.2972(−2) |
| 0.9 | | 1.9324(−3) | 3.9132(−3) | 7.9385(−3) | 1.9873(−2) | 2.9053(−2) | 3.7075(−2) |
| 1 | | 5.0711(−3) | 1.0119(−2) | 2.0044(−2) | 4.7608(−2) | 6.7349(−2) | 8.3758(−2) |

TABLE 9

INTENSITY $I_*(\tau, \mu, \phi)$ FOR THE CLOUD $C_1$ PHASE FUNCTION WITH $\tau_0 = 64$, $\omega = 1$, AND $\mu_0 = 1$

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 1.0637 | 1.0062 | 9.6321(−1) | 8.5824(−1) | 5.2453(−1) | 2.4600(−1) | |
| −0.9 | 9.5309(−1) | 9.9566(−1) | 9.6972(−1) | 8.6939(−1) | 5.3599(−1) | 2.5740(−1) | |
| −0.8 | 9.5408(−1) | 9.9828(−1) | 9.7777(−1) | 8.8053(−1) | 5.4744(−1) | 2.6884(−1) | |
| −0.7 | 8.8254(−1) | 9.8851(−1) | 9.8352(−1) | 8.9157(−1) | 5.5890(−1) | 2.8028(−1) | |
| −0.6 | 8.2471(−1) | 9.7910(−1) | 9.8890(−1) | 9.0256(−1) | 5.7036(−1) | 2.9173(−1) | |
| −0.5 | 7.7261(−1) | 9.6977(−1) | 9.9399(−1) | 9.1350(−1) | 5.8181(−1) | 3.0319(−1) | |
| −0.4 | 7.1144(−1) | 9.5800(−1) | 9.9833(−1) | 9.2438(−1) | 5.9327(−1) | 3.1465(−1) | |
| −0.3 | 6.4031(−1) | 9.4343(−1) | 1.0018 | 9.3518(−1) | 6.0472(−1) | 3.2611(−1) | |
| −0.2 | 5.5848(−1) | 9.2583(−1) | 1.0042 | 9.4589(−1) | 6.1618(−1) | 3.3756(−1) | |
| −0.1 | 4.5873(−1) | 9.0459(−1) | 1.0055 | 9.5651(−1) | 6.2763(−1) | 3.4902(−1) | |
| −0.0 | 2.5 (−1) | 8.7952(−1) | 1.0055 | 9.6701(−1) | 6.3909(−1) | 3.6048(−1) | |
| 0.0 | | 8.7952(−1) | 1.0055 | 9.6701(−1) | 6.3909(−1) | 3.6048(−1) | 3.9 (−2) |
| 0.1 | | 8.5070(−1) | 1.0042 | 9.7741(−1) | 6.5054(−1) | 3.7194(−1) | 7.2039(−2) |
| 0.2 | | 8.1872(−1) | 1.0019 | 9.8770(−1) | 6.6200(−1) | 3.8339(−1) | 8.8949(−2) |
| 0.3 | | 7.8522(−1) | 9.9902(−1) | 9.9794(−1) | 6.7345(−1) | 3.9485(−1) | 1.0414(−1) |
| 0.4 | | 7.5460(−1) | 9.9679(−1) | 1.0082 | 6.8491(−1) | 4.0631(−1) | 1.1838(−1) |
| 0.5 | | 7.3516(−1) | 9.9760(−1) | 1.0188 | 6.9636(−1) | 4.1776(−1) | 1.3199(−1) |
| 0.6 | | 7.3765(−1) | 1.0060 | 1.0300 | 7.0782(−1) | 4.2922(−1) | 1.4514(−1) |
| 0.7 | | 7.7792(−1) | 1.0301 | 1.0428 | 7.1927(−1) | 4.4068(−1) | 1.5796(−1) |
| 0.8 | | 8.8705(−1) | 1.0859 | 1.0589 | 7.3073(−1) | 4.5213(−1) | 1.7054(−1) |
| 0.9 | | 1.1539 | 1.2141 | 1.0827 | 7.4219(−1) | 4.6359(−1) | 1.8295(−1) |
| 1 | | 8.0746(+1) | 1.1786(+1) | 1.2606 | 7.5366(−1) | 4.7504(−1) | 1.9523(−1) |

effort, a solution for the radiation intensity that is continuous in all variables, that is accurate generally to five significant figures, and that is valid for essentially all $\tau_0$ and all $\omega$. As discussed elsewhere (Benassi, Cotta, and Siewert 1983), the method readily accommodates more general boundary conditions and internal emission. We note that in contrast to the findings of Garcia and Siewert (1982) in regard to a problem less challenging than those solved here, we found that the Marshak (Dave 1975; Davison 1957) projection scheme yielded good results in high order whereas the projection scheme based on the functions $\mu P_\alpha(2\mu - 1)$ became computationally unstable in high order; we also found that the particular solution used here yielded better numerical results than the exact particular solution reported earlier (Garcia and Siewert 1982).

We have focused our effort here on showing how the classical spherical harmonics method can be used in high order to solve particularly accurately five challenging test problems (Lenoble 1977). We conclude by pointing out that the method has been used (Benassi, Cotta and Siewert 1983) to solve problems in radiative transfer that include both specularly and diffusely reflecting boundaries and internal emission. The method can also readily be used, as was the $F_N$ method (Devaux *et al.* 1979), to solve radiative transfer problems in many-layer systems that can be used to represent inhomogeneous media.

TABLE 10

INTENSITY $I_*(\tau, \mu, \phi)$ FOR THE CLOUD $C_1$ PHASE FUNCTION WITH $\tau_0 = 64$, $\omega = 0.9$, AND $\mu_0 = 1$

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 2.0977(−1) | 8.6613(−2) | 4.1344(−2) | 9.5111(−3) | 1.0827(−4) | 2.5786(−6) | |
| −0.9 | 1.3306(−1) | 7.9916(−2) | 4.1642(−2) | 9.8886(−3) | 1.1300(−4) | 2.6919(−6) | |
| −0.8 | 1.5586(−1) | 8.4979(−2) | 4.3752(−2) | 1.0415(−2) | 1.1927(−4) | 2.8417(−6) | |
| −0.7 | 1.2248(−1) | 8.3045(−2) | 4.5315(−2) | 1.1071(−2) | 1.2735(−4) | 3.0346(−6) | |
| −0.6 | 1.0613(−1) | 8.3817(−2) | 4.7706(−2) | 1.1909(−2) | 1.3756(−4) | 3.2784(−6) | |
| −0.5 | 1.0037(−1) | 8.7263(−2) | 5.1103(−2) | 1.2966(−2) | 1.5030(−4) | 3.5825(−6) | |
| −0.4 | 9.3636(−2) | 9.1882(−2) | 5.5403(−2) | 1.4275(−2) | 1.6603(−4) | 3.9577(−6) | |
| −0.3 | 8.6110(−2) | 9.7787(−2) | 6.0730(−2) | 1.5883(−2) | 1.8530(−4) | 4.4173(−6) | |
| −0.2 | 7.8467(−2) | 1.0521(−1) | 6.7252(−2) | 1.7841(−2) | 2.0877(−4) | 4.9770(−6) | |
| −0.1 | 6.7611(−2) | 1.1404(−1) | 7.5125(−2) | 2.0215(−2) | 2.3724(−4) | 5.6560(−6) | |
| −0.0 | 4.1 (−2) | 1.2443(−1) | 8.4571(−2) | 2.3084(−2) | 2.7169(−4) | 6.4777(−6) | |
| 0.0 | | 1.2443(−1) | 8.4571(−2) | 2.3084(−2) | 2.7169(−4) | 6.4777(−6) | 7.7 (−8) |
| 0.1 | | 1.3660(−1) | 9.5875(−2) | 2.6545(−2) | 3.1332(−4) | 7.4704(−6) | 1.3642(−7) |
| 0.2 | | 1.5095(−1) | 1.0942(−1) | 3.0718(−2) | 3.6359(−4) | 8.6693(−6) | 1.7437(−7) |
| 0.3 | | 1.6827(−1) | 1.2573(−1) | 3.5756(−2) | 4.2432(−4) | 1.0118(−5) | 2.1526(−7) |
| 0.4 | | 1.9022(−1) | 1.4562(−1) | 4.1853(−2) | 4.9778(−4) | 1.1869(−5) | 2.6167(−7) |
| 0.5 | | 2.2019(−1) | 1.7042(−1) | 4.9271(−2) | 5.8678(−4) | 1.3992(−5) | 3.1578(−7) |
| 0.6 | | 2.6371(−1) | 2.0243(−1) | 5.8373(−2) | 6.9491(−4) | 1.6570(−5) | 3.7993(−7) |
| 0.7 | | 3.3127(−1) | 2.4609(−1) | 6.9714(−2) | 8.2671(−4) | 1.9712(−5) | 4.5693(−7) |
| 0.8 | | 4.4604(−1) | 3.1087(−1) | 8.4269(−2) | 9.8803(−4) | 2.3557(−5) | 5.5024(−7) |
| 0.9 | | 6.7672(−1) | 4.2268(−1) | 1.0417(−1) | 1.1865(−3) | 2.8285(−5) | 6.6425(−7) |
| 1 | | 6.9479(+1) | 8.9757 | 2.2309(−1) | 1.4327(−3) | 3.4129(−5) | 8.0462(−7) |

TABLE 11

THE FLUXES

| Case | Quantity | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $q_+(\tau)$ | 3.1416 | 3.1379 | 3.1335 | 3.1229 | 3.0795 | 3.0309 | 2.9684 |
| | $q_-(\tau)$ | 1.7322(−1) | 1.6957(−1) | 1.6512(−1) | 1.5449(−1) | 1.1111(−1) | 6.2499(−2) | 0.0 |
| | $q(\tau)$ | 2.9684 | 2.9684 | 2.9684 | 2.9684 | 2.9684 | 2.9684 | 2.9684 |
| 2 | $q_+(\tau)$ | 3.1416 | 3.1215 | 3.1007 | 3.0578 | 2.9206 | 2.7992 | 2.6713 |
| | $q_-(\tau)$ | 1.2367(−1) | 1.2090(−1) | 1.1760(−1) | 1.0984(−1) | 7.8869(−2) | 4.4545(−2) | 0.0 |
| | $q(\tau)$ | 3.0179 | 3.0006 | 2.9831 | 2.9479 | 2.8418 | 2.7547 | 2.6713 |
| 3 | $q_+(\tau)$ | 1.5708 | 1.5448 | 1.5168 | 1.4580 | 1.2806 | 1.1432 | 1.0159 |
| | $q_-(\tau)$ | 2.2549(−1) | 2.1915(−1) | 2.1095(−1) | 1.9160(−1) | 1.2385(−1) | 6.4082(−2) | 0.0 |
| | $q(\tau)$ | 1.3453 | 1.3257 | 1.3059 | 1.2664 | 1.1568 | 1.0791 | 1.0159 |
| 4 | $q_+(\tau)$ | 3.1416 | 3.5574 | 3.5693 | 3.2918 | 2.2477 | 1.3724 | 4,7985(−1) |
| | $q_-(\tau)$ | 2.6617 | 3.0776 | 3.0894 | 2.8119 | 1.7678 | 8.9257(−1) | 0.0 |
| | $q(\tau)$ | 4.7985(−1) | 4.7985(−1) | 4.7985(−1) | 4.7985(−1) | 4.7985(−1) | 4.7985(−1) | 4.7985(−1) |
| 5 | $q_+(\tau)$ | 3.1416 | 1.8757 | 9.8091(−1) | 2.3398(−1) | 2.6776(−3) | 6.3835(−5) | 1.4733(−6) |
| | $q_-(\tau)$ | 3.7531(−1) | 2.7048(−1) | 1.5105(−1) | 3.7425(−2) | 4.3182(−4) | 1.0290(−5) | 0.0 |
| | $q(\tau)$ | 2.7663 | 1.6052 | 8.2987(−1) | 1.9656(−1) | 2.2458(−3) | 5.3544(−5) | 1.4733(−6) |

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M. BENASSI and C. E. SIEWERT: Mathematics Department, North Carolina State University, Raleigh, NC 27650

R. D. M. GARCIA: Divisão de Física de Reatores, Instituto de Pesquisas Energéticas e Nucleares, Cidade Universitária, São Paulo, Brasil

A. H. KARP: IBM Scientific Center, 1530 Page Mill Road, Palo Alto, CA 94304