ABSTRACT

MAIORINO, JOSÉ RUBENS. The F_N Method for Solving Radiation Transport Problems (Under the direction of CHARLES EDWARD SIEWERT).

The F_N method is applied to basic problems in radiative transfer and neutron transport theory.

The monochromatic equation of transfer for isotropic scattering in spherical geometry with a point source located at the center, is solved by combining "exact" analysis and the F_N method. Numerical results for the total flux, as a function of the optical variable, are reported for various orders of the F_N approximation and compared with "exact" and "Monte Carlo" results.

The net radiative heat flux relevant to radiative transfer in an anisotropically scattering plane-parallel medium with specularly and diffusely reflecting boundaries is obtained. Numerical results for quantities basic to compute the net radiative heat flux in a Mie scattering medium with constant heat generation are reported for various orders of the F_N approximation.

Critical problems for a slab reactor with a finite reflector (two-region reactor), and for a slab reactor with a blanket and a finite reflector (three-region reactor) are considered. The critical thicknesses, for these two considered problems, are determined for different values of the mean number of secondary neutrons per collision, and reflector and blanket thicknesses using various orders of the F_N approximation.

The thermal disadvantage factor calculation, required in the study of thermal utilization in heterogeneous reactor cells, is considered. Numerical results for basic cells in plane geometry with isotropic scattering in the

fuel and anisotropic scattering in the moderator are reported, and compared with other computational techniques.

The solution of the azimuth-independent vector equation of transfer for polarized light in a finite plane parallel atmosphere with a mixture of Rayleigh and isotropic scattering is discussed. Numerical results for the Stokes parameters at the boundaries, and for the albedo and transmission factor are reported.

Finally, the complete solution for the scattering of polarized light with azimuthal dependence in a Rayleigh and isotropically scattering atmosphere with ground reflection is considered. Numerical results for the Stokes parameters are reported.

THE F_N METHOD FOR SOLVING

RADIATION TRANSPORT PROBLEMS

Orientador: Charles Edward Siewert

by

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BIOGRAPHY

The author was born in São Paulo, Brazil, on April 29, 1951. He received his elementary and secondary education in São Vicente, Brazil, and graduated from Martim Afonso High School in 1968. He entered Campinas State University (UNICAMP), Campinas, Brazil, in March 1969 and received a Bachelor of Science degree with a major in Physics in December 1973. At that time, the author worked as Research Assistant in the Cosmic Radiation Physics Department at UNICAMP.

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1. INTRODUCTION

Transport theory is the branch of mathematical physics devoted to the study of the integro-differential equation introduced by Boltzmann [9], in 1872, to describe the evolution of the velocity distribution function of particles in space-time. The Boltzmann equation in its general form is a non-linear integro-differential equation, but assumptions based on physical grounds, such as neglecting collisions between identical particles, make it possible to linearize this equation. The assumptions used to linearize the Boltzmann equation are adequate to describe the transport of uncharged particles, such as neutrons and photons, and the motion of molecules in a rarefied medium. Thus, the linearized Boltzmann equation is the basic equation in the field of rarefied gas dynamics, with applications in the study of motion of a gas in a vacuum apparatus, the propagation of sound in gases and the structure of shockwaves, as well as in neutron transport theory, with applications in the design of nuclear reactors, and in the theory of radiative transfer, with applications in heat transfer, radiation shielding, and calculations of stellar and planetary atmospheres.

The early theoretical investigations of the linearized transport equation were in the theory of radiative transfer as those of Shuster [80], Schwarzschild [77] and Eddington [34]. They were interested in the astrophysical information obtained by solving the radiative transfer equation rather than in the rigor of the mathematical solutions. The earliest exact solution of the linearized Boltzmann equation is credited to Hopf [41], who solved a very idealized astrophysical problem, the so-called Milne problem, through the use of the Wiener-Hopf technique. However, since the Wiener-Hopf technique for solving the radiative transfer equation was limited to very idealized physical problems, most of the work in the field of radiative transfer was based on approximate techniques, such as the discrete ordinates method, introduced by Wick [106]. Chandrasekhar [23,24] used a discrete ordinates method to formulate an approximate solution of the polarized light problem. Another technique, based on principles of invariance governing the diffuse reflection and transmission of the light, was introduced by Ambarzumian [1], and an excellent review of these principles with applications in astrophysical problems can be found in the classical radiative transfer book by Chandrasekhar [25].

With the discovery of the neutron, in 1932, of nuclear fission, in 1938, and consequently with the interest in the development of nuclear reactors, many researchers were attracted to the field of transport theory. Since the equation describing the diffusion of neutrons is essentially the same as the basic equation of radiative transfer, early work in neutron transport theory employed the methods already used in radiative transfer. The earliest approximation used in the field of neutron diffusion, was the so-called "diffusion theory". In this theory, a preferential direction of the neutron current is imposed through Fick's law, and details concerning the direction of the scattered neutrons are neglected. A general description of diffusion theory and its application in reactor physics is given in the classical text books of Lamarsh [50], Murray [64], and Zweifel [113]. Other approximate techniques used in neutron transport theory are the S_N method [17], which is a generalization of the discrete ordinates method, and the P_N [53] and DP_N [110] methods, both based on a Legendre expansion of the angular distribution. A synopsis of these and other approximate techniques as they relate to reactor problems is provided by Bell and Glasstone [5] and

Davison [30]. Also, a monograph by Case, de Hoffman and Placzek [19] contains an excellent review of the early works in neutron transport theory.

Analytical solutions of the neutron transport equation were obtained by Case [20], in 1960, using an idea developed earlier by Van Kampen [105]. In short, the method used by Case follows the classical eigenfunction expansion method used in the study of boundary-value problems. The solution of the transport equation is written as a linear combination of eigenfunctions, which can be obtained by a suitable separation of variables. This linear combination contains two discrete eigenfunctions and a continuous set of eigenfunctions. The continuous eigenfunctions are not functions in the usual sense, but distributions, or generalized functions [38]. Also, the linear combination of eigenfunctions contains a set of arbitrary expansion coefficients to be determined from the boundary conditions, through the use of orthogonality properties of the eigenfunctions.

The contribution of Case was not in obtaining a normal mode expansion as solution of the transport equation, which has been obtained before by Van Kampen [105] and Davison [29], but in using the techniques of Muskhelishvili [65], for solving singular integral equations, to demonstrate completeness of the normal mode expansion, and to prove formal orthogonality of the eigenfunctions. An excellent review of Case's technique is provided in the classical book by Case and Zweifel [21], and by McCormick and Kuščer [58]. It should be noted that the singular eigenfunction expansion technique (as Case's technique is known) is not the only analytical solution of the transport equation. The Fourier transform technique [107], the invariant imbedding technique [6], which is a generalization of the Chandrasekhar's technique, and the transfer-matrix technique [3] also provide solutions of the transport equation. Case's technique has been called an "exact" method: however, the word is rather misleading because, with an increase in the capacity of the modern computer machines, numerical techniques, such as the doubling method [40], and the F_N method [39] are capable of producing numerical results just as accurate as the "exact" method. What should be said about Case's technique is that it is capable of producing results in closed analytical form, and is an excellent (not unique) technique to provide accurate numerical results which can be used as "benchmark" for approximate techniques. For example, the inaccuracies of the numerical methods may be assessed, and transport-corrected boundary conditions, such as extrapolated distance boundary condition, may be obtained.

Recently, Siewert and coworkers [39,93] introduced a new approximated method, called the F_N method (the capital letter F stands for the French word "Facile" which is translated in English as "easy"). The F_N method follows closely the early C_N method of Benoist [7], but it yields more concise equations that can be solved numerically even more efficiently than the C_N method. The F_N method, uses partially Case's technique in order to derive a set of singular integral equations for the angular distributions at the boundaries, and then it approximates these angular distributions by a polynomial of order N to derive a set of linear algebraic equations for the coefficients of the polynomial approximation. In short the F_N method can be summarized in the following steps:

1) For each specific problem we write the transport equation and the appropriate boundary conditions (usually conditions for the angular distributions at the boundaries).
2) Write the formal solution as given by Case's technique, i.e., the normal mode expansion.

3) Derive singular integral equations relating the angular distributions at the boundaries by using the full-range orthogonality properties of the Case's eigenfunctions.

4) Approximate the angular distributions at the boundaries by a polynomial of order N.

5) Substitute the approximation, described in 4, into the singular integral equations, derived in 3. Then apply the boundary conditions and obtain a system of linear algebraic equations for the coefficients of the polynomial approximation.

In this work, we apply the F_N method to various problems in radiative transfer and neutron transport theory in order to extend the range of possible applications, and to demonstrate the numerical accuracy of the method. In chapter 2, a general survey of the literature of transport theory, with emphasis on Case's technique and the F_N method, is presented.

In chapter 3, we use "exact" analysis together with the F_N method to compute the radiation field due to a point source of radiation located at the center of a finite sphere. We note that this problem has applications in radiative transfer, as well as in neutron transport theory. In radiative transfer, a point source in a finite sphere, can be considered a reasonable physical model for a "star", and in neutron diffusion the applications can be in the study of reactors with spherical fuel elements, like AVR reactor, and the so-called pebble bed reactors.

In chapter 4, we consider the application of the F_N method to a radiative transfer problem in an anisotropically scattering plane parallel medium with specularly and diffusely reflecting boundaries, and internal heat sources. Numerical results, for quantities basic to compute the net radiative heat flux, are reported. We note that this problem has application in heat transfer problems when conduction and convection are negligible, as in rocket applications, where the exhaust gas containing anisotropically scattering particles flows at high speeds and very high temperatures.

In chapters 5 and 6 we use the F_N method to solve basic problems in reactor physics, such as the critical problem for multiregion reactors (chapter 5), and the thermal disadvantage factor calculation (chapter 6), basic in the study of thermal utilization in heterogeneous reactor cells. We report accurate results for the critical half-thickness for various test problems, as well as the disadvantage factor for various heterogeneous cells.

In chapter 7 and 8 we apply the F_N method for solving a classical problem, basic to astrophysics and atmospherical sciences, namely, the transfer of polarized light in a mixture of Rayleigh and isotropic scattering plane parallel atmosphere, with ground reflection -- the so-called "planetary problem". We note that this problem is basic to study the propagation of light in planetary atmospheres.

Chapter 11 contains the solution of the inverse problem for a finite Rayleigh scattering atmospheres, the methods we used to compute the discrete eigenvalues, and some discussion about basic functions used in previous chapters.

2. REVIEW OF LITERATURE

In this chapter we wish to review briefly the contributions of previous workers, mainly those who used the singular eigenfunction expansion for solving radiation transport problems. Thus, our literature review begins with the fundamental work of Case [20], which introduced an elegant technique providing analytical solutions to the one-speed, steady-state, neutron transport equation.

The work of Case was rapidly extended: the critical problem for a slab reactor was considered by Zelazny [111]; solutions of the one speed neutron transport problems in slab geometry were given by Pahor [71] and by McCormick and Mendelson [54]. The time-dependent one-speed neutron transport equation was first considered by Bowden [12], and later by Kuščer and Zweifel [47]. Mitsis [63] extended Case's technique for other geometries, and Erdmann and Siewert [35] also considered the one-speed transport equation in spherical geometry. The one-speed transport equation with anisotropic scattering was first solved by Mika [62], and later by McCormick and Kuščer [55], and by Shure and Natelson [78]. Recently, Siewert and Williams [89] studied the effect of the scattering law on the critical thickness of a slab reactor, using a kernel which consists of a linear combination of backward, forward and isotropic scattering. Particular solutions of the one speed neutron transport equation were obtained by Özişik and Siewert [69]. Multiregion problems in one-speed neutron transport theory were first solved by Kuszell [48], and later by Mendelson and Summerfield [59]; however, the works of McCormick [56] and McCormick and Doyas [57] are considered the most significant contribution to two-media problems. Recently, Burkart, Ishiguro and Siewert [14], combining the principles of invariance, as developed by Chandrasekhar [25], with Case's technique were able to solve various one-speed neutron transport problems in two dissimilar media with anisotropic scattering.

The energy-dependent transport equation was considered by Bednarg and Mika [4], who treated the energy as a continuous variable. The multigroup approach for solving the energy-dependent transport equation was first reported by Zelazny and Kuszell [112], using the two-group model in plane geometry with isotropic scattering. Some years later, Siewert and Shieh [84], proved the full-range completeness and orthogonality theorems for the two-group transport theory, and were able to obtain the infinite medium Green's function. Half-space and slab problems in two-group transport theory were solved by Siewert and Ishiguro [85] using the H-matrix introduced earlier by Siewert, Burniston and Kriese [85]. Two media problems in two-group neutron transport theory were solved by Ishiguro and Maiorino [42], considering two half-spaces, and by Ishiguro and Garcia [43], considering finite adjacent media. Two-group transport theory with anisotropic scattering was considered by Reith and Siewert [74], and a comparison of two-group Case's technique with approximated techniques, like P_1, P_3 and DP_1, was made by Metcalf and Zweifel [60,61]. A generalization, from two-group to multigroup transport theory, was discussed by Yoshimura and Katsuragi [108].

The similarity between the neutron transport equation and the radiative transfer equation made possible the use of Case's method in solving radiative transfer problems. Siewert and Zweifel [81,82] and Ferziger and Simons [37] applied the singular eigenfunction expansion method to radiative transfer problems. Özişik and Siewert [68] solved radiative heat transfer problems using a gray model, and Reith, Siewert and Özişik [73] using a non-gray model. A discussion of the application of Case's method for solving radiative transfer in planetary atmosphere is provided in chapters 7 and 8.

An application of Case's method for solving problems in kinetic theory of gases was made by Siewert and Burniston [90], and by Siewert and Kriese [91]. In these works, half-range orthogonality relations concerning the elementary solutions of the time-dependent linearized BGK model of the Boltzmann equation, were developed. An excellent survey of the applications in kinetic theory is provided by Cercignani [22].

The F_N method was introduced by two companion papers: the first to them, by Siewert and Benoist [93], introduces the theory of the F_N method, and the second of them, by Grandjean and Siewert [39], applied the F_N method for solving basic problems, like the half-space albedo and constant source problems, two-media problems, the albedo problem for a finite slab, and the critical problem for a bare slab. After these works, Siewert [92] extended the F_N method for solving radiative transfer problems for plane-parallel media with anisotropic scattering, and Devaux, Grandjean, Ishiguro and Siewert [31] used the F_N method for solving radiative transfer problems, based on the general anisotropically scattering model, in multi-layer atmospheres.

3. A POINT SOURCE IN A FINITE SPHERE[1]

3.1. Introduction

In a recent series of papers [31,39,92,93] the F_N method basic to radiative transfer and neutron-transport theory was introduced and used to solve concisely and accurately numerous basic problems. To date, however, the F_N method has been used primarily to compute surface quantities such as the albedo and the transmission factor. Here we wish to apply the method in order to establish the mean intensity J, as a function of the optical variable, interior to a finite sphere.

We consider the equation of transfer for isotropic scattering in the monochromatic form, or one-speed model in the case of neutron-transport theory.

$$\mu \frac{\partial}{\partial r} I(r,\mu) + \frac{(1-\mu^2)}{r} \frac{\partial}{\partial \mu} I(r,\mu) + I(r,\mu) = \frac{\omega}{2} \int_{-1}^{1} I(r,\mu) d\mu + \frac{\delta(r)}{8\pi r^2} . \quad (3.1)$$

Here, the isotropically emitting source term

$$S(r) = \frac{\delta(r)}{8\pi r^2} \quad (3.2)$$

is normalized so that

$$4\pi \int_{-1}^{1} \int_{0}^{\infty} r^2 S(r) dr d\mu = 1. \quad (3.3)$$

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[1]This chapter is based on a paper published in JQSRT [96].

We thus seek a solution of Eq. (3.1) for $r \in (0,R)$, R is the radius of the sphere, subject to the condition of no entering radiation:

$$I(R,-\mu) = 0, \ \mu > 0. \tag{3.4}$$

The solution to this problem was formulated by Erdmann and Siewert [35] some years ago and recently, the method of elementary solutions [21] was used to evaluate the solution numerically [94]. We thus have available results with which to compare the solution obtained here by the F_N method.

As noted by Davison [30], Eq. (3.1), along with the boundary condition given by Eq. (3.4), can be converted to the equivalent integral form

$$\rho(r) = \frac{1}{r}\int_{-R}^{R} r'E_1(|r-r'|)\left[\frac{\omega}{2}\rho(r') + S(r')\right]dr' \ , \ r \in [-R,R] \ . \tag{3.5}$$

Here

$$\rho(r) = 2J(r) = \int_{-1}^{1} I(r,\mu)d\mu \tag{3.6}$$

and we have extended the range of r to $r \in [-R,R]$. We have also defined $\rho(-r) = \rho(r)$ and $S(-r) = S(r)$. The first exponential integral function is denoted by $E_1(x)$. Of course, once $\rho(r)$ is known, the complete radiation intensity $I(r,\mu)$ can readily be obtained from Eq. (3.1).

3.2. Basic Analysis

Following the paper of Wu and Siewert [108], we now define two transform functions,

$$\Phi(r,\mu) = \frac{1}{\mu} \int_{-R}^{r} dr'r'e^{-(r-r')/\mu}\left[S(r') + \frac{\omega}{2}\rho(r')\right] , \mu\in(0,1), \tag{3.7a}$$

and

$$\Phi(r,-\mu) = \frac{1}{\mu} \int_{r}^{R} dr'r'e^{-(r'-r)/\mu}\left[S(r') + \frac{\omega}{2}\rho(r')\right] , \mu\in(0,1), \tag{3.7b}$$

to find that we can express $\rho(r)$ as

$$\rho(r) = \frac{1}{r} \int_{-1}^{1} \Phi(r,\mu)d\mu, \tag{3.8}$$

where, differentiation of Eqs. (3.7) may be used to show that $\Phi(r,\mu)$ is a solution of the pseudoslab problem defined by

$$\mu \frac{\partial}{\partial r} \Phi(r,\mu) + \Phi(r,\mu) = \frac{\omega}{2} \int_{-1}^{1} \Phi(r,\mu')d\mu', \quad r \neq 0, \tag{3.9}$$

with the conditions

$$\Phi(-r,-\mu) = -\Phi(r,\mu), \tag{3.10a}$$

$$4\pi\mu^2 \left[\Phi(0^+,\mu) - \Phi(0^-,\mu)\right] = 1, \ \mu \ (-1,1), \tag{3.10b}$$

and

$$\Phi(R,-\mu) = 0, \ \mu > 0. \tag{3.10c}$$

To solve the pseudo-problem defined by Eqs. (3.9) and (3.10), we first write

$$\Phi(r,\mu) = A(\nu_o)\phi(\nu_o,\mu)e^{-r/\nu_o} + \int_0^1 A(\nu)\phi(\nu,\mu)e^{-r/\nu}d\nu$$

$$+ \Phi_c(r,\mu), \quad r > 0, \tag{3.11a}$$

and

$$\Phi(r,\mu) = -A(\nu_o)\phi(-\nu_o,\mu)e^{r/\nu_o} - \int_0^1 A(\nu)\phi(-\nu,\mu)e^{r/\nu}d\nu$$

$$+ \Phi_c(r,\mu), \quad r < 0, \tag{3.11b}$$

where

$$\Phi_c(r,\mu) = B(\nu_o)\left[\phi(\nu_o,\mu)e^{-r/\nu_o} - \phi(-\nu_o,\mu)e^{r/\nu_o}\right]$$

$$+ \int_0^1 B(\nu)\left[\phi(\nu,\mu)e^{-r/\nu} - \phi(-\nu,\mu)e^{r/\nu}\right]d\nu \tag{3.12}$$

is a correction term to account for the fact that we are considering a finite sphere. Here, Case's elementary solutions [21] are

$$\phi(\nu_o,\mu) = \frac{\omega\nu_o}{2}\frac{1}{\nu_o - \mu} \tag{3.13}$$

where

$$\Lambda(\nu_o) = 1 + \frac{\omega\nu_o}{2} \int_{-1}^{1} \frac{d\mu}{\mu - \nu_o} = 0 \tag{3.14}$$

and

$$\phi(\nu,\mu) = \frac{\omega\nu}{2} \, Pv \left(\frac{1}{\nu - \mu}\right) + \left[1 - \omega\nu\tanh^{-1}\nu\right]\delta(\nu - \mu), \tag{3.15}$$

where P_v denotes the Cauchy principal value.

If we substitute Eqs. (3.11) into the jump condition, Eq. (3.10b), we can use Case's full-range orthogonality theorem [21] to find

$$A(\nu_o) = \frac{1}{4\pi\nu_o N(\nu_o)} \tag{3.16a}$$

and

$$A(\nu) = \frac{1}{4\pi\nu N(\nu)} \tag{3.16b}$$

where

$$N(\nu_o) = \frac{\omega}{2} \nu_o^3 \left(\frac{\omega}{\nu_o^2 - 1} - \frac{1}{\nu_o^2}\right) \tag{3.17a}$$

and

$$N(\nu) = \nu \left\{ \left[1 - \omega\nu\tanh^{-1}\nu\right]^2 + \left[\frac{\omega\nu\pi}{2}\right]^2 \right\} . \tag{3.17b}$$

Considering Eqs. (3.8), (3.11) and (3.16), we see that we can now write

$$\rho(r) = \rho_\infty(r) + \rho_c(r),$$

where

$$\rho_\infty(r) = \frac{1}{4\pi r}\left[\frac{1}{\nu_o N(\nu_o)}\, e^{-r/\nu_o} + \int_0^1 \frac{1}{\nu N(\nu)}\, e^{-r/\nu} d\nu\right], \tag{3.18}$$

and

$$\rho_c(r) = \frac{1}{r}\int_{-1}^{1} \Phi_c(r,\mu)\,d\mu , \tag{3.19a}$$

or, using Eq. (3.12)

$$\rho_c(r) = \frac{2}{r}\left\{B(\nu_o)\sinh(r/\nu_o) + \int_0^1 B(\nu)\sinh(r/\nu)\,d\nu\right\}. \tag{3.19b}$$

Since $\Phi_c(r,\mu)$ satisfies Eq. (3.9) subject to

$$\Phi_c(-r,-\mu) = -\Phi_c(r,\mu) \tag{3.20a}$$

and

$$\Phi_c(-R,\mu) = K(\mu),\ \mu > 0, \tag{3.20b}$$

with

$$K(\mu) = A(\nu_o)\phi(-\nu_o,\mu)e^{-R/\nu_o} + \int_0^1 A(\nu)\phi(-\nu,\mu)e^{-R/\nu}d\nu , \tag{3.21}$$

we can use the F_N method to compute this correction term and thus to complete the desired solution. We start using Eq. (3.12) along with Case's full-range orthogonality theorem, to deduce

$$\int_{-1}^{1} \Phi_c(-R,\mu)\phi(-\xi,\mu)\mu d\mu = B(\xi)N(\xi)e^{-R/\xi} , \qquad (3.22a)$$

and

$$\int_{-1}^{1} \Phi_c(-R,\mu)\phi(\xi,\mu)\mu d\mu = B(\xi)N(\xi)e^{R/\xi} , \qquad (3.22b)$$

where $\xi \in \left[\nu_o U(0,1)\right]$. Now, if we eliminate $B(\xi)N(\xi)$ between Eqs. (3.22) we can find the singular integral equation

$$\int_{0}^{1} \Phi_c(-R,-\mu)\phi(\xi,\mu)\mu d\mu - e^{-2R/\xi}\int_{0}^{1} \Phi_c(-R,-\mu)\phi(-\xi,\mu)\mu d\mu =$$
$$\int_{0}^{1} \Phi_c(-R,\mu)\phi(-\xi,\mu)\mu d\mu - e^{-2R/\xi}\int_{0}^{1} \Phi_c(-R,\mu)\phi(\xi,\mu)\mu d\mu ,$$
$$\xi \in \left[\nu_o U(0,1)\right]. \qquad (3.23)$$

Following the paper of Grandjean and Siewert [39], we can now approximate the exit distribution by writing

$$\Phi_c(-R,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \ \mu > 0 , \qquad (3.24)$$

and then we can insert this approximation into Eq. (3.23) and use the boundary condition given by Eq. (3.20b) to find the following system of linear algebraic equations:

$$\sum_{\alpha=0}^{N} a_\alpha \left[B_\alpha(\xi_\beta) - e^{-2R/\xi_\beta} A_\alpha(\xi_\beta)\right] = \Delta(\xi_\beta) \ , \tag{3.25}$$

with

$$\Delta(\xi_\beta) = \frac{2}{\omega\xi_\beta}\left[\int_0^1 \phi(-\xi_\beta,\mu)K(\mu)\mu d\mu - e^{-2R/\xi_\beta}\int_0^1 \phi(\xi_\beta,\mu)K(\mu)\mu d\mu\right]. \tag{3.26}$$

For the F_o approximation, we use only $\xi_o = \nu_o$ in Eq. (3.25); for the F_1 approximation, we use $\xi_o = \nu_o$ and $\xi_1 = 0$; for the F_2-approximation, we use $\xi_o = \nu_o$, $\xi_1 = 0$ and $\xi_2 = 1$ and, for higher-order approximation, we use additional values of ξ_β spaced equally in the interval $[0,1]$. We note that the R.H.S. of Eq. (3.25) can be expressed as

$$\Delta(\xi) = \Delta_1(\xi) - e^{-2R/\xi}\Delta_2(\xi) \tag{3.27}$$

with

$$\Delta_1(\xi) = \frac{\omega}{8\pi}\left\{\frac{e^{-R/\nu_o}}{N(\nu_o)}\left[A_o(\xi) - A_o(\nu_o)\right]\frac{1}{\nu_o - \xi} + \int_0^1 \frac{e^{-R/\nu}}{N(\nu)}\left[A_o(\xi) - A_o(\nu)\right]\frac{d\nu}{\nu - \xi}\right\} \tag{3.28a}$$

and

$$\Delta_2(\xi) = \frac{\omega}{8\pi}\left\{\frac{e^{-R/\nu_o}}{N(\nu_o)}\left[B_o(\xi) + A_o(\nu_o)\right]\frac{1}{\nu_o + \xi}\right.$$

$$+ \int_0^1 \frac{e^{-R/\nu}}{N(\nu)} \left[B_o(\xi) + A_o(\nu) \right] \frac{d\nu}{\nu + \xi} \Bigg\} . \tag{3.28b}$$

The functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ appearing in Eqs. (3.25) and (3.28) are given by

$$A_\alpha(\xi) = \frac{2}{\omega\xi} \int_0^1 \mu^{\alpha+1} \phi(-\xi,\mu) d\mu \tag{3.29a}$$

and

$$B_\alpha(\xi) = \frac{2}{\omega\xi} \int_0^1 \mu^{\alpha+1} \phi(\xi,\mu) d\mu , \tag{3.29b}$$

or, by the following recurrence relations

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha + 1} , \alpha \geq 1. \tag{3.30a}$$

with

$$A_o(\xi) = 1 - \xi \log \left(1 + \frac{1}{\xi}\right) , \tag{3.30b}$$

and

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \frac{1}{\alpha + 1} , \alpha \geq 1 , \tag{3.31a}$$

with

$$B_o(\xi) = \frac{2}{\omega} - 1 - \xi \log \left(1 + \frac{1}{\xi}\right) . \tag{3.31b}$$

Once we have solved the system of linear algebraic equations given by Eqs. (3.25) to obtain the constants a_α, $\alpha = 0, 1, 2, \ldots, N$, we can use Eqs. (3.23) along with the approximation given by Eq. (3.24) to find the expansion coefficients $B(\nu_o)$ and $B(\nu)$ required in Eq. (3.19b) to complete the solution. We thus find that our final result can be expressed as

$$\rho(r) = \rho_\infty(r) - \frac{1}{4\pi r}\left[E(\nu_o)e^{-R/\nu_o}\sinh(r/\nu_o) + \int_0^1 E(\nu)e^{-R/\nu}\sinh(r/\nu)d\nu\right], \qquad (3.32)$$

where

$$E(\xi) = \frac{4\pi\omega\xi}{N(\xi)}\left[\Delta_2(\xi) - \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi)\right]. \qquad (3.33)$$

3.3. Numerical Results and Conclusions

After solving ν_o from Eq. (3.14), we approximate the integral in Eq. (3.18) by a Gaussian quadrature scheme in order to evaluate $\rho_\infty(r)$. We also evaluated Eqs. (3.28) in a similar manner and solved Eqs. (3.25) for the constants a_α required in Eq. (3.33), and thus we were able for various orders of the F_N approximation to evaluate Eq. (3.32) and establish $\rho(r)$. In Tables 3.1 and 3.2 we list our results for the F_N calculation of $\rho(r)$ along with $\rho_\infty(r)$, the "exact" results taken from Ref. 94 and the Monte Carlo results [103] obtained by Dunn [32].

It is clear from Tables 3.1 and 3.2 that the F_N method yields better results for $\omega = 0.9$ than for $\omega = 0.3$. This result is obtained because the approximation given by Eq. (3.24) has greater validity when there is less absorption. We note also that the F_N results improve as r is diminished from r = R. This can be explained by the fact that we have used the F_N approximation only for the correction term, $\rho_c(r)$, and as the interior of the sphere is approached this term becomes less important when compared with the "exact" infinite term, $\rho_\infty(r)$.

We note that the F_N method is remarkably simple in terms of computation requirements, since it involves only the inversion of a matrix whose elements are very elementary functions, whereas the "exact" solution of this problem requires the iterative numerical solution of a set of Fredholm integral equations. We note also that the Monte Carlo method requires a quite large number of histories in order to get a degree of accuracy comparable to the F_N method.

Finally, we note that the method of solution used here can be applied in solving space-dependent stationary problems related to the diffusion of test particles in a random distribution of field particles, and in the presence of an external conservative force. This problem was discussed by Boffi, Molinari, Pescatore, and Pizzio [8], and the equation they obtained to describe this physical situation was identical to that of the point-source problem discussed here.

Table 3.1 Numerical results for ω = 0.3 and R = 1

r	$4\pi r^2\rho_\infty(r)$	$4\pi r^2\rho(r)$					
	Exact	F_3	F_4	F_5	F_6	Exact	Monte Carlo[2]
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.1	0.96440	0.96418	0.96418	0.96418	0.96418	0.96417	0.9678
0.2	0.91965	0.91873	0.91872	0.91872	0.91872	0.91871	0.9266
0.3	0.87101	0.86888	0.86887	0.86886	0.86886	0.86885	0.8530
0.4	0.82091	0.81697	0.81695	0.81694	0.81693	0.81692	0.7953
0.5	0.77076	0.76429	0.76426	0.76424	0.76423	0.76421	0.7760
0.6	0.72147	0.71155	0.71151	0.71147	0.71146	0.71143	0.7040
0.7	0.67364	0.65900	0.65894	0.65889	0.65888	0.65883	0.6450
0.8	0.62765	0.60636	0.60628	0.60626	0.60624	0.60620	0.5985
0.9	0.58374	0.55282	0.55270	0.55258	0.55256	0.55247	0.5513
1.0	0.54204	0.49206	0.49192	0.49172	0.49169	0.49154	0.4850

[2] The number of histories to obtain these results was 1000 [32].

Table 3.2 Numerical results for $\omega = 0.9$ and $R = 1$

r	$4\pi r^2 \rho_\infty(r)$	$4\pi r^2 \rho(r)$					
	Exact	F_3	F_4	F_5	F_6	Exact	Monte Carlo[3]
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.1	1.1208	1.1147	1.1147	1.1147	1.1147	1.1147	1.1224
0.2	1.2342	1.2099	1.2099	1.2099	1.2099	1.2099	1.1965
0.3	1.3383	1.2832	1.2832	1.2832	1.2832	1.2832	1.2837
0.4	1.4326	1.3336	1.3336	1.3336	1.3336	1.3336	1.3229
0.5	1.5169	1.3603	1.3602	1.3602	1.3602	1.3602	1.3610
0.6	1.5914	1.3621	1.3620	1.3620	1.3620	1.3620	1.3614
0.7	1.6567	1.3374	1.3373	1.3373	1.3372	1.3372	1.3391
0.8	1.7129	1.2830	1.2829	1.2829	1.2828	1.2828	1.2800
0.9	1.7607	1.1917	1.1916	1.1915	1.1915	1.1914	1.1884
1.0	1.8006	1.0263	1.0260	1.0259	1.0259	1.0258	1.0236

[3]The number of histories to obtain these results was 10^5 [32].

4. ON RADIATIVE TRANSFER PROBLEMS WITH REFLECTIVE BOUNDARY CONDITIONS[4]

4.1. Introduction

In this chapter we introduce the manner in which the F_N method is used to compute the net radiative heat flux in an anisotropically scattering plane parallel medium with specularly and diffusely reflecting boundaries. We consider the equation of transfer [25,70],

$$\mu \frac{\partial}{\partial\tau} I(\tau,\mu) + I(\tau,\mu) = \omega \sum_{\ell=0}^{L} \left(\frac{2\ell + 1}{2}\right) f_\ell P_\ell(\mu) \int_{-1}^{1} P_\ell(\mu')I(\tau,\mu')d\mu'$$

$$+ (1 - \omega)\frac{\sigma T^4(\tau)}{\pi} , \tag{4.1}$$

which includes anisotropic scattering of order L. Here τ is the optical variable, μ is the direction cosine of the propagating radiation (as measured from the positive τ axis), ω is the single scattering albedo, $T(\tau)$ is the temperature distribution in the medium and the constants f_ℓ, $\ell = 0, 1, 2, \ldots, L$, with $f_o = 1$, are the coefficients in a Legendre expansion of the phase function. For a plate of thickness Δ we seek solution of Eq. (4.1) subject to a boundary condition of the form

$$I(L,\mu) = \varepsilon_1\left(\frac{\sigma T_1^4}{\pi}\right) + \rho_1^s I(L,-\mu) + 2\rho_1^d \int_{-1}^{1} I(L,-\mu')\mu' d\mu', \quad \mu > 0, \tag{4.2a}$$

and

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[4] This chapter is based on a paper accepted for publication in JQSRT [99].

$$I(R,-\mu) = \varepsilon_2 \left(\frac{\sigma T_2^4}{\pi}\right) + \rho_2^s I(R,\mu) + 2\rho_2^d \int_{-1}^{1} I(R,\mu')\mu' d\mu', \quad \mu > 0, \quad (4.2b)$$

where $\tau = L$ and $\tau = R$ refer respectively to the left and right boundaries of the plate. Here T_1 and T_2 refer to the left and right surface temperatures, σ is the Stefan-Boltzmann constant, ρ_α^s and ρ_α^d, $\alpha = 1$ or 2, are respectively the specular and diffuse reflectivities and ε_1 and ε_2 are the emissivities.

4.2. Analysis

We begin by expressing the radiation intensity in terms of the known elementary solutions [58,62] and a particular solution $I_p(\tau,\mu)$ in the form

$$I(\tau,\mu) = \sum_{\beta=0}^{\kappa-1} \left[A(\nu_\beta)\phi(\nu_\beta,\mu)e^{-\tau/\nu_\beta} + A(-\nu_\beta)\phi(-\nu_\beta,\mu)e^{\tau/\nu_\beta} \right]$$

$$+ \int_{-1}^{1} A(\nu)\phi(\nu,\mu)e^{-\tau/\nu}d\nu + I_p(\tau,\mu) \quad . \quad (4.3)$$

Here

$$\phi(\nu,\mu) = \tfrac{1}{2}\omega\nu g(\nu,\mu)\mathrm{Pv}\left(\frac{1}{\nu - \mu}\right) + \lambda(\nu)\delta(\nu - \mu), \quad (4.4a)$$

$$g(\nu,\mu) = \sum_{\ell=0}^{L} (2\ell + 1)f_\ell g_\ell(\nu)P_\ell(\mu), \quad (4.4b)$$

$$\lambda(\nu) = 1 + \nu P \int_{-1}^{1} \psi(x)\frac{dx}{x - \nu} , \quad (4.4c)$$

$$\psi(x) = \tfrac{1}{2}\omega g(x,x), \quad (4.4d)$$

and the polynomials $g_\ell(\nu)$ defined as

$$g_\ell(\nu) = \int_{-1}^{1} P_\ell(\mu)\phi(\nu,\mu)d\mu \ , \tag{4.5}$$

can be generated from the recursion formula

$$\nu h_\ell g_\ell(\nu) = (\ell + 1)g_{\ell+1}(\nu) + \ell g_{\ell-1}(\nu) \tag{4.6a}$$

with

$$g_o(\nu) = 1 \text{ and } g_1(\nu) = \nu(1 - \omega). \tag{4.6b and c}$$

In addition

$$h_\ell = (2\ell + 1)(1 - \omega f_\ell), \tag{4.6d}$$

$$\phi(\nu_\beta,\mu) = \frac{1}{2}\omega\nu_\beta g(\nu_\beta,\mu) \left(\frac{1}{\nu_\beta - \mu}\right) \tag{4.7}$$

and the ν_β, $\beta = 0, 1, 2, \ldots, \kappa - 1$, denote the positive zeros of

$$\Lambda(z) = 1 + z \int_{-1}^{1} \psi(x)\frac{dx}{x - z} \tag{4.8}$$

in the complex plane cut from -1 to 1 along the real axis. The expansion coefficients $A(\pm\nu_\beta)$ and $A(\nu)$, $\nu \in (-1,1)$ appearing in Eq. (4.3) are to be determined by the boundary conditions.

As discussed by Grandjean and Siewert [39], the full-range orthogonality relations concerning the functions $\phi(\xi,\mu)$ can be used to develop a system of singular integral equations and constraints for the distribution of radiation at the surfaces of the considered plate (the same singular integral equations and constraints were reported by Bowden, McCrosson, and Rhodes [13]). Thus we let $I^*(\tau,\mu) = I(\tau,\mu) - I_p(\tau,\mu)$ and consider

$$\int_0^1 \mu\phi(\xi,\mu)I^*(L,-\mu)d\mu + e^{-\Delta/\xi}\int_0^1 \mu\phi(-\xi,\mu)I^*(R,\mu)d\mu$$
$$= \int_0^1 \mu\phi(-\xi,\mu)I^*(L,\mu)d\mu + e^{-\Delta/\xi}\int_0^1 \mu\phi(\xi,\mu)I^*(R,-\mu)d\mu,$$
$$\xi \in P, \tag{4.9a}$$

and

$$\int_0^1 \mu\phi(\xi,\mu)I^*(R,\mu)d\mu + e^{-\Delta/\xi}\int_0^1 \mu\phi(-\xi,\mu)I^*(L,-\mu)d\mu$$
$$= \int_0^1 \mu\phi(-\xi,\mu)I^*(R,-\mu)d\mu + e^{-\Delta/\xi}\int_0^1 \mu\phi(\xi,\mu)I^*(L,\mu)d\mu,$$
$$\xi \in P, \tag{4.9b}$$

where $\xi \in P \Rightarrow \xi \in \{\nu_\beta\} U(0,1)$. Equations (4.9) are exact; however, we wish now to introduce the F_N method. We let

$$I(L,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu > 0, \tag{4.10a}$$

and

$$I(R,\mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \quad \mu > 0. \tag{4.10b}$$

If we substitute Eqs. (4.10) into Eqs. (4.9) and use boundary conditions given by Eqs. (4.2) we find we can evaluate analytically the required integrals to obtain

$$\sum_{\alpha=0}^{N} a_\alpha \left[B_\alpha(\xi) - \rho_1^s A_\alpha(\xi) - 2\rho_1^d\left(\frac{1}{\alpha+2}\right)A_o(\xi)\right] + e^{-\Delta/\xi}\sum_{\alpha=0}^{N} b_\alpha\left[A_\alpha(\xi) - \rho_2^s B_\alpha(\xi) - 2\rho_2^d\left(\frac{1}{\alpha+2}\right)B_o(\xi)\right] = K_1(\xi), \quad \xi \in P, \tag{4.11a}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha \left[B_\alpha(\xi) - \rho_2^s A_\alpha(\xi) - 2\rho_2^d\left(\frac{1}{\alpha+2}\right)A_o(\xi)\right] + e^{-\Delta/\xi}\sum_{\alpha=0}^{N} a_\alpha\left[A_\alpha(\xi) - \rho_1^s B_\alpha(\xi) - 2\rho_1^d\left(\frac{1}{\alpha+2}\right)B_o(\xi)\right] = K_2(\xi), \quad \xi \in P, \tag{4.11b}$$

where

$$K_1(\xi) = \varepsilon_1\left(\frac{\sigma T_1^4}{\pi}\right)A_o(\xi) - \frac{2}{\omega\xi}\int_{-1}^{1}\mu\phi(-\xi,\mu)I_p(L,\mu)d\mu$$

$$+ e^{-\Delta/\xi}\left[\varepsilon_2\left(\frac{\sigma T_2^4}{\pi}\right)B_o(\xi) + \frac{2}{\omega\xi}\int_{-1}^{1}\mu\phi(-\xi,\mu)I_p(R,\mu)d\mu\right] \qquad (4.12a)$$

and

$$K_2(\xi) = \varepsilon_2\left(\frac{\sigma T_2^4}{\pi}\right)A_o(\xi) + \frac{2}{\omega\xi}\int_{-1}^{1}\mu\phi(\xi,\mu)I_p(R,\mu)d\mu$$

$$+ e^{-\Delta/\xi}\left[\varepsilon_1\left(\frac{\sigma T_1^4}{\pi}\right)B_o(\xi) - \frac{2}{\omega\xi}\int_{-1}^{1}\mu\phi(\xi,\mu)I_p(L,\mu)d\mu\right]. \qquad (4.12b)$$

We note that the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ can be readily computed from [92]

$$A_{\alpha+1}(\xi) = -\xi A_\alpha(\xi) + \sum_{\ell=0}^{L}(2\ell+1)(-1)^\ell f_\ell g_\ell(\xi)\Delta_{\alpha,\ell} \quad , \qquad (4.13a)$$

with

$$A_o(\xi) = 1 - \frac{2}{\omega}\xi\psi(\xi)\log(1+1/\xi)$$

$$+ \sum_{\ell=1}^{L}(2\ell+1)f_\ell g_\ell(\xi)\Pi_\ell(\xi), \qquad (4.13b)$$

and

$$B_{\alpha+1}(\xi) = \xi B_\alpha(\xi) - \sum_{\ell=0}^{L}(2\ell+1)f_\ell g_\ell(\xi)\Delta_{\alpha,\ell} \; , \qquad (4.14a)$$

with

$$B_o(\xi) = \frac{2}{\omega} - 2 + A_o(\xi) \ . \tag{4.14b}$$

In addition

$$\Delta_{\alpha,\ell} = \int_0^1 \mu^{\alpha+1} P_\ell(\mu) d\mu \tag{4.15}$$

is readily available from

$$\Delta_{\alpha,\ell+2} = \left(\frac{1 + \alpha - \ell}{4 + \alpha + \ell}\right)\Delta_{\alpha,\ell} \tag{4.16a}$$

with

$$\Delta_{\alpha,0} = \frac{1}{\alpha + 2} \tag{4.16b}$$

and

$$\Delta_{\alpha,1} = \frac{1}{\alpha + 3} \ . \tag{4.16c}$$

The polynomials $\Pi_\ell(\xi)$ required in Eq. (4.13b) can be generated from

$$(2\ell + 1)\xi\Pi_\ell(\xi) = (-1)^\ell(2\ell + 1)\Delta_{o,\ell} + (\ell + 1)\Pi_{\ell+1}(\xi)$$

$$+ \ell\Pi_{\ell-1}(\xi) \ , \tag{4.17a}$$

with

$$\Pi_o(\xi) = 1, \tag{4.17b}$$

$$\Pi_1(\xi) = \xi - \frac{1}{2} \tag{4.17c}$$

and

$$\Pi_2(\xi) = \frac{3}{2}\xi(\xi - \frac{1}{2}). \tag{4.17d}$$

Although Eqs. (4.11) cannot be satisfied for all $\xi \in P$, we can select N + 1 different values of $\xi \in P$, say $\{\xi_j\}$, and solve the following system of 2(N + 1) linear algebraic equations for a_α and b_α, $\alpha = 0, 1, 2, \ldots, N$:

$$\sum_{\alpha=0}^{N} a_\alpha\left[B_\alpha(\xi_j) - \rho_1^s A_\alpha(\xi_j) - 2\rho_1^d\left(\frac{1}{\alpha + 2}\right)A_o(\xi_j)\right] + e^{-\Delta/\xi_j}\sum_{\alpha=0}^{N} b_\alpha\left[A_\alpha(\xi_j) - \rho_2^s B_\alpha(\xi_j) - 2\rho_2^d\left(\frac{1}{\alpha + 2}\right)B_o(\xi_j)\right] = K_1(\xi_j) \tag{4.18a}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha\left[B_\alpha(\xi_j) - \rho_2^s A_\alpha(\xi_j) - 2\rho_2^d\left(\frac{1}{\alpha + 2}\right)A_o(\xi_j)\right] + e^{-\Delta/\xi_j}\sum_{\alpha=0}^{N} a_\alpha\left[A_\alpha(\xi_j) - \rho_1^s B_\alpha(\xi_j) - 2\rho_1^d\left(\frac{1}{\alpha + 2}\right)B_o(\xi_j)\right] = K_2(\xi_j) \quad . \tag{4.18b}$$

We note that the F_N method yields first of all the exit distributions of radiation $I(L,-\mu)$, and $I(R,\mu)$, $\mu > 0$. However once these quantities are established the complete solution is given by Eq. (4.3) and

$$A(\pm\xi)e^{\mp L/\xi} = \pm\frac{1}{N(\xi)} \int_0^1 \mu\phi(\pm\xi,\mu)\left[I(L,\mu) - I_p(L,\mu)\right]d\mu, \; \xi\in P, \quad (4.19a)$$

or

$$A(\pm\xi)e^{\mp R/\xi} = \pm\frac{1}{N(\xi)}\int_0^1 \mu\phi(\pm\xi,\mu)\left[I(R,\mu) - I_p(R,\mu)\right]d\mu, \; \xi\in P, \quad (4.19b)$$

where

$$N(\nu_\beta) = \frac{1}{2}\,\omega\nu_\beta^2 g(\nu_\beta,\nu_\beta)\Lambda'(\nu_\beta) \quad (4.20)$$

and

$$N(\nu) = \nu\left[\lambda^2(\nu) + \frac{1}{4}\pi^2\omega^2\nu^2 g^2(\nu,\nu)\right] . \quad (4.21)$$

4.3. Numerical Results and Conclusions

In order to demonstrate the accuracy of the F_N method we consider now the specific case of constant heat generation, i.e. $T(\tau) = T$, a constant (since there is only heat generation by radiation). Thus the appropriate particular solution is

$$I_p(\tau,\mu) = \frac{\sigma T^4}{\pi} , \quad (4.22)$$

and Eqs. (4.12) become

$$K_1(\xi) = \varepsilon_1 \left(\frac{\sigma T_1^4}{\pi}\right) A_o(\xi) + \frac{2}{\omega}(1-\omega)\left(\frac{\sigma T^4}{\pi}\right)$$
$$+ e^{-\Delta/\xi}\left[\varepsilon_2\left(\frac{\sigma T_2^4}{\pi}\right) B_o(\xi) - \frac{2}{\omega}(1-\omega)\left(\frac{\sigma T^4}{\pi}\right)\right], \tag{4.23a}$$

and

$$K_2(\xi) = \varepsilon_2 \left(\frac{\sigma T_2^4}{\pi}\right) A_o(\xi) + \frac{2}{\omega}(1-\omega)\left(\frac{\sigma T^4}{\pi}\right)$$
$$+ e^{-\Delta/\xi}\left[\varepsilon_1\left(\frac{\sigma T_1^4}{\pi}\right) B_o(\xi) - \frac{2}{\omega}(1-\omega)\left(\frac{\sigma T^4}{\pi}\right)\right]. \tag{4.23b}$$

We also now consider that the two surfaces have the same reflecting properties, i.e. $\rho_1^s = \rho_2^s$ and $\rho_1^d = \rho_2^d$ with $\varepsilon + \rho^s + \rho^d \leq 1$, thus it is apparent that if we choose $L = -\Delta/2$ we can express the desired solution as

$$I(\tau,\mu) = \left(\frac{\sigma T^4}{\pi}\right)\Phi(\tau,\mu) + \varepsilon_1\left(\frac{\sigma T_1^4}{\pi}\right)\Theta(\tau,\mu) + \varepsilon_2\left(\frac{\sigma T_2^4}{\pi}\right)\Theta(-\tau,-\mu). \tag{4.24}$$

Here $\Phi(\tau,\mu)$ satisfies Eq. (4.1), with $\sigma T^4/\pi = 1$, $\Phi(\tau,\mu) = \Phi(-\tau,-\mu)$ and the boundary condition

$$\Phi(-\Delta/2,\mu) = \rho^s\Phi(-\Delta/2,-\mu) + 2\rho^d\int_0^1 \Phi(-\Delta/2,-\mu')\mu'd\mu', \quad \mu > 0. \tag{4.25}$$

Also $\Theta(\tau,\mu)$ satisfies Eq. (4.1) with $\sigma T^4/\pi = 0$,

$$\Theta(-\Delta/2,\mu) = 1 + \rho^s\Theta(-\Delta/2,-\mu) + 2\rho^d\int_0^1 \Theta(-\Delta/2,-\mu')\mu'd\mu', \quad \mu>0, \tag{4.26a}$$

$$\Theta(\Delta/2,-\mu) = \rho^s\Theta(\Delta/2,\mu) + 2\rho^d\int_0^1 \Theta(\Delta/2,\mu')\mu'd\mu', \quad \mu > 0. \tag{4.26b}$$

The first basic problem clearly is symmetric and thus to establish $\Phi(\tau,\mu)$ we need only consider Eq. (4.18a) and solve, in the F_N approximation, (N + 1) linear algebraic equations. For the second problem we have the choice of solving 2(N + 1) simultaneous equations or, after we express $\Theta(\tau,\mu)$ in terms of symmetric and antisymmetric components, solving two independent system of (N + 1) equations.

For our numerical calculation we consider a scattering law, shown in Table 4.1, deduced from Mie scattering theory [26,70], and relevant to $n = 1.2$ and $x = 3$, where n is the index of refraction and x is the size parameter, defined as $x = \pi D/\lambda$, where D is the diameter of the scatter particle and λ is the radiation wave length. We consider three cases for the single-scattering albedo, $\omega = 0.2$, $\omega = 0.8$ and $\omega = 0.95$, and in Table 4.2 we list the discrete eigenvalues basic to these parameters (Appendix 11.1 discusses the computation of the eigenvalues).

Table 4.1: Scattering law

ℓ	$(2\ell + 1)f_\ell$
0	1.00000
1	2.35789
2	2.76628
3	2.20142
4	1.24514
5	0.51215
6	0.16096
7	0.03778
8	0.00667
9	0.00081
10	0.00000

Table 4.2: Discrete Eigenvalues

ω	ν_o	ν_1
0.2	1.06303332	--
0.8	2.43716171	1.05196601
0.95	6.34762059	1.14901459

As discussed, we solved two problems, one for $\Phi(\tau,\mu)$ and the other for $\Theta(\tau,\mu)$. The F_N equations for the first problem are given by

$$\sum_{\alpha=0}^{N} a_\alpha \left\{ B_\alpha(\xi_j)\left[1 - \rho^s e^{-\Delta/\xi_j}\right] + A_\alpha(\xi_j)\left[e^{-\Delta/\xi_j} - \rho^s\right] - \frac{2\rho^d}{\alpha+2}\left[A_o(\xi_j) + B_o(\xi_j)e^{-\Delta/\xi_j}\right]\right\}$$

$$= (1 - e^{-\Delta/\xi_j})\frac{2}{\omega}(1-\omega) \tag{4.27}$$

and for the second problem by

$$\sum_{\alpha=0}^{N} a_\alpha^* \left[B_\alpha(\xi_j) - \rho^s A_\alpha(\xi_j) - \frac{2\rho^d}{\alpha+2} A_o(\xi_j)\right]$$

$$+ e^{-\Delta/\xi_j} \sum_{\alpha=0}^{N} b_\alpha^* \left[A_\alpha(\xi_j) - \rho^s B_\alpha(\xi_j) - \frac{2\rho^d}{\alpha+2} B_o(\xi_j)\right] = A_o(\xi_j)$$

(4.28a)

and

$$\sum_{\alpha=0}^{N} b_\alpha^* \left[B_\alpha(\xi_j) - \rho^s A_\alpha(\xi_j) - \frac{2\rho^d}{\alpha + 2} A_o(\xi_j) \right]$$
$$+ e^{-\Delta/\xi_j} \sum_{\alpha=0}^{N} a_\alpha^* \left[A_\alpha(\xi_j) - \rho^s B_\alpha(\xi_j) - \frac{2\rho^d}{\alpha + 2} B_o(\xi_j) \right] = e^{-\Delta/\xi_j} B_o(\xi_j). \qquad (4.29b)$$

where we have used the approximations

$$\Phi(-\Delta/2,-\mu) = \Phi(\Delta/2,\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu > 0 \qquad (4.30)$$

and

$$\Theta(-\Delta/2,-\mu) = \sum_{\alpha=0}^{N} a_\alpha^* \mu^\alpha, \quad \mu > 0, \qquad (4.31a)$$

and

$$\Theta(\Delta/2,\mu) = \sum_{\alpha=0}^{N} b_\alpha^* \mu^\alpha, \quad \mu > 0. \qquad (4.31b)$$

To solve the above linear system of equations, we have chosen a different scheme of selecting points from that used in the previous chapter. We use here $\xi_\beta = \nu_{1,\beta}$, $\beta = 0, 1, 2, \ldots, \kappa - 1$, and the remaining given by

$$\xi_{j + \kappa - 1} = \frac{2j - 1}{2(N - \kappa + 1)}, \quad j = 1, 2, \ldots (N - \kappa + 1), \qquad (4.32)$$

where N is the order of approximation.

Here we wish to report the net radiative heat flux at the boundaries, i.e.

$$q(\pm\Delta/2) = \int_{-1}^{1} I(\pm\Delta/2,\mu)\mu d\mu \ , \tag{4.33}$$

or, in terms of the forward and backward partial fluxes, we can write

$$q(\pm\Delta/2) = q^{+}(\pm\Delta/2) - q^{-}(\pm\Delta/2) \ . \tag{4.34}$$

Using Eq. (4.24), we note that

$$q^{\pm}(\pm\Delta/2) = \left(\frac{\sigma T^4}{\pi}\right)\varphi^{\pm}(\pm\Delta/2) + \varepsilon_1\left(\frac{\sigma T_1^4}{\pi}\right)\vartheta^{\pm}(\pm\Delta/2)$$

$$+ \varepsilon_2\left(\frac{\sigma T_2^4}{\pi}\right)\vartheta^{\mp}(\mp\Delta/2) \tag{4.35}$$

where

$$\varphi^{\pm}(\pm\Delta/2) = \int_{0}^{1} \Phi(\pm\Delta/2, \pm\mu)\mu d\mu \tag{4.36a}$$

and

$$\vartheta^{\pm}(\pm\Delta/2) = \int_{0}^{1} \Theta(\pm\Delta/2,\pm\mu)\mu d\mu \ . \tag{4.36b}$$

In terms of the F_N approximation, we can write

$$\varphi^{-}(-\Delta/2) = \varphi^{+}(\Delta/2) = \sum_{\alpha=0}^{N}\left(\frac{a_\alpha}{\alpha + 2}\right) , \tag{4.37a}$$

$$\varphi^{+}(-\Delta/2) = \varphi^{-}(\Delta/2) = (\rho^s + \rho^d) \sum_{\alpha=0}^{N}\left(\frac{a_\alpha}{\alpha + 2}\right) , \tag{4.37b}$$

$$\vartheta^{-}(-\Delta/2) = \sum_{\alpha=0}^{N}\left(\frac{a_\alpha^*}{\alpha + 2}\right) , \tag{4.37c}$$

$$\vartheta^{+}(\Delta/2) = \sum_{\alpha=0}^{N}\left(\frac{b_\alpha^*}{\alpha + 2}\right) , \tag{4.37d}$$

$$\vartheta^{-}(\Delta/2) = (\rho^s + \rho^d) \sum_{\alpha=0}^{N}\left(\frac{b_\alpha^*}{\alpha + 2}\right) , \tag{4.37e}$$

and

$$\vartheta^{+}(-\Delta/2) = \frac{1}{2} + (\rho^s + \rho^d) \sum_{\alpha=0}^{N}\left(\frac{a_\alpha^*}{\alpha + 2}\right) . \tag{4.37f}$$

The basic quantities $\varphi^{-}(-\Delta/2)$, $\vartheta^{-}(-\Delta/2)$ and $\vartheta^{+}(\Delta/2)$ clearly can be used with Eq. (4.37) to deduce the net radiative heat flux, as given by Eq. (4.35), and thus these quantities are reported in Tables 4.3, 4.4 and 4.5 for the considered cases. In order to illustrate the effectiveness of the F_N method for this problem, we list in Tables 4.3 - 4.5 the results predicted by the approximation as N varied from 0 to 5. We also included the "exact" results deduced from the F_N method as N varied between 10 and 20. Clearly the considered cases of $\omega = 0.8$ and $\omega = 0.95$ can be solved by the F_o approximation to within 10% of error, which is adequate for some engineering applications.

We note that in contrast to the Eddington approximation [34,70], the F_o approximation, though particularly concise, includes the effect of the complete scattering law. For the cases considered the F_3 approximation is generally accurate to at least three significant figures, and for all considered cases of $\omega = 0.8$ and $\omega = 0.95$ the F_3 approximation is accurate to four significant figures. We consider this excellent especially since the computation time, on the IBM 370/165 machine, for the F_3 approximation of $\varphi^-(-\Delta/2)$, $\vartheta^-(-\Delta/2)$ and $\vartheta^+(\Delta/2)$ for a set of given values of ρ^s, ρ^d, Δ and ω is less than 10 seconds, which includes the calculation of the required discrete eigenvalues. We also note that we have chosen a different scheme for selecting the points that satisfy the F_N equations, from that used in the previous chapter. Since the choice of the points in the interval (0,1) is arbitrary, we have tried three different schemes: i) equally spaced points, as that discussed in chapter 3; ii) Gaussian quadrature nodes; and iii) the scheme used here. Although all these schemes converge to the same result, the scheme we used here is the one which gives the fastest convergence, and thus we concluded that it would be the best choice of points.

In conclusion, we note that once the constants a_α, a^*_α and b^*_α have been established the complete intensity $I(\tau,\mu)$ can be found in the manner discussed in the previous chapter.

Table 4.3: Partial Heat Fluxes for $\omega = 0.2$ and $\Lambda = 1$

Wall Reflectivity ρ^s	Wall Reflectivity ρ^d	Partial Heat Fluxes	F_o	F_1	F_2	F_3	F_4	F_5	"Exact"
0.0	0.5	$\varphi^-(L)$	0.3776	0.4186	0.4161	0.4163	0.4164	0.4164	0.4164
		$\vartheta^-(L)$	0.04128	0.02323	0.02475	0.02461	0.02459	0.02458	0.02458
		$\vartheta^+(R)$	0.2035	0.1395	0.1430	0.1427	0.1426	0.1426	0.1426
0.5	0.0	$\varphi^-(L)$	0.3776	0.4158	0.4137	0.4139	0.4140	0.4140	0.4140
		$\vartheta^-(L)$	0.04128	0.02763	0.02886	0.02861	0.02862	0.02863	0.02863
		$\vartheta^+(R)$	0.2035	0.1408	0.1438	0.1436	0.1435	0.1434	0.1434
0.25	0.25	$\varphi^-(L)$	0.3776	0.4173	0.4150	0.4152	0.4152	0.4153	0.4153
		$\vartheta^-(L)$	0.04128	0.02537	0.02677	0.02657	0.02657	0.02656	0.02656
		$\vartheta^+(R)$	0.2035	0.1400	0.1433	0.1430	0.1429	0.1429	0.1429

Table 4.4: Partial Heat Fluxes for ω = 0.8 and Δ = 1

Wall Reflectivity ρ^s	Wall Reflectivity ρ^d	Partial Heat Fluxes	F_0	F_1	F_2	F_3	F_4	F_5	"Exact"
0.0	0.5	$\varphi^-(L)$	0.2203	0.2341	0.2348	0.2342	0.2341	0.2341	0.2341
		$\vartheta^-(L)$	0.1641	0.1609	0.1633	0.1639	0.1639	0.1639	0.1639
		$\vartheta^+(R)$	0.3953	0.3710	0.3671	0.3678	0.3679	0.3678	0.3678
0.5	0.0	$\varphi^-(L)$	0.2203	0.2330	0.2328	0.2323	0.2323	0.2323	0.2323
		$\vartheta^-(L)$	0.1640	0.1635	0.1690	0.1696	0.1695	0.1695	0.1696
		$\vartheta^+(R)$	0.3953	0.3705	0.3654	0.3659	0.3659	0.3659	0.3659
0.25	0.25	$\varphi^-(L)$	0.2203	0.2336	0.2339	0.2333	0.2333	0.2333	0.2333
		$\vartheta^-(L)$	0.1640	0.1622	0.1661	0.1667	0.1666	0.1666	0.1666
		$\vartheta^+(R)$	0.3953	0.3706	0.3661	0.3667	0.3668	0.3668	0.3668

Table 4.5: Partial Heat Fluxes for $\omega = 0.95$ and $\Delta = 1$

Wall Reflectivity		Partial Heat Fluxes	F_0	F_1	F_2	F_3	F_4	F_5	"Exact"
ρ^s	ρ^d								
0.0	0.5	$\varphi^-(L)$	0.08285	0.08496	0.08502	0.08494	0.08494	0.08494	0.08494
		$\vartheta^-(L)$	0.2890	0.2956	0.2982	0.2982	0.2982	0.2982	0.2982
		$\vartheta^+(R)$	0.5453	0.5345	0.5318	0.5319	0.5319	0.5319	0.5319
0.5	0.0	$\varphi^-(L)$	0.08285	0.08474	0.08468	0.08462	0.08462	0.08462	0.08462
		$\vartheta^-(L)$	0.2890	0.2979	0.3029	0.3031	0.3030	0.3030	0.3030
		$\vartheta^+(R)$	0.5453	0.5326	0.5277	0.5277	0.5277	0.5277	0.5277
0.25	0.25	$\varphi^-(L)$	0.08285	0.08487	0.08487	0.08480	0.08480	0.08480	0.08480
		$\vartheta^-(L)$	0.2890	0.2969	0.3006	0.3007	0.3006	0.3006	0.3006
		$\vartheta^+(R)$	0.5453	0.5334	0.5297	0.5297	0.5298	0.5298	0.5298

5. ON THE CRITICAL PROBLEM FOR MULTIREGION REACTORS

5.1. Two Region Reactor[5]

The critical problem for a slab reactor with infinite reflector has been solved by Siewert and Burkart [88] through the use of a combination of Case's technique [21] with Chandrasekhar's invariance principles [25]. The case of the finite reflector was solved by Neshat, Siewert, and Ishiguro [66] using the principles of invariance and the P_N method; Ishiguro and Garcia [43] also solved this problem using a new method of regularization. Recently [28], this problem was reviewed and equations for the expansion coefficients were summarized.

Here, we wish to apply the F_N method, to the above problem in order to demonstrate the simplicity of the method and the computation merits of the technique in solving critical problems.

We consider the one-speed transport equation for the core, $-\tau \leq x \leq \tau$, and the reflector, $\tau < |x| < b$, written in the familiar manner [21]

$$\mu \frac{\partial}{\partial x} \Psi_\alpha(x,\mu) + \Psi_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} \Psi_\alpha(x,\mu')d\mu', \quad \alpha = 1 \text{ and } 2, \tag{5.1}$$

where $\alpha = 1$ implies the core and $\alpha = 2$ the reflector. We take $c_1 > 1$ and $c_2 \leq 1$ and seek solutions of Eq. (5.1) subject to the boundary conditions

$$\Psi_1(\tau,\mu) = \Psi_2(\tau,\mu), \ \mu > 0, \ \mu < 0, \tag{5.2a}$$

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[5]This section is based on a paper accepted for publication in Annals of Nucl. Energy [67].

$$\Psi_2(b,-\mu) = 0, \ \mu > 0, \tag{5.2b}$$

and

$$\Psi_2(-b,\mu) = 0, \ \mu > 0. \tag{5.2c}$$

Here we consider c_1 and c_2, as well as the reflector thickness $\Delta = b - \tau$, to be given and thus seek the critical half-thickness τ.

We begin by writing the solution of Eq. (5.1) in terms of the elementary solutions [21]

$$\Psi_1(x,\mu) = A(\nu_o)\left[\phi_1(\nu_o,\mu)e^{-x/\nu_o} + \phi_1(-\nu_o,\mu)e^{x/\nu_o}\right] + \int_0^1 A(\nu)\left[\phi_1(\nu,\mu)e^{-x/\nu} + \phi_1(-\nu,\mu)e^{x/\nu}\right]d\nu \tag{5.3}$$

and

$$\Psi_2(x,\mu) = B(\eta_o)\phi_2(\eta_o,\mu)e^{-x/\eta_o} + B(-\eta_o)\phi_2(-\eta_o,\mu)e^{x/\eta_o} + \int_0^1 B(\eta)\phi_2(\eta,\mu)e^{-x/\eta}d\eta + \int_0^1 B(-\eta)\phi_2(-\eta,\mu)e^{x/\eta}d\eta \, . \tag{5.4}$$

Using the full-range orthogonality properties of the eigenfunctions, we note that

$$\int_{-1}^{1} \mu\phi_1(-\xi,\mu)\Psi_1(x,\mu)d\mu = N_1(-\xi)A(-\xi)e^{x/\xi}, \xi \in P_1, \tag{5.5a}$$

and

$$\int_{-1}^{1} \mu\phi_2(-\hat{\xi},\mu)\Psi_2(x,\mu)d\mu = N_2(-\hat{\xi})B(-\hat{\xi})e^{x/\hat{\xi}}, \hat{\xi}\subset P_2, \tag{5.5b}$$

where N $(-\xi)$ are the full-range normalization factor and P_1 and P_2 are the intervals defined by $P_1 = \nu_o \cup (0,1)$ and $P_2 = \eta_o \cup (0,1)$.

If we write Eq. (5.5b) for $x = \tau$ and $x = b$, and then eliminate $N_2(-\hat{\xi})B(-\hat{\xi})$ between the resulting equation we obtain

$$\int_0^1 \mu\phi_2(-\hat{\xi},\mu)\Psi_2(\tau,\mu)d\mu - \int_0^1 \mu\phi_2(\hat{\xi},\mu)\Psi_2(\tau,-\mu)d\mu$$
$$= e^{-\Delta/\hat{\xi}} \int_0^1 \mu\phi_2(-\hat{\xi},\mu)\Psi_2(b,\mu)d\mu \ , \tag{5.6}$$

and writing Eq. (5.5b) for $x = -\tau$ and $x = -b$, we find

$$-\int_0^1 \mu\phi_2(\hat{\xi},\mu)\Psi_2(-b,-\mu)d\mu = e^{-\Delta/\hat{\xi}}\left[\int_0^1 \mu\phi_2(-\hat{\xi},\mu)\Psi_2(-\tau,\mu)d\mu \right.$$
$$\left. -\int_0^1 \mu\phi_2(\hat{\xi},\mu)\Psi_2(-\tau,-\mu)d\mu\right]. \tag{5.7}$$

In the same manner, we write Eq. (5.5a) for $x = \tau$ and $x = -\tau$, to obtain

$$\int_0^1 \mu\phi_1(-\xi,\mu)\Psi_1(-\tau,\mu)d\mu - \int_0^1 \mu\phi_1(\xi,\mu)\Psi_1(-\tau,-\mu)d\mu =$$
$$e^{-2\tau/\xi}\left[\int_0^1 \mu\phi_1(-\xi,\mu)\Psi_1(\tau,\mu)d\mu - \int_0^1 \mu\phi_1(\xi,\mu)\Psi_1(\tau,-\mu)d\mu\right]. \tag{5.8}$$

Now, we can introduce the F_N approximation, as reported by Grandjean and Siewert [39],

$$\Psi_1(\tau,\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \ \mu > 0, \tag{5.9a}$$

$$\Psi_1(\tau,-\mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \ \mu > 0, \tag{5.9b}$$

and

$$\Psi_2(b,\mu) = \Psi_2(-b,-\mu) = \sum_{\alpha=0}^{N} e_\alpha \mu^\alpha, \ \mu > 0, \tag{5.9c}$$

into Eqs. (5.6), (5.7) and (5.8). If we make use of the boundary conditions, Eqs. (5.2), we obtain the F_N equations for this problem: i.e.

$$\sum_{\alpha=0}^{N} a_\alpha A_\alpha(\hat{\xi}) - \sum_{\alpha=0}^{N} b_\alpha B_\alpha^{(2)}(\hat{\xi}) = e^{-\Delta/\hat{\xi}} \sum_{\alpha=0}^{N} e_\alpha A_\alpha(\hat{\xi}), \tag{5.10}$$

$$\sum_{\alpha=0}^{N} b_\alpha A_\alpha(\hat{\xi}) - \sum_{\alpha=0}^{N} a_\alpha B_\alpha^{(2)}(\hat{\xi}) = -e^{-\Delta/\hat{\xi}} \sum_{\alpha=0}^{N} e_\alpha B_\alpha^{(2)}(\hat{\xi}), \tag{5.11}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi) - \sum_{\alpha=0}^{N} a_\alpha B_\alpha^{(1)}(\xi) = e^{-2\tau/\xi} \left[\sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi) - \sum_{\alpha=0}^{N} b_\alpha B_\alpha^{(1)}(\xi) \right] \tag{5.12}$$

where $A_\alpha(\xi)$ and $B_\alpha^{(i)}(\xi)$ are those quantities given by Eqs. (3.30) and (3.31).

To find the critical half-thickness we let $a_o = 1$ and give as an initial value of τ as computed by the F_o approximation; i.e.

$$\tau^{(0)} = -\frac{\nu_o}{2}\log\left[\frac{b_oA_o(\nu_o) - B_o^{(1)}(\nu_o)}{A_o(\nu_o) - b_oB_o^{(1)}(\nu_o)}\right] \tag{5.13}$$

with

$$b_o = \frac{A_o(\eta_o)B_o^{(2)}(\eta_o)\left[1 - e^{-2\Delta/\eta_o}\right]}{\left[B_o^{(2)}(\eta_o)\right]^2 - \left[A_o(\eta_o)\right]^2 e^{-2\Delta/\eta_o}} . \tag{5.14}$$

We then solve Eqs. (5.10), (5.11) with $\hat{\xi}\in P_2$ and Eq. (5.12) with $\xi\in(0,1)$ for the coefficients a_α, b_α, and e_α. Finally we insert these coefficients into

$$e^{-2\tau/\nu_o}\sum_{\alpha=0}^{N}\left[b_\alpha B_\alpha^{(1)}(\nu_o) - a_\alpha A_\alpha(\nu_o)\right]$$
$$= \sum_{\alpha=0}^{N}\left[a_\alpha B_\alpha^{(1)}(\nu_o) - b_\alpha A_\alpha(\nu_o)\right] \tag{5.15}$$

in order to find a new value for τ. The iteration is repeated until a converged result is obtained.

To solve Eqs. (5.10), (5.11) and (5.12) for the coefficients, we pick values of $\hat{\xi}\in P_2$ and $\xi\in(0,1)$ according to the scheme suggested by Grandjean and Siewert [39]; i.e., for $\hat{\xi}$ we pick $\hat{\xi}_o = \eta_o$, $\hat{\xi}_1 = 0$, $\hat{\xi}_2 = 1$, and the remaining $\hat{\xi}_\beta$ spaced equally in [0,1], and for ξ we pick $\xi_o = 0$, $\xi_1 = 1$, and the remaining also equally spaced in [0,1]. For each iteration we solve (3N + 2) linear system of equations. The scheme generally converges in three or four iterations.

In Table 5.1, we show the cases studied, and in Table 5.2, we report the values of the half-thickness for various orders of approximation, and the "exact" value reported by Burkart [15].

We note that the F_N method yields results that are accurate to three or four significant figures for $N \leq 5$, which is remarkably good considering the simplicity of the method. The computation time on the IBM 370/165 machine is on the order of 3 seconds per case for $N \leq 8$. We also note that to solve the liner algebraic system of equations we have used a Gauss-Seidel scheme.

Table 5.1. Cases studied for a two-region reactor

Case	c_1	c_2	$\Delta = b - \tau$
1	1.01	0.09	0.5
2	1.01	0.90	1.0
3	1.30	0.09	0.5
4	1.30	0.90	1.0
5	1.50	0.09	0.5
6	1.50	0.90	1.0
7	1.91	0.09	0.5
8	1.91	0.90	1.0

Table 5.2. Critical Half-thickness for a two-region reactor

Case	F_0	F_3	F_4	F_5	F_6	F_7	"Exact"
1	8.3587	8.3106	8.3106	8.3107	8.3107	8.3107	8.3107
2	7.7346	7.6776	7.6777	7.6777	7.6778	7.6778	7.6778
3	0.9323	0.9246	0.9245	0.9245	0.9245	0.9246	0.9246
4	0.6230	0.6025	0.6026	0.6026	0.6026	0.6027	0.6027
5	0.5891	0.5939	0.5943	0.5943	0.5943	0.5943	0.5943
6	0.3671	0.3590	0.3594	0.3597	0.3597	0.3597	0.3597
7	0.3171	0.3324	0.3334	0.3343	0.3346	0.3346	0.3346
8	0.1841	0.1871	0.1883	0.1886	0.1892	0.1893	0.1893

5.2. Three Region Reactor[6]

Here, we wish to consider the application of the F_N method to solve the critical problem for a slab reactor consisting of a core, a blanket region and a reflector. In terms of the optical variable x, we consider the core material between x = -a and x = a to have $c_1 > 1$ as the mean number of secondary neutrons per collision. We take x = 0 to be a plane of symmetry and thus let $x \in [a,b]$, with $c_2 \geq 1$ or $c_2 < 1$, define the blanket region. For the reflector, we take $x \in [b,c]$, and let $c_3 < 1$. We consider that c_1, c_2 and c_3 are given and thus seek the critical half-thickness a which corresponds to a given blanket thickness, $\Delta_b = b - a$, and reflector thickness, $\Delta_r = c - b$.

We use here the one group model of the neutron-transport equation and thus seek a real, non-negative solution of

$$\mu \frac{\partial}{\partial x} \Psi_\alpha(x,\mu) + \Psi_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} \Psi_\alpha(x,\mu')d\mu' ,$$

$$\alpha = 1,\ 2 \text{ and } 3, \tag{5.16}$$

which obey the conditions.

$$\Psi_1(x,\mu) = \Psi_1(-x,-\mu),\ x \in (0,a), \tag{5.17a}$$

$$\Psi_1(a,\mu) = \Psi_2(a,\mu),\ \mu \in (-1,1), \tag{5.17b}$$

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[6] This section is partially based on a paper accepted for publication in Annals of Nuclear Energy [51].

$$\Psi_2(b,\mu) = \Psi_3(b,\mu), \ \mu \in (-1,1), \tag{5.17c}$$

$$\Psi_3(c,-\mu) = 0, \ \mu \in (0,1), \tag{5.17d}$$

and

$$\Psi_3(-c,\mu) = 0, \ \mu \in (0,1). \tag{5.17e}$$

The solution of Eq. (5.16) can be written in terms of elementary solutions as

$$\Psi_1(x,\mu) = A(\nu_1)\left[\phi_1(\nu_1,\mu)e^{-x/\nu_1} + \phi_1(-\nu_1,\mu)e^{x/\nu_1}\right] + \int_0^1 A_1(\nu)\left[\phi_1(\nu,\mu)e^{-x/\nu} + \phi_1(-\nu,\mu)e^{x/\nu}\right]d\nu, \tag{5.18a}$$

which includes the symmetry condition, Eq. (5.12a),

$$\Psi_2(x,\mu) = A(\nu_2)\phi_2(\nu_2,\mu)e^{-x/\nu_2} + A(-\nu_2)\phi_2(-\nu_2,\mu)e^{x/\nu_2} + \int_{-1}^1 A_2(\nu)\phi_2(\nu,\mu)e^{-x/\nu}d\nu, \tag{5.18b}$$

and

$$\Psi_3(x,\mu) = A(\nu_3)\phi_3(\nu_3,\mu)e^{-x/\nu_3} + A(-\nu_3)\phi_3(-\nu_3,\mu)e^{x/\nu_3} + \int_{-1}^1 A_3(\nu)\phi_3(\nu,\mu)e^{-x/\nu}d\nu \tag{5.18c}$$

where $\phi_\alpha(\xi,\mu)$ are the Case's eigenfunctions and ν_α the discrete eigenvalues. Now, using the full-range orthogonality properties of the eigenfunctions we note that

$$\int_{-1}^{1} \mu\phi_1(-\xi,\mu)\Psi_1(x,\mu)d\mu = N_1(-\xi)A_1(\xi)e^{x/\xi}, \quad \xi \in P_1, \tag{5.19a}$$

$$\int_{-1}^{1} \mu\phi_2(-\hat{\xi},\mu)\Psi_2(x,\mu)d\mu = N_2(-\hat{\xi})A_2(-\hat{\xi})e^{x/\hat{\xi}}, \quad \hat{\xi} \in P_2, \tag{5.19b}$$

and

$$\int_{-1}^{1} \mu\phi_3(-\xi^*,\mu)\Psi_3(x,\mu)d\mu = N_3(-\xi^*)A_3(-\xi^*)e^{x/\xi^*}, \quad \xi^* \in P_3, \tag{5.19c}$$

where $N_\alpha(-\xi)$ are the full-range normalization factor and P_1, P_2 and P_3 are the intervals defined by $P_1 = \nu_1 \cup (0,1)$, $P_2 = \nu_2 \cup (0,1)$ and $P_3 = \nu_3 \cup (0,1)$.

If we write Eq. (5.19a) for x = a and x = -a, and then eliminate $N_1(-\xi)A_1(\xi)$ between the resulting equations, we obtain

$$\int_{0}^{1} \mu\phi_1(-\xi,\mu)\Psi_1(-a,\mu)d\mu - \int_{0}^{1} \mu\phi_1(\xi,\mu)\Psi_1(a,\mu)d\mu$$
$$= e^{-2a/\xi}\left\{\int_{0}^{1} \mu\phi_1(-\xi,\mu)\Psi_1(a,\mu)d\mu - \int_{0}^{1} \mu\phi_1(\xi,\mu)\Psi_1(a,-\mu)d\mu\right\}. \tag{5.20a}$$

In the same manner, we write Eq. (5.19b) for x = a and x = b, x = -a and x = -b, and Eq. (5.19c) for x = b and x = c, and x = -b and x = -c, to obtain

$$\int_0^1 \mu\phi_2(-\hat{\xi},\mu)\Psi_1(a,\mu)d\mu - \int_0^1 \mu\phi_2(\hat{\xi},\mu)\Psi_1(a,-\mu)d\mu$$

$$= e^{-\Delta_b/\hat{\xi}} \left\{ \int_0^1 \mu\phi_2(-\hat{\xi},\mu)\Psi_2(b,\mu)d\mu - \int_0^1 \mu\phi_2(\hat{\xi},\mu)\Psi_2(b,-\mu)d\mu \right\}, \tag{5.20b}$$

$$\int_0^1 \mu\phi_3(-\xi^*,\mu)\Psi_2(b,\mu)d\mu - \int_0^1 \mu\phi_3(\xi^*,\mu)\Psi_2(b,-\mu)d\mu$$

$$= e^{-\Delta_r/\xi^*} \int_0^1 \mu\phi_3(-\xi^*,\mu)\Psi_3(c,\mu)d\mu, \tag{5.20c}$$

$$\int_0^1 \mu\phi_2(-\hat{\xi},\mu)\Psi_2(-b,\mu)d\mu - \int_0^1 \mu\phi_2(\hat{\xi},\mu)\Psi_2(-b,-\mu)d\mu$$

$$= e^{-\Delta_b/\hat{\xi}} \left\{ \int_0^1 \mu\phi_2(-\hat{\xi},\mu)\Psi_1(-a,\mu)d\mu - \int_0^1 \mu\phi_2(\hat{\xi},\mu)\Psi_1(a,\mu)d\mu \right\}, \tag{5.20d}$$

and

$$e^{-\Delta_r/\xi^*} \left\{ \int_0^1 \mu\phi_3(\xi^*,\mu)\Psi_2(-b,-\mu)d\mu - \int_0^1 \mu\phi_3(-\xi^*,\mu)\Psi_2(-b,\mu)d\mu \right\}$$

$$= \int_0^1 \mu\phi_3(\xi^*,\mu)\Psi_3(-c,-\mu)d\mu, \tag{5.20e}$$

which consist of a set of singular integral equations for the angular fluxes at the boundaries. We note that Eqs. (5.20) are "exact", however, here we want to introduce the F_N approximations

$$\Psi_1(a,\mu) = \Psi_1(-a,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu \in (0,1), \tag{5.21a}$$

$$\Psi_1(-a,\mu) = \Psi_1(a,-\mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \mu \in (0,1), \tag{5.21b}$$

$$\Psi_2(b,\mu) = \Psi_2(-b,-\mu) = \sum_{\alpha=0}^{N} d_\alpha \mu^\alpha, \quad \mu \in (0,1), \tag{5.21c}$$

$$\Psi_2(b,-\mu) = \Psi_2(-b,\mu) = \sum_{\alpha=0}^{N} e_\alpha \mu^\alpha, \quad \mu \in (0,1), \tag{5.21d}$$

and

$$\Psi_3(c,\mu) = \Psi_3(-c,-\mu) \sum_{\alpha=0}^{N} f_\alpha \mu^\alpha, \quad \mu \in (0,1), \tag{5.21e}$$

to obtain the F_N equations for this problem, i.e.

$$\sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi) - \sum_{\alpha=0}^{N} a_\alpha B_\alpha^{(1)}(\xi) = e^{-2a/\xi} \left\{ \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi) - \sum_{\alpha=0}^{N} b_\alpha B_\alpha^{(1)}(\xi) \right\}, \tag{5.22a}$$

$$\sum_{\alpha=0}^{N} a_\alpha A_\alpha(\hat{\xi}) - \sum_{\alpha=0}^{N} b_\alpha B_\alpha^{(2)}(\hat{\xi}) = e^{-\Delta_b/\hat{\xi}} \left\{ \sum_{\alpha=0}^{N} d_\alpha A_\alpha(\hat{\xi}) - \sum_{\alpha=0}^{N} e_\alpha B_\alpha^{(2)}(\hat{\xi}) \right\}, \tag{5.22b}$$

$$\sum_{\alpha=0}^{N} d_\alpha A_\alpha(\xi^*) - \sum_{\alpha=0}^{N} e_\alpha B_\alpha^{(3)}(\xi^*) = e^{-\Delta_r/\xi^*} \sum_{\alpha=0}^{N} f_\alpha A_\alpha(\xi^*), \tag{5.22c}$$

$$\sum_{\alpha=0}^{N} e_\alpha A_\alpha(\hat{\xi}) - \sum_{\alpha=0}^{N} d_\alpha B_\alpha^{(2)}(\hat{\xi}) = e^{-\Delta_b/\hat{\xi}} \sum_{\alpha=0}^{N} b_\alpha A_\alpha(\hat{\xi}) - \sum_{\alpha=0}^{N} a_\alpha B_\alpha^{(2)}(\hat{\xi}) \quad , \tag{5.22d}$$

and

$$e^{-\Delta_r/\xi^*}\left\{\sum_{\alpha=0}^{N} d_\alpha B_\alpha^{(3)}(\xi^*) - \sum_{\alpha=0}^{N} e_\alpha A_\alpha(\xi^*)\right\} = \sum_{\alpha=0}^{N} f_\alpha B_\alpha^{(3)}(\xi^*), \tag{5.22e}$$

where the superscript in the functions $B_\alpha(\xi)$ indicates the region where we should evaluate these functions.

To find the critical half-thickness we let $a_o = 1$ and give as an initial value, the one computed by the F_o-approximation, i.e.

$$a^{(0)} = -\frac{\nu_1}{2} \log\left[\frac{b_o A_o(\nu_1) - B_o^{(1)}(\nu_1)}{A_o(\nu_1) - b_o B_o^{(1)}(\nu_1)}\right] , \tag{5.23}$$

where

$$b_o = \frac{G(\nu_2,\nu_3) A_o(\nu_2) + e^{-2\Delta_b/\nu_2} B_o^{(2)}(\nu_2)}{e^{-2\Delta_b/\nu_2} A_o(\nu_2) + B_o^{(2)}(\nu_2) G(\nu_2,\nu_3)} \quad , \tag{5.24}$$

with

$$G(\nu_2,\nu_3) = \frac{A_o(\nu_2) - B_o^{(2)}(\nu_2) F(\nu_3)}{A_o(\nu_2) F(\nu_3) - B_o^{(2)}(\nu_2)} \quad , \tag{5.25}$$

and

$$F(\nu_3) = \frac{[B_o^{(3)}(\nu_3)]^2 - e^{-2\Delta_r/\nu_3}[A_o(\nu_3)]^2}{A_o(\nu_3)B_o^{(3)}(\nu_3)[1 - e^{-2\Delta_r/\nu_3}]} , \tag{5.26}$$

We then insert the first guess, $a^{(0)}$, into Eqs. (5.22) with $\hat{\xi} \in P_2$ for Eqs. (5.22b and d), $\xi^* \in P_3$ for Eqs. (5.22c and e) and $\xi \in (0,1)$ for Eq. (5.22a), thus solving for the coefficients a_α, b_α, d_α and f_α. Finally, we insert these coefficients into

$$e^{-2a/\nu_1} \sum_{\alpha=0}^{N} \left[b_\alpha B_\alpha^{(1)}(\nu_1) - a_\alpha A_\alpha(\nu_1) \right] =$$

$$\sum_{\alpha=0}^{N} \left[a_\alpha B_\alpha^{(1)}(\nu_1) - b_\alpha A_\alpha(\nu_1) \right] , \tag{5.27}$$

in order to find a new value for a. The iteration is repeated until a converged result is obtained.

In order to demonstrate the numerical accuracy of the F_N method for solving the critical problem for a three-region reactor, we select the F_N points, ξ, $\hat{\xi}$ and ξ^*, accordingly with the scheme described in chapter 4, and we were able to solve the iterative scheme described above. In Table 5.4 we list the critical half-thickness for the cases specified in Table 5.3. To have an idea of the simplicity and speed of the method, a typical calculation of the critical half-thickness involves the solution of system of (5N + 4) linear equations per iteration (N is the order of the approximation), and usually the scheme converges in three to five iterations. The time of computation to perform these calculations, from N = 0 up to

N = 8, is of the order of 40 seconds per case, on an IBM 370/165 machine, which includes the evaluation of the discrete eigenvalues for the three regions considered in each case.

We note that the F_o approximation gives results within 10% of the converged F_N-approximation, which can be considered remarkably good considering that this approximation can be obtained by solving explicitly Eq. (5.24).

Table 5.3: Cases Studied for a Three-Region Reactor

Case	c_1	c_2	c_3	Δ_b	Δ_r
1	1.3	0.95	0.9	1.0	∞
2	1.3	0.95	0.9	2.0	∞
3	1.3	1.10	0.9	1.0	∞
4	1.3	1.10	0.9	0.5	∞
5	1.3	0.95	0.9	1.0	3.0
6	1.3	0.95	0.9	2.0	3.0
7	1.3	1.10	0.9	1.0	3.0
8	1.3	1.10	0.9	0.5	3.0
9	1.3	0.95	0.9	1.0	1.0
10	1.3	0.95	0.9	2.0	1.0
11	1.3	1.10	0.9	1.0	1.0
12	1.3	1.10	0.9	0.5	1.0

Table 5.4: Critical Half-Thickness for a Three-Region Reactor

Case	F_o	F_3	F_5	F_7	"Exact"[6]
1	0.410	0.40731	0.40733	0.40734	0.40734
2	0.380	0.38029	0.38031	0.38032	0.38032
3	0.112	0.10629	0.10626	0.10625	0.10625
4	0.278	0.27109	0.27114	0.27114	0.27114
5	0.420	0.41296	0.41299	0.41299	0.41299
6	0.385	0.38278	0.38281	0.38282	0.38282
7	0.129	0.11612	0.11609	0.11607	0.11607
8	0.298	0.28203	0.28207	0.28207	0.28207
9	0.483	0.45542	0.45544	0.45545	0.45545
10	0.412	0.40091	0.40094	0.40095	0.40095
11	0.233	0.19024	0.19018	0.19018	0.19018
12	0.423	0.36781	0.36783	0.36783	0.36783

[6] By "Exact" we mean the converged F_N result, as N varies between 10 and 20.

6. ON THE CALCULATION OF THE THERMAL DISADVANTAGE FACTOR FOR A CELL WITH GENERAL ANISOTROPIC SCATTERING[7]

6.1. Introduction

We consider in this chapter the application of the F_N method to compute the thermal disadvantage factor required in the calculation of thermal utilization in heterogeneous reactor cells. It is well known, from the standard reactor physics books [50,64,113], that the disadvantage factor is a measure of the differences in the neutron fluxes in the fuel and moderator, being defined in plane geometry as

$$\zeta = \frac{a}{\Delta} \frac{\int_a^b dx \int_{-1}^{1} \Psi_2(x,\mu)\,d\mu}{\int_0^a dx \int_{-1}^{1} \Psi_1(x,\mu)\,d\mu}\,, \tag{6.1}$$

where the subscripts 1 and 2 denote the fuel and moderator regions of a basic reactor cell, and 'a' and '$\Delta = b - a$' their optical half-thickness, respectively.

Most of the early calculations of this parameter were based on diffusion-theory [72], Amouyal-Benoist-Horowitz (A-B-H) method [2,104] and approximated transport-techniques [18,104,75]. Ferziger and Robinson [36] were the first in using the method of singular eigenfunctions to

[7]This chapter is partially based on a paper accepted for publication in Annals of Nuclear Energy [51].

calculate ζ for an isotropically scattering cells. Bond and Siewert [10], hereafter referred as B & S, extended this method and solved a more realistic case of linear anisotropic scattering in the moderator. A more general approach for this problem was made by Eccleston and McCormick [33], hereafter referred as E & M, who intended to calculate ζ for a cell with arbitrary anisotropic scattering in the moderator, however the numerical results of this work were questioned by the work of Laletin, Sultanov, Vlasov, and Koniaev [49], hereafter referred as L-S-V-K. Here, we wish to report some numerical results, in order to demonstrate the computation merits of the F_N method, and to decide the controversy between the works of E & M and L-S-V-K.

6.2. Analysis

The appropriate one-group neutron transport equations for the fuel and moderator of a basic reactor cell are given by

$$\mu \frac{\partial}{\partial x} \Psi_1(x,\mu) + \Psi_1(x,\mu) = \frac{1}{2} c_1 \sum_{\ell=0}^{L_1} (2\ell + 1) f_{1,\ell} P_\ell(\mu) \int_{-1}^{1} P_\ell(\mu') \Psi_1(x,\mu') d\mu',$$

$$0 \leq x \leq a, \qquad (6.2a)$$

and

$$\mu \frac{\partial}{\partial x} \Psi_2(x,\mu) + \Psi_2(x,\mu) = \frac{1}{2} c_2 \sum_{\ell=0}^{L_2} (2\ell + 1) f_{2,\ell} P_\ell(\mu) \int_{-1}^{1} P_\ell(\mu') \Psi_2(x,\mu') d\mu'$$

$$+ \frac{s}{\sigma_2}, \quad a \leq x \leq b. \qquad (6.2b)$$

Here we allow general anisotropic scattering in both regions, however, it is well known that the assumption of isotropic scattering in the heavy fuel region is reasonable, and thus we could consider $L_1 = 0$, whereas in the light moderator region the scattering is anisotropic, and thus $L_2 > 0$. The term S represents a constant and isotropic source resulting from neutrons slowing-down in the moderator, σ_2 is the total macroscopic cross-section in the moderator, and the relevant boundary conditions are

$$\Psi_1(0,\mu) = \Psi_1(0,-\mu) \; , \; \mu > 0, \tag{6.3a}$$

$$\Psi_2(b,\mu) = \Psi_2(b,-\mu) \; , \; \mu > 0, \tag{6.3b}$$

$$\Psi_1(a,\mu) = \Psi_2(a,\mu) \; , \; \mu > 0, \tag{6.3c}$$

$$\Psi_1(a,-\mu) = \Psi_2(a,-\mu) \; , \; \mu > 0, \tag{6.3d}$$

$$\int_{-1}^{1} \mu\Psi_1(0,\mu)d\mu = 0, \tag{6.3e}$$

and

$$\int_{-1}^{1} \mu\Psi_1(b,\mu)d\mu = 0. \tag{6.3f}$$

Since $\Psi_1(x,\mu)$ and

$$\Psi_2^*(x,\mu) = \Psi_2(x,\mu) - \frac{S}{\sigma_2(1 - c_2)} \tag{6.4}$$

can be expressed in terms of elementary solutions [62] of Eqs. (6.2), we can use the full-range orthogonality of the eigenfunctions along with boundary conditions, given by Eqs. (6.3), to deduce a set of singular integral equations:

$$\int_0^1 \mu \left[\phi_1(\xi,\mu) - \phi_1(-\xi,\mu) \right] \Psi(0,\mu) d\mu$$
$$+ e^{-a/\xi} \int_0^1 \mu \left[\phi_1(-\xi,\mu)\Psi(a,\mu) - \phi_1(\xi,\mu)\Psi(a,-\mu) \right] d\mu = 0, \tag{6.5a}$$

$$\int_0^1 \mu \left[\phi_1(\xi,\mu)\Psi(a,\mu) - \phi_1(-\xi,\mu)\Psi(a,-\mu) \right] d\mu$$
$$+ e^{-a/\xi} \int_0^1 \mu \left[\phi_1(-\xi,\mu) - \phi_1(\xi,\mu) \right] \Psi(0,\mu) d\mu = 0, \tag{6.5b}$$

$$\int_0^1 \mu \left[\phi_2(\xi,\mu)\Psi(a,-\mu) - \phi_2(-\xi,\mu)\Psi(a,\mu) \right] d\mu$$
$$+ e^{-\Delta/\xi} \int_0^1 \mu \left[\phi_2(-\xi,\mu) - \phi_2(\xi,\mu) \right] \Psi(b,\mu) d\mu$$
$$= \frac{S\xi}{\sigma_2} \left[1 - e^{-\Delta/\xi} \right], \tag{6.6a}$$

and

$$\int_0^1 \mu \left[\phi_2(\xi,\mu) - \phi_2(-\xi,\mu) \right] \Psi(b,\mu) d\mu$$
$$+ e^{-\Delta/\xi} \int_0^1 \mu \left[\phi_2(-\xi,\mu)\Psi(a,-\mu) - \phi_2(\xi,\mu)\Psi(a,\mu) \right] d\mu$$
$$= \frac{S\xi}{\sigma_2} \left[1 - e^{-\Delta/\xi} \right], \tag{6.6b}$$

where $\phi_j(\xi,\mu)$, $j = 1, 2$, are the eigenfunctions given by Eqs. (4.4a) and (4.7), and we omit the subscript in the angular distributions. We note that Eqs. (6.5) require $\xi \in P_1 = \{\nu_{1,\beta}\} \cup (0,1)$, and Eqs. (6.6) require $\xi \in P_2 = \{\nu_{2,\beta}\} \cup (0,1)$. If we now introduce, for $\mu > 0$, the approximations

$$\Psi(0,\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha , \tag{6.7a}$$

$$\Psi(a,-\mu) = \sum_{\alpha=0}^{N} e_\alpha \mu^\alpha , \tag{6.7b}$$

$$\Psi(a,\mu) = \sum_{\alpha=0}^{N} f_\alpha \mu^\alpha , \tag{6.7c}$$

and

$$\Psi(b,\mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha \tag{6.7d}$$

into Eqs. (6.5 and 6) and consider the resulting equations for selected values of ξ_β, then our F_N results can be obtained here by solving the following system of $4(N + 1)$ linear algebraic equations:

$$\sum_{\alpha=0}^{N} \left\{ a_\alpha \left[B_\alpha^{(1)}(\xi_\beta) - A_\alpha^{(1)}(\xi_\beta) \right] + e^{-a/\xi_\beta} \left[f_\alpha A_\alpha^{(1)}(\xi_\beta) - e_\alpha B_\alpha^{(1)}(\xi_\beta) \right] \right\} = 0, \tag{6.8a}$$

$$\sum_{\alpha=0}^{N} \left\{ \left[f_\alpha B_\alpha^{(1)}(\xi_\beta) - e_\alpha A_\alpha^{(1)}(\xi_\beta) \right] \right.$$

$$+ e^{-a/\xi_\beta} a_\alpha \left[A_\alpha^{(1)}(\xi_\beta) - B_\alpha^{(1)}(\xi_\beta) \right] \Bigg\} = 0, \tag{6.8b}$$

$$\sum_{\alpha=0}^{N} \Bigg\{ \left[e_\alpha B_\alpha^{(2)}(\xi_\beta) - f_\alpha A_\alpha^{(2)}(\xi_\beta) \right]$$

$$+ e^{-\Delta/\xi_\beta} b_\alpha \left[A_\alpha^{(2)}(\xi_\beta) - B_\alpha^{(2)}(\xi_\beta) \right] \Bigg\}$$

$$= \frac{2S}{c_2 \sigma_2} \left[1 - e^{-\Delta/\xi_\beta} \right], \tag{6.9a}$$

and

$$\sum_{\alpha=0}^{N} \Bigg\{ b_\alpha \left[B_\alpha^{(2)}(\xi_\beta) - A_\alpha^{(2)}(\xi_\beta) \right]$$

$$+ e^{-\Delta/\xi_\beta} \left[e_\alpha A_\alpha^{(2)}(\xi_\beta) - f_\alpha B_\alpha^{(2)}(\xi_\beta) \right] \Bigg\}$$

$$= \frac{2S}{c_2 \sigma_2} \left[1 - e^{-\Delta/\xi_\beta} \right]. \tag{6.9b}$$

The functions $A_\alpha^{(i)}(\xi)$ and $B_\alpha^{(i)}(\xi)$ appearing in Eqs. (6.8) and (6.9) are those reported earlier in chapter 4, and thus we do not repeat all the required definitions. Now, if we integrate Eqs. (6.2) from $\mu = -1$ to $\mu = 1$, we can deduce the following equations of continuity:

$$\frac{d}{dx} \int_{-1}^{1} \mu \Psi_1(x,\mu) d\mu + (1 - c_1) \int_{-1}^{1} \Psi_1(x,\mu) d\mu = 0, \tag{6.10a}$$

and

$$\frac{d}{dx} \int_{-1}^{1} \mu \Psi_2(x,\mu) d\mu + (1 - c_2) \int_{-1}^{1} \Psi_2(x,\mu) d\mu = \frac{2S}{\sigma_2}. \tag{6.10b}$$

Now, we integrate in x, Eq. (6.10a), from 0 to a, and Eq. (6.10b) from a to b, and make use of the boundary conditions, Eqs. (6.3e and f), to obtain

$$(1 - c_1)\int_0^a dx \int_{-1}^{1} \Psi_1(x,\mu)d\mu = -\int_{-1}^{1} \Psi(a,\mu)\mu d\mu \ , \tag{6.11a}$$

and

$$(1 - c_2)\int_a^b dx \int_{-1}^{1} \Psi_2(x,\mu)d\mu = \frac{2S}{\sigma_2}(b - a)\int_{-1}^{1} \Psi(a,\mu)\mu d\mu \ . \tag{6.11b}$$

Therefore, we can express the disadvantage factor, given by Eq. (6.1), as

$$\zeta = \frac{a}{\Delta}\left(\frac{1 - c_1}{1 - c_2}\right)\left\{ - 1 - \frac{2S\Delta}{\sigma_2}\left[\int_{-1}^{1} \Psi(a,\mu)\mu d\mu\right]^{-1}\right\} , \tag{6.12}$$

or, if we use the approximations given by Eqs. (6.7b and c), and let, without loss of generality, $2S\Delta = \sigma_2$, we obtain

$$\zeta = \frac{a}{\Delta}\left(\frac{1 - c_1}{1 - c_2}\right)\left\{\left[\sum_{\alpha=0}^{N}\left(\frac{e_\alpha - f_\alpha}{\alpha + 2}\right)\right]^{-1} - 1\right\} . \tag{6.13}$$

Thus solving the F_N equations given by Eqs. (6.8) and (6.9) for the coefficients e_α and f_α, we can compute the disadvantage factor, by using Eq. (6.13).

6.3. Numerical Results and Conclusions

In order to have a simple scheme for selecting the points ξ_β to be used in Eq. (6.8) we use $\xi_\beta^{(1)} = \nu_{1,\beta}$, $\beta = 0, 1, 2, \ldots, \kappa_1 - 1$,

$\xi^{(1)}_{j+\kappa_1-1} = (2j - 1)/[2(N - \kappa_1 + 1)]$, $j = 1, 2, \ldots, (N - \kappa_1 + 1)$, where N is the order of approximation, and κ_1 is the number of pairs of discrete eigenvalues. We choose the points $\xi^{(2)}_{\beta}$ to be used in Eq. (6.9) in a similar manner.

We compute the disadvantage factor for the same basic cells considered by B & S, E & M and L-S-V-K, and thus we consider $c_1 = 0.55370$ and $c_2 = 0.99163$. In order to have a comparison among various computational methods, we first consider an isotropic scattering in both regions, and in Table 6.1 we report the results of the F_N approximation along with various other results published in the literature. We also considered the cases of linear anisotropic scattering in the moderator studies by B & S, and in Table 6.2 we list the results from the F_N method along with the B & S results. Table 6.3 contains the basic data for the scattering law studied by E & M and L-S-V-K, and in Table 6.4 we report the results obtained by the F_N method along with the results of E & M and L-S-V-K. It is apparent that the calculations of E & M are in error for the cases 3, 4 and 6. Finally, we note that our results confirm the accepted physical conclusion that high-order terms in the scattering law have little effect on the disadvantage factor.

Table 6.1: Disadvantage Factor for Isotropic Scattering in the Moderator and Fuel. Comparison of various methods

Computational Method	Cell 1[8]	Cell 2[8]	Cell 3[8]	Cell 4[8]
P-1 theory (10)	1.028	1.113	1.253	1.447
Asymptotic Diffusion (72)	1.06	1.18	1.34	1.56
Modified A-B-H (104)	1.08	1.20	1.36	1.58
S-8 (75)	1.090	1.231	1.410	1.632
Integral Transp. Theory (18)	1.0979	1.2318	1.408	1.629
Ferziger and Robison (36)	1.094	1.227	1.401	1.623
B & S (10)	1.0978	1.2317	1.4077	1.6284
F_3	1.0969	1.2305	1.4072	1.6286
F_5	1.0971	1.2317	1.4077	1.6285
F_7	1.0975	1.2318	1.4076	1.6284
F_9	1.0977	1.2318	1.4075	1.6284

[8] Dimensions of the Cells are the same as those reported in Table 6.2.

Table 6.2: Disadvantage Factor for Linearly Anisotropic Scattering in the Moderator

Calculational Model	Anisotropy Coefficient ($f_{2,1}$)	ζ- Disadvantage Factor Cell 1[9]	Cell 2[9]	Cell 3[9]	Cell 4[9]
B & S	0.0333...	1.0970	1.2283	1.4001	1.6151
F_3		1.0960	1.2271	1.3996	1.6153
F_5		1.0962	1.2283	1.4001	1.6153
F_7		1.0967	1.2284	1.4000	1.6151
B & S	0.1	1.0953	1.2215	1.3849	1.5885
F_3		1.0943	1.2203	1.3844	1.5887
F_5		1.0945	1.2214	1.3849	1.5887
F_7		1.0950	1.2216	1.3848	1.5885
B & S	0.2	1.0927	1.2113	1.3621	1.5485
F_3		1.0917	1.2101	1.3616	1.5487
F_5		1.0919	1.2111	1.3621	1.5486
F_7		1.0924	1.2114	1.3620	1.5485
B & S	0.3	1.0901	1.2010	1.3392	1.5083
F_3		1.0892	1.1998	1.3388	1.5085
F_5		1.0894	1.2010	1.3393	1.5085
F_7		1.0898	1.2011	1.3392	1.5084

[9]Cell 1: a = 0.0717 (0.10 cm), b = 0.8872 (0.45 cm); Cell 2: a = 0.1434 (0.20 cm), b = 1.7744 (0.90 cm); Cell 3: a = 0.2150 (0.30 cm), b = 2.6615 (1.35 cm); Cell 4: a = 0.2868 (0.40 cm), b = 3.5488 (1.80 cm).

Table 6.3: Basic Data[10]

Case	$3f_{2,1}$	$5f_{2,2}$	$7f_{2,3}$	$9f_{2,4}$
1	0	0	0	0
2	2	0	0	0
3	2	1.25	0	0
4	2	1.25	0	-0.375
5	0.97088	0	0	0
6	0.97088	0.24428	0	0

[10]For all cases $f_{1,\ell} = \delta_{0,\ell}$, $f_{2,0} = 1$, $f_{2,\ell} = 0$, $\ell > 4$

Table 6.4: Disadvantage Factor. Comparison of E & M, L-S-V-K and F_N Results.[11]

Case	E & M		L-S-V-K		F_3		F_5		F_7	
	Cell A	Cell B	Cell A	Cell B	Cell A	Cell B	Cell A	Cell B	Cell A	Cell B
1	1.2317	1.6284	1.2314	1.6284	1.2305	1.6286	1.2317	1.6286	1.2318	1.6284
2	1.1634	1.3599	1.1630	1.3599	1.1622	1.3601	1.1633	1.3600	1.1635	1.3599
3	1.0664	1.1924	1.1679	1.3682	1.1672	1.3685	1.1683	1.3684	1.1684	1.3683
4	1.0554	1.1752	1.1678	1.3682	1.1671	1.3684	1.1681	1.3683	1.1683	1.3682
5	1.1986	1.4988	1.1982	1.4989	1.1974	1.4990	1.1985	1.4990	1.1986	1.4988
6	1.1405	1.4049	1.1991	1.5002	1.1983	1.5004	1.1994	1.5003	1.1995	1.5002

[11] Cell dimensions are: Cell A - a = 0.1434 (0.2 cm), Δ = 1.631 (0.7 cm)
Cell B - a = 0.2868 (0.4 cm), Δ = 3.262 (1.4 cm)

7. ON POLARIZATION STUDIES FOR PLANE PARALLEL ATMOSPHERES[12]

7.1. Introduction

The equation of transfer for two components of a polarized radiation field in a plane parallel, free electron atmosphere, was formulated by Chandrasekhar [23,24]. Most of the early studies of the scattering of polarized light were based on Chandrasekhar's approach [27,78,83] and were restricted to the case of a conservative (no true absorption, or $\omega = 1$) Rayleigh-Scattering atmosphere. Some years ago a great deal of analysis, based on Case's singular expansion technique [21], was used to obtain solutions for a semi-infinite Rayleigh-Scattering, with true absorption (non-conservative) atmosphere [16,76]. Bond and Siewert [11], using the H-matrix [86], obtained numerical results for the Albedo and Milne problems in a non-conservative mixture of Rayleigh- and Isotropic-Scattering laws. In a recent paper, hereafter referred as I, Siewert [95] used the F_N method to establish an approximate semianalytical solution of the equation of transfer in a finite plane-parallel atmosphere with a combination of a Rayleigh and isotropic scattering. Here, we wish to use the analytical approach used in I, although with some modifications, to obtain some numerical results that show the computational merit of the F_N method.

We consider the vector equation of transfer, as formulated by Chandrasekhar [25],

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[12] This chapter is based on a paper accepted for publication in JQSRT [52].

$$\mu \frac{\partial}{\partial\tau} \underset{\sim}{I}(\tau,\mu) + \underset{\sim}{I}(\tau,\mu) = \frac{1}{2}\, \omega \underset{\sim}{Q}(\mu) \int_{-1}^{1} \underset{\sim}{Q}^{T}(\mu')\underset{\sim}{I}(\tau,\mu')d\mu',$$

$$\tau \in [0,\tau_o], \tag{7.1}$$

where, $\underset{\sim}{I}(\tau,\mu)$ denotes a vector whose two components $I_\ell(\tau,\mu)$ and $I_r(\tau,\mu)$ are the azimuth-independent angular intensities for the two polarization states, τ is the optical variable, μ is the direction cosine (as measured from the positive τ-axis), ω is the albedo for single-scattering, and by writing

$$\underset{\sim}{Q}(\mu) = \frac{3}{2}\,(c+2)^{-\frac{1}{2}} \begin{vmatrix} c\mu^2 + \frac{2}{3}\,(1-c) & (2c)^{\frac{1}{2}}(1-\mu^2) \\ \frac{1}{3}\,(c+2) & 0 \end{vmatrix}, \tag{7.2}$$

we allow a combination of Rayleigh and isotropic scattering, i.e., for $c = 1$ we have pure Rayleigh scattering and for $c = 0$ pure isotropic scattering. Further, the superscript T indicates the transpose matrix.

The general boundary conditions for this problem are

$$\underset{\sim}{I}(0,\mu) = \underset{\sim}{F}_1(\mu)\ ,\ \mu > 0, \tag{7.3a}$$

and

$$\underset{\sim}{I}(\tau_o,-\mu) = \underset{\sim}{F}_2(\mu),\ \mu > 0, \tag{7.3b}$$

where, $\underset{\sim}{F}_1(\mu)$ and $\underset{\sim}{F}_2(\mu)$ are considered given. As reported in I, we can write

$$\underset{\sim}{I}(\tau,\mu) = A(\eta_o)\underset{\sim}{\Phi}(\eta_o,\mu)e^{-\tau/\eta_o} + A(-\eta_o)\underset{\sim}{\Phi}(-\eta_o,\mu)e^{\tau/\eta_o}$$

$$+ \int_{-1}^{1} \underset{\sim}{\Phi}(\eta,\mu)\underset{\sim}{A}(\eta)e^{-\tau/\eta}d\eta, \tag{7.4}$$

where

$$\underset{\sim}{\Phi}(\pm\eta_o,\mu) = \frac{1}{2}\,\omega\eta_o\left(\frac{1}{\eta_o \mp \mu}\right)\underset{\sim}{Q}(\mu)\underset{\sim}{M}(\eta_o), \tag{7.5}$$

$$\underset{\sim}{\Phi}(\eta,\mu) = \frac{1}{2}\,\omega\eta P_v\left(\frac{1}{\eta-\mu}\right)\underset{\sim}{Q}(\mu) + \delta(\eta-\mu)\underset{\sim}{Q}^{-T}(\eta)\underset{\sim}{\lambda}(\eta), \tag{7.6}$$

and $\pm\eta_o$ are the discrete eigenvalues, being the zeros of $\Lambda(z) = \det\underset{\sim}{\Lambda}(z)$. Here

$$\underset{\sim}{\Lambda}(z) = \underset{\sim}{I} + \frac{1}{2}\,\omega z\int_{-1}^{1}\underset{\sim}{Q}^T(\mu)\underset{\sim}{Q}(\mu)\,\frac{d\mu}{\mu-z}\ , \tag{7.7}$$

$$\underset{\sim}{\lambda}(\eta) = \underset{\sim}{I} + \frac{1}{2}\,\omega\eta P_v\int_{-1}^{1}\underset{\sim}{Q}^T(\mu)\underset{\sim}{Q}(\mu)\,\frac{d\mu}{\mu-z}\ ,\eta \quad (-1,1), \tag{7.8}$$

and $\underset{\sim}{M}(\eta_o)$ is a vector given by

$$\underset{\sim}{M}(\eta_o) = \begin{vmatrix} \Lambda_{22}(\eta_o) \\ \\ -\Lambda_{12}(\eta_o) \end{vmatrix} . \tag{7.9}$$

We can use the full-range orthogonality [11] properties of the functions $\underset{\sim}{\Phi}(\xi,\mu)$ to deduce the following system of singular integral equations for

for the surface intensities

$$\int_0^1 \mu \underset{\sim}{\Phi}^T(\xi,\mu)\underset{\sim}{I}(0,-\mu)d\mu + e^{-\tau_o/\xi}\int_0^1 \mu\underset{\sim}{\Phi}^T(-\xi,\mu)\underset{\sim}{I}(\tau_o,\mu)d\mu$$

$$= \underset{\sim}{L}_1(\xi), \quad \xi \in P, \tag{7.10a}$$

and

$$\int_0^1 \mu \underset{\sim}{\Phi}^T(\xi,\mu)\underset{\sim}{I}(\tau_o,\mu)d\mu + e^{-\tau_o/\xi}\int_0^1 \mu\underset{\sim}{\Phi}^T(-\xi,\mu)\underset{\sim}{I}(0,-\mu)$$

$$= \underset{\sim}{L}_2(\xi), \quad \xi \in P, \tag{7.10b}$$

where the known functions are

$$\underset{\sim}{L}_1(\xi) = \int_0^1 \mu\underset{\sim}{\Phi}^T(-\xi,\mu)\underset{\sim}{F}_1(\mu)d\mu + e^{-\tau_o/\xi}\int_0^1 \mu\underset{\sim}{\Phi}^T(\xi,\mu)\underset{\sim}{F}_2(\mu)d\mu, \tag{7.11a}$$

and

$$\underset{\sim}{L}_2(\xi) = \int_0^1 \mu\underset{\sim}{\Phi}^T(-\xi,\mu)\underset{\sim}{F}_2(\mu)d\mu + e^{-\tau_o/\xi}\int_0^1 \mu\underset{\sim}{\Phi}^T(\xi,\mu)\underset{\sim}{F}_1(\mu)d\mu, \tag{7.11b}$$

and $\xi \in P \Rightarrow \xi \in \eta_o \cup (0,1)$.

7.2. Analysis

We first consider the planetary problem formulated by Chandrasekhar [25], i.e., the illumination of an atmosphere of optical thickness τ_o by a beam of polarized light. At the "ground" location, $\tau = \tau_o$, we allow reflection according to Lambert's law [25] with an albedo λ_o, and thus for this application Eqs. (7.3) become

$$\underset{\sim}{I}(0,\mu) = \frac{1}{2}\,\delta(\mu - \mu_o)\underset{\sim}{F}, \quad \mu, \mu_o > 0, \tag{7.12a}$$

and

$$\underset{\sim}{I}(\tau_o,-\mu) = \lambda_o \begin{vmatrix} 1 & 1 \\ & \\ 1 & 1 \end{vmatrix} \int_0^1 \underset{\sim}{I}(\tau_o,\mu')\mu' d\mu', \quad \mu > 0, \tag{7.12b}$$

where the constant $\underset{\sim}{F}$ has components F_ℓ and F_r. Equation (7.12b) clearly indicates that the "ground" introduces a component of unpolarized light into the atmosphere. We recall that the matrix $\underset{\sim}{Q}(\mu)$ given by Eq. (7.2) already allows (if $c \neq 1$) a combination of Rayleigh and depolarizing isotropic scattering within the atmosphere.

We now introduce the F_N method, and thus we approximate the unknown surface intensities by

$$\underset{\sim}{I}(0,-\mu) = \underset{\sim}{Q}(\mu) \sum_{\alpha=0}^{N} \underset{\sim}{a}_\alpha \mu^\alpha, \quad \mu > 0, \tag{7.13a}$$

and

$$\underset{\sim}{I}(\tau_o,\mu) = \frac{1}{2}\,\delta(\mu - \mu_o) e^{-\tau_o/\mu_o}\underset{\sim}{F} + \underset{\sim}{Q}(\mu) \sum_{\alpha=0}^{N} \underset{\sim}{b}_\alpha \mu^\alpha, \quad \mu > 0, \tag{7.13b}$$

where the constants $\underset{\sim}{a}_\alpha$ and $\underset{\sim}{b}_\alpha$ are to be determined. Note that in Eqs. (7.13) we have included a factor $\underset{\sim}{Q}(\mu)$ that was not used in I. If we substitute Eqs. (7.13) into Eqs. (7.10) and use the boundary conditions given by Eqs. (7.12) we find

$$\sum_{\alpha=0}^{N} \underset{\sim}{\Delta}_\alpha(\xi)\underset{\sim}{a}_\alpha + e^{-\tau_o/\xi}\sum_{\alpha=0}^{N}\left\{\underset{\sim}{\Gamma}_\alpha(\xi) - \lambda_o \underset{\sim}{B}_o^T(\xi)\underset{\sim}{D}\left[\underset{\sim}{Q}_o\left(\frac{1}{\alpha+2}\right) + \underset{\sim}{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\right\}\underset{\sim}{b}_\alpha = \underset{\sim}{K}_1(\xi) \tag{7.14a}$$

and

$$\sum_{\alpha=0}^{N}\left\{\underset{\sim}{\Delta}_\alpha(\xi) - \lambda_o \underset{\sim}{A}_o^T(\xi)\underset{\sim}{D}\left[\underset{\sim}{Q}_o\left(\frac{1}{\alpha+2}\right) + \underset{\sim}{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\right\}\underset{\sim}{b}_\alpha + e^{-\tau_o/\xi}\sum_{\alpha=0}^{N}\underset{\sim}{\Gamma}_\alpha(\xi)\underset{\sim}{a}_\alpha = \underset{\sim}{K}_2(\xi) \tag{7.14b}$$

where

$$\underset{\sim}{D} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \tag{7.15}$$

$$\underset{\sim}{A}_o(\xi) = \left[\underset{\sim}{Q}_o + \frac{1}{3}\underset{\sim}{Q}_2 + \xi(\xi-\frac{1}{2})\underset{\sim}{Q}_2 - \xi\underset{\sim}{Q}(\xi)\log(1+1/\xi)\right]\underset{\sim}{M}(\xi), \tag{7.16a}$$

$$\underset{\sim}{B}_o(\xi) = \left[\frac{2}{\omega}\underset{\sim}{\Delta}(\xi) - \underset{\sim}{Q}_o - \frac{1}{3}\underset{\sim}{Q}_2 + \xi(\xi-\frac{1}{2})\underset{\sim}{Q}_2 - \xi\underset{\sim}{Q}(\xi)\log(1+\frac{1}{\xi})\right]\underset{\sim}{M}(\xi), \tag{7.16b}$$

$$\underset{\sim}{\Delta}(\xi) = \underset{\sim}{Q}^{-T}(\xi)\left[\underset{\sim}{I} + \omega\xi^2\underset{\sim}{Q}_2^T(\underset{\sim}{Q}_o + \frac{1}{3}\underset{\sim}{Q}_2)\right], \tag{7.16c}$$

and the known functions are

$$\underset{\sim}{K}_1(\xi) = \frac{1}{2}\mu_o\left\{\left[1 - e^{-\tau_o(\frac{1}{\mu_o}+\frac{1}{\xi})}\right]\frac{2}{\omega\xi}\underset{\sim}{\Phi}^T(-\xi,\mu_o)\right.$$

$$\left. + \lambda_o e^{-\tau_o(\frac{1}{\mu_o}+\frac{1}{\xi})}\underset{\sim}{B}_o^T(\xi)\underset{\sim}{D}\right\}\underset{\sim}{F} \tag{7.17a}$$

and

$$\underset{\sim}{K}_2(\xi) = \frac{1}{2}\mu_o\left\{\left[e^{-\tau_o/\xi} - e^{-\tau_o/\mu_o}\right]\frac{2}{\omega\xi}\underset{\sim}{\Phi}^T(\xi,\mu_o)\right.$$

$$\left. + \lambda_o e^{-\tau_o/\mu_o}\underset{\sim}{A}_o^T(\xi)\underset{\sim}{D}\right\}\underset{\sim}{F}. \tag{7.17b}$$

The functions $\underset{\sim}{\Gamma}_\alpha(\xi)$ and $\underset{\sim}{\Delta}_\alpha(\xi)$ appearing in Eqs. (7.14) are defined by

$$\underset{\sim}{\Gamma}_\alpha(\xi) = \frac{2}{\omega\xi}\int_0^1 \mu^{\alpha+1}\underset{\sim}{\Phi}^T(-\xi,\mu)\underset{\sim}{Q}(\mu)\,d\mu \tag{7.18a}$$

and

$$\underset{\sim}{\Delta}_\alpha(\xi) = \frac{2}{\omega\xi}\int_0^1 \mu^{\alpha+1}\underset{\sim}{\Phi}^T(\xi,\mu)\underset{\sim}{Q}(\mu)\,d\mu \tag{7.18b}$$

and can readily be generated from

$$\underset{\sim}{\Gamma}_\alpha(\xi) = -\xi\underset{\sim}{\Gamma}_{\alpha-1}(\xi) + \underset{\sim}{M}^T(\xi)\underset{\sim}{R}_\alpha \tag{7.19a}$$

with

$$\underset{\sim}{\Gamma}_o(\xi) = \underset{\sim}{M}^T(\xi)\left[\underset{\sim}{R}_o - \xi\underset{\sim}{K}(\xi)\right] \tag{7.19b}$$

and

$$\Delta_\alpha(\xi) = \xi\Delta_{\alpha-1}(\xi) - M^T(\xi)R_\alpha \tag{7.20a}$$

with

$$\Delta_o(\xi) = M^T(\xi)\left[\frac{2}{\omega}I - R_o - \xi K(\xi)\right] . \tag{7.20b}$$

Here

$$R_\alpha = Q_o^TQ_o\left(\frac{1}{\alpha+1}\right) + (Q_2^TQ_o + Q_o^TQ_2)\left(\frac{1}{\alpha+3}\right) + Q_2^TQ_2\left(\frac{1}{\alpha+5}\right) \tag{7.21}$$

and

$$K(\xi) = Q_o^TQ_o\log\left(1+\frac{1}{\xi}\right) + (Q_2^TQ_o + Q_o^TQ_2)\left[\xi^2\log\left(1+\frac{1}{\xi}\right) + \frac{1}{2} - \xi\right] + \left[\xi^4\log\left(1+\frac{1}{\xi}\right) + \frac{1}{4} - \frac{\xi}{3} + \frac{\xi^2}{2} - \xi^3\right]Q_2^TQ_2 \tag{7.22}$$

where we have written

$$Q(\mu) = Q_o + \mu^2 Q_2. \tag{7.23}$$

We use $M(\xi) = I$, $\xi \in (0,1)$, whereas $M(\eta_o)$ is a null vector of the dispersion matrix, as given by Eq. (7.9).

It is apparent that once the desired constants a_α and b_α are found the surface intensities become known and thus other quantities of physical interest are readily available. For example, the albedo

$$A^* = 2\left[\mu_o(F_\ell + F_r)\right]^{-1}\int_0^1\left[I_\ell(0,-\mu) + I_r(0,-\mu)\right]\mu d\mu \tag{7.24a}$$

and the transmission factor

$$B^* = 2\left[\mu_o(F_\ell + F_r)\right]^{-1}\int_0^1\left[I_\ell(\tau_o,\mu) + I_r(\tau_o,\mu)\right]\mu d\mu \tag{7.24b}$$

can be expressed, in the F_N approximation, by

$$A^* = 2\left[\mu_o(F_\ell + F_r)\right]^{-1}\begin{vmatrix}1\\1\end{vmatrix}^T\sum_{\alpha=0}^{N}\left[\left(\frac{1}{\alpha+2}\right)\underset{\sim}{Q}_o + \left(\frac{1}{\alpha+4}\right)\underset{\sim}{Q}_2\right]\underset{\sim}{a}_\alpha \tag{7.25a}$$

and

$$B^* = 2\left[\mu_o(F_\ell + F_r)\right]^{-1}\left\{\frac{1}{2}\mu_o(F_\ell + F_r)e^{-\tau_o/\mu_o} + \begin{vmatrix}1\\1\end{vmatrix}^T\sum_{\alpha=0}^{N}\left[\left(\frac{1}{\alpha+2}\right)\underset{\sim}{Q}_o + \left(\frac{1}{\alpha+4}\right)\underset{\sim}{Q}_2\right]\underset{\sim}{b}_\alpha\right\} . \tag{7.25b}$$

In a similar manner, the Stokes parameters [25] at the two surfaces

$$I(0,-\mu) = I_\ell(0,-\mu) + I_r(0,-\mu), \quad \mu > 0, \tag{7.26a}$$

$$I^*(\tau_o,\mu) = I_\ell^*(\tau_o,\mu) + I_r^*(\tau_o,\mu), \quad \mu > 0, \tag{7.26b}$$

$$Q(0,-\mu) = I_\ell(0,-\mu) - I_r(0,-\mu), \quad \mu > 0, \tag{7.26c}$$

$$Q^*(\tau_o,\mu) = I^*_\ell(\tau_o,\mu) - I^*_r(\tau_o,\mu), \ \mu > 0, \tag{7.26d}$$

where

$$\underset{\sim}{I}^*(\tau,\mu) = \underset{\sim}{I}(\tau,\mu) - \frac{1}{2}\,\delta(\mu - \mu_o)e^{-\tau/\mu}\,\underset{\sim}{F}, \tag{7.27}$$

can be expressed in the F_N approximation as

$$I(0,-\mu) = \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \underset{\sim}{Q}(\mu) \sum_{\alpha=0}^{N} \underset{\sim}{a}_\alpha \mu^\alpha, \ \mu > 0, \tag{7.28a}$$

$$I^*(\tau_o,\mu) = \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \underset{\sim}{Q}(\mu) \sum_{\alpha=0}^{N} \underset{\sim}{b}_\alpha \mu^\alpha, \ \mu > 0, \tag{7.28b}$$

$$Q(0,-\mu) = \begin{vmatrix} 1 \\ -1 \end{vmatrix}^T \underset{\sim}{Q}(\mu) \sum_{\alpha=0}^{N} \underset{\sim}{a}_\alpha \mu^\alpha, \ \mu > 0, \tag{7.28c}$$

and

$$Q^*(\tau_o,\mu) = \begin{vmatrix} 1 \\ -1 \end{vmatrix}^T \underset{\sim}{Q}(\mu) \sum_{\alpha=0}^{N} \underset{\sim}{b}_\alpha \mu^\alpha, \ \mu > 0. \tag{7.28d}$$

Finally we wish to discuss the Classical Albedo Problem, and thus we seek a bounded solution of Eq. (7.1) in a semi-infinite atmosphere such that the incident distribution may be specified. Thus for this application, we write

$$\underset{\sim}{I}(0,\mu) = \underset{\sim}{I}_{inc}(\mu), \ \mu \in (0,1). \tag{7.29}$$

The exit distribution $\underset{\sim}{I}(0,-\mu)$ can be approximated by Eq. (7.13a), and the constants $\underset{\sim}{a}_\alpha$ can be found by the F_N equations,

$$\sum_{\alpha=0}^{N} \underset{\sim}{\Delta}_\alpha(\xi)\underset{\sim}{a}_\alpha = \frac{2}{\omega\xi}\int_0^1 \mu\underset{\sim}{\Phi}^T(-\xi,\mu)\underset{\sim}{I}\text{inc}(\mu)d\mu, \quad \xi\in P. \tag{7.30}$$

Thus the albedo A*, defined as the fraction of the incident radiation reflected from the atmosphere, can be computed from

$$A^* = \left[\int_0^1 \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \underset{\sim}{I}\text{inc}(\mu)\mu d\mu\right]^{-1} \int_0^1 \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \underset{\sim}{I}(0,-\mu)\mu d\mu. \tag{7.31}$$

For the particular case of an isotropic incident distribution, i.e.

$$\underset{\sim}{I}\text{inc}(\mu) = \begin{vmatrix} \ell \\ r \end{vmatrix}, \quad (\ell,r) = (0,1) \text{ and } (1,0), \tag{7.32}$$

Eq. (7.31) yields, in the F_N approximation,

$$A^*(\ell,r) = 2(\ell + r)^{-1} \sum_{\alpha=0}^{N} \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \left[\underset{\sim}{Q}_o\left(\frac{1}{\alpha+2}\right) + \underset{\sim}{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\underset{\sim}{a}_\alpha, \tag{7.33}$$

where the constants $\underset{\sim}{a}_\alpha$ can be found from solving

$$\sum_{\alpha=0}^{N} \underset{\sim}{\Delta}_\alpha(\xi)\underset{\sim}{a}_\alpha = \underset{\sim}{A}_o(\xi)\begin{vmatrix} \ell \\ r \end{vmatrix} \tag{7.34}$$

for selected values of $\xi\in P$. In the next section we report our numerical results for the two considered problems.

7.3. Numerical Results

In order to find the required constants $\underset{\sim}{a}_\alpha$ and $\underset{\sim}{b}_\alpha$ we now consider the following set of linear algebraic equations generated by evaluating Eqs. (7.14) at selected values, say $\{\xi_j\}$, of $\xi \in P$:

$$\sum_{\alpha=0}^{N} \underset{\sim}{\Delta}_\alpha(\xi_j)\underset{\sim}{a}_\alpha + e^{-\tau_o/\xi_j} \sum_{\alpha=0}^{N} \left\{ \underset{\sim}{\Gamma}_\alpha(\xi_j) - \lambda \underset{\sim}{B}_o^T(\xi_j)\underset{\sim}{D}\left[\underset{\sim}{Q}_o\left(\frac{1}{\alpha+2}\right) + \underset{\sim}{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\right\} \underset{\sim}{b}_\alpha = \underset{\sim}{K}_1(\xi_j) \tag{7.35a}$$

and

$$\sum_{\alpha=0}^{N} \left\{ \underset{\sim}{\Delta}_\alpha(\xi_j) - \lambda \underset{\sim}{A}_o^T(\xi_j)\underset{\sim}{D}\left[\underset{\sim}{Q}_o\left(\frac{1}{\alpha+2}\right) + \underset{\sim}{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\right\} \underset{\sim}{b}_\alpha + e^{-\tau_o/\xi_j} \sum_{\alpha=0}^{N} \underset{\sim}{\Gamma}_\alpha(\xi_j)\underset{\sim}{a}_\alpha = \underset{\sim}{K}_2(\xi_j) \tag{7.35b}$$

In I a particular scheme was suggested for selecting the points $\{\xi_j\}$, and a restriction was placed on the form of $\underset{\sim}{a}_o$ and $\underset{\sim}{b}_o$. We have tried the scheme suggested in I and have concluded that the restriction placed on $\underset{\sim}{a}_o$ and $\underset{\sim}{b}_o$ impedes the numerical accuracy of the method. In addition, the point $\xi = 1$ cannot be used as suggested in I, since $\underset{\sim}{Q}(1)$ is singular. Here we use the approximations given by Eqs. (7.12), and to have a simple (and effective) scheme for choosing the points we use, for the F_o approximation, $\xi_o = \eta_o$ in Eqs. (7.35) to establish two scalar equations and for $\xi_1 = 1$ we use only two of the four scalar equations obtained from Eqs. (7.35); in particular we use for $\xi = 1$ the upper (scalar) equations in Eqs. (7.35). In addition we define, for higher order F_N approximations, $\xi_j = (2j - 3)/(2N)$, $j = 2, 3, 4, \ldots (N + 1)$, and use each of the four scalar equations resulting, for each ξ_j, $j > 1$, from Eqs. (7.35). Thus, in general, we must solve a system of $4(N + 1)$ linear algebraic equations to establish the desired constants $\{\underset{\sim}{a}_\alpha\}$ and $\{\underset{\sim}{b}_\alpha\}$.

From a computational point of view, the first thing we wish is to establish the discrete eigenvalue, η_o. In appendix 11.1 we discuss how we found η_o. Once the discrete eigenvalue was established we solved the system of linear algebraic equations given by Eq. (7.35), with $F_\ell = F_r = \frac{1}{2}$, for the data cases given in Table 7.1. In Tables 7.3 and 7.4 we report the albedo and transmission factor as computed from various orders of the F_N approximation. We also list "exact" results deduced from "converged" F_N computations and confirmed, for the half-space, with a previous work [11]. Tables 7.5 - 7.9 show the "converged" F_N results for the Stokes parameters for data cases considered. For the half-space the results of Table 7.9 agree with the exact analysis of Bond and Siewert [11]. For the finite atmosphere our "converged" results for $I(0,-\mu)$ and $Q(0,-\mu)$ shown in Tables 7.5 and 7.7 agree to the given accuracy with the work of Kawabata [44] and Hansen [40].

For the classical Albedo Problem, we list in Table 7.10 the albedo, $A^*(\ell,r)$ for various orders of the F_N approximation along with the "exact" result taken from Bond and Siewert, for the data cases shown in Table 7.2.

We note that our F_N calculation have generally "converged" to five significant figures, except near $\mu = 0$ and $\mu = 1$, for the angular dependent Stokes parameters. The method yielded not suprisingly, even better results for the integrated quantities A^* and B^*. We note that $\eta_o \to \infty$ as $\omega \to 1$ and that the calculation of the quantities $\underset{\sim}{\Gamma}_\alpha(\eta_o)$ and $\underset{\sim}{\Delta}_\alpha(\eta_o)$ requires some care, for

$\omega \simeq 1$, to avoid a loss of accuracy. Finally we have observed that the method becomes less accurate for small thickness ($\tau_o \simeq 0.1$); however a modification in Eqs. (7.13) can improve the method for very thin media.

Finally, we note that the problem of diffuse reflection and transmission by a plane parallel atmosphere scattering radiation in accordance with Rayleigh's law is a basic one in the theory of the illumination and polarization of the sky. The problem is also of interest for the illumination of other planets by the sun.

Table 7.1: Cases Studied for the Planetary Problem

Case	ω	c	λ_o	τ_o	μ_o
1	0.9	1.0	0.0	1.0	1.0
2	0.9	1.0	0.1	1.0	1.0
3	0.9	0.8	0.1	5.0	1.0
4	0.9	0.8	0.2	5.0	1.0
5	0.9	0.8	-	∞	1.0

Table 7.2: Cases Studied for the Albedo Problem

Case	ω	c
1	0.9	0.4
2	0.9	1.0
3	0.8	0.4
4	0.8	1.0

Table 7.3: The Albedo - A*

Case	F_3	F_5	F_7	"Exact"
1	0.27016	0.27022	0.27023	0.27023
2	0.29950	0.29956	0.29957	0.29957
3	0.41753	0.41754	0.41754	0.41754
4	0.41802	0.41803	0.41803	0.41803
5	0.41960	0.41961	0.41961	0.41961

Table 7.4: The Transmission Factor - B*

Case	F_3	F_5	F_7	"Exact"
1	0.59591	0.59618	0.59617	0.59617
2	0.61772	0.61800	0.61800	0.61799
3	0.082906	0.082955	0.082958	0.082958
4	0.087331	0.087334	0.087334	0.087334
5	-	-	-	0.0

Table 7.5: "Converged" Results for Case 1. (Planetary Problem)

μ	$I(0,-\mu)$	$-Q(0,-\mu)$	$I^*(\tau_o,\mu)$	$-Q^*(\tau_o,\mu)$
0.02	0.2812	0.2184	0.1549	0.1003
0.06	0.2856	0.2155	0.1671	0.1062
0.10	0.2875	0.2105	0.1776	0.1109
0.16	0.2879	0.2006	0.1919	0.1165
0.20	0.2869	0.1928	0.1998	0.1187
0.28	0.2829	0.1749	0.2111	0.1183
0.32	0.2803	0.1652	0.2148	0.1159
0.40	0.2751	0.1450	0.2198	0.1078
0.52	0.2688	0.1142	0.2241	0.09020
0.64	0.2655	0.08401	0.2277	0.06914
0.72	0.2651	0.06449	0.2305	0.05417
0.84	0.2667	0.03616	0.2358	0.03110
0.92	0.2692	0.01786	0.2401	0.01556
0.96	0.2708	0.00888	0.2425	0.00778
0.98	0.2717	0.0044	0.2438	0.0039
1.00	0.2726	0.0	0.2450	0.0

Table 7.6: "Converged" Results for Case 2 (Planetary Problem)

μ	$I(0,-\mu)$	$-Q(0,-\mu)$	$I^*(\tau_o,\mu)$	$-Q^*(\tau_o,\mu)$
0.02	0.2926	0.2197	0.1933	0.0997
0.06	0.2980	0.2168	0.2038	0.1058
0.10	0.3009	0.2117	0.2128	0.1107
0.16	0.3029	0.2017	0.2252	0.1165
0.20	0.3030	0.1939	0.2318	0.1187
0.28	0.3015	0.1758	0.2408	0.1184
0.32	0.3002	0.1661	0.2434	0.1161
0.40	0.2975	0.1457	0.2463	0.1080
0.52	0.2949	0.1147	0.2479	0.09043
0.64	0.2949	0.08440	0.2493	0.06933
0.72	0.2964	0.06479	0.2508	0.05433
0.84	0.3006	0.03632	0.2544	0.03120
0.92	0.3047	0.01794	0.2577	0.01561
0.96	0.3070	0.00892	0.2597	0.00780
0.98	0.3083	0.00444	0.2607	0.0039
1.00	0.3094	0.0	0.2617	0.0

Table 7.7: "Converged" Results for Case 3 (Planetary Problem)

μ	$I(0,-\mu)$	$-Q(0,-\mu)$	$I^*(\tau_o,\mu)$	$-Q^*(\tau_o,\mu)\times 100$
0.02	0.3624	0.1660	0.03444	0.4432
0.06	0.3730	0.1617	0.03686	0.4457
0.10	0.3801	0.1567	0.03912	0.4499
0.16	0.3876	0.1484	0.04242	0.4568
0.20	0.3913	0.1424	0.04462	0.4613
0.28	0.3971	0.1300	0.04911	0.4685
0.32	0.3995	0.1235	0.05144	0.4709
0.40	0.4037	0.1101	0.05630	0.4725
0.52	0.4096	0.08928	0.06419	0.4630
0.64	0.4156	0.06772	0.07288	0.4289
0.72	0.4199	0.05303	0.07907	0.3843
0.84	0.4269	0.03057	0.08888	0.2699
0.92	0.4321	0.01537	0.09571	0.1537
0.96	0.4348	0.00770	0.09919	0.08181
0.98	0.4362	0.00386	0.1010	0.04216
1.00	0.4376	0.0	0.1027	0.0

Table 7.8: "Converged" Results for Case 4 (Planetary Problem)

μ	$I(0,-\mu)$	$-Q(0,-\mu)$	$I^*(\tau_o,\mu)$	$-Q^*(\tau_o,\mu)\times 100$
0.02	0.3627	0.1660	0.04058	0.4388
0.06	0.3732	0.1618	0.04280	0.4438
0.10	0.3803	0.1567	0.04488	0.4496
0.16	0.3878	0.1484	0.04796	0.4584
0.20	0.3916	0.1425	0.05002	0.4637
0.28	0.3974	0.1300	0.05428	0.4721
0.32	0.3998	0.1235	0.05650	0.4749
0.40	0.4041	0.1101	0.06115	0.4770
0.52	0.4100	0.08930	0.06879	0.4675
0.64	0.4160	0.06774	0.07724	0.4329
0.72	0.4204	0.05304	0.08330	0.3880
0.84	0.4275	0.03058	0.09292	0.2720
0.92	0.4328	0.01537	0.09964	0.1548
0.96	0.4355	0.00770	0.1031	0.08238
0.98	0.4369	0.00386	0.1048	0.04247
1.00	0.4384	0.0	0.1065	0.0

Table 7.9: "Converged" Results for Case 5 (Planetary Problem)

μ	$I(0,-\mu)$	$-Q(0,-\mu)$
0.02	0.3633	0.1661
0.06	0.3739	0.1618
0.10	0.3810	0.1568
0.16	0.3886	0.1485
0.20	0.3924	0.1425
0.28	0.3983	0.1301
0.32	0.4008	0.1236
0.40	0.4052	0.1102
0.52	0.4112	0.08936
0.64	0.4175	0.06780
0.72	0.4220	0.05309
0.84	0.4294	0.03061
0.92	0.4348	0.01539
0.96	0.4377	0.007715
0.98	0.4392	0.003862
1.00	0.4407	0.0

Table 7.10: The Half-Space Albedo

Case	A*(0,1)				A*(1,0)			
	F_3	F_5	F_7	"Exact"	F_3	F_5	F_7	"Exact"
1	0.47933	0.47933	0.47933	0.47933	0.47730	0.47731	0.47731	0.47731
2	0.48460	0.48455	0.48455	0.48455	0.47424	0.47425	0.47425	0.47425
3	0.34408	0.34408	0.34408	0.34408	0.34059	0.34060	0.34060	0.34060
4	0.35100	0.35097	0.35096	0.35096	0.33660	0.33660	0.33660	0.33660

8. COMPLETE SOLUTION FOR THE SCATTERING OF POLARIZED LIGHT IN A RAYLEIGH AND ISOTROPICALLY SCATTERING ATMOSPHERE[13]

8.1. Introduction and Formulation of the Problem

In chapter 7 we discussed the transport of polarized light without considering the azimuthal dependence of the radiation field, however to describe a general radiation field, with azimuthal dependence, four parameters should be specified which will give the intensity, the degree of polarization, the plane of polarization and the ellipticity of the radiation at each point and in any given direction. As observed by Chandrasekhar [25], in his classical treatment of the scattering of polarized light, the most convenient representation of polarized light is by a set of four parameters, I, Q, U and V or I_ℓ, I_r, U and V called the Stokes parameters [25]. Here, we wish to apply the F_N method to solve the complete problem, for the Stokes parameters, concerning the diffusion of polarized light in a plane parallel Rayleigh and isotropically scattering atmosphere with Lambert's reflection at the ground.

As noted by Chandrasekhar [25], the parameters I_ℓ, I_r and U can be solved independently of the parameter V, and then we first consider the transfer equation for the three vector $\underset{\sim}{I}(\tau,\mu,\varphi)$ with components $I_\ell(\tau,\mu,\varphi)$, $I_r(\tau,\mu,\varphi)$ and $U(\tau,\mu,\varphi)$

$$\mu \frac{\partial}{\partial\tau} \underset{\sim}{I}(\tau,\mu,\varphi) + \underset{\sim}{I}(\tau,\mu,\varphi) =$$

[13]This chapter is based on a paper submitted for publication in J. Atmospheric Science [102].

$$\frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1} \underset{\sim}{P}(\mu,\varphi:\mu',\varphi')\underset{\sim}{I}(\tau,\mu',\varphi')d\mu'd\varphi' \tag{8.1}$$

and, boundary conditions of the form

$$\underset{\sim}{I}(0,\mu,\varphi) = \pi\delta(\mu - \mu_o)\delta(\varphi - \varphi_o)\underset{\sim}{F}, \ \mu, \ \mu_o > 0, \ \ , \ \varphi_o \in [0,2\pi], \tag{8.2a}$$

and

$$\underset{\sim}{I}(\tau_o,-\mu,\varphi) = \frac{\lambda_o}{\pi}\underset{\sim}{E}\int_0^{2\pi}\int_0^{1}\underset{\sim}{I}(\tau_o,\mu',\varphi')\mu'd\mu'd\varphi', \ \mu > 0, \ \varphi \in [0,2\pi]. \tag{8.2b}$$

Here $\tau \in [0,\tau_o]$ is the optical variable, μ is the direction cosine as measured from the <u>positive</u> τ axis, $\underset{\sim}{F}$ with components F_ℓ, F_r and F_u is a prescribed constant, λ_o is the coefficient for Lambert reflection at the ground, and

$$\underset{\sim}{E} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{8.3}$$

We note that we have changed μ to $-\mu$ in regard to Chandrasekhar's notation and that here $\underset{\sim}{I}(\tau,\mu,\varphi)$ is the complete field and not just the diffuse component. Thus we use Chandrasekhar's phase matrix, to describe a Rayleigh and isotropically scattering atmosphere, in the form

$$\underset{\sim}{P}(\mu,\varphi:\mu',\varphi') = \underset{\sim}{Q}\Big[c\underset{\sim}{P}^{(o)}(\mu:\mu') + (1 - c)\ \underset{\sim}{E}$$

$$+ c(1 - \mu^2)^{\frac{1}{2}}(1 - \mu'^2)^{\frac{1}{2}}\ \underset{\sim}{P}^{(1)}(\mu,\varphi:\mu',\varphi') + c\underset{\sim}{P}^{(2)}(\mu,\varphi:\mu',\varphi')\Big] \tag{8.4}$$

where, c is a factor which gives the fraction of the isotropic scattering. In addition

$$\underset{\sim}{P}^{(0)}(\mu:\mu') = \frac{3}{4} \begin{vmatrix} 2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2 & 0 \\ \mu'^2 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \quad (8.5)$$

$$\underset{\sim}{P}^{(1)}(\mu,\varphi:\mu',\varphi') = \frac{3}{4} \begin{vmatrix} 4\mu\mu'\cos(\varphi'-\varphi) & 0 & -2\mu\sin(\varphi'-\varphi) \\ 0 & 0 & 0 \\ 2\mu'\sin(\varphi'-\varphi) & 0 & \cos(\varphi'-\varphi) \end{vmatrix}, \quad (8.6)$$

and

$$\underset{\sim}{P}^{(2)}(\mu,\varphi:\mu',\varphi') = \frac{3}{4} \begin{vmatrix} \mu^2\mu'^2\cos2(\varphi'-\varphi) & -\mu^2\cos2(\varphi'-\varphi) & -\mu^2\mu'\sin2(\varphi'-\varphi) \\ -\mu'^2\cos2(\varphi'-\varphi) & \cos2(\varphi'-\varphi) & \mu'\sin2(\varphi'-\varphi) \\ \mu\mu'^2\sin2(\varphi'-\varphi) & -\mu\sin2(\varphi'-\varphi) & \mu\mu'\cos2(\varphi'-\varphi) \end{vmatrix} \quad (8.7)$$

and

$$\underset{\sim}{Q} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} \quad (8.8)$$

As discussed before the fourth component of the radiation field $V(\tau,\mu,\varphi)$ is uncoupled from the three elements of $\underset{\sim}{I}(\tau,\mu,\varphi)$. In fact to establish $V(\tau,\mu,\varphi)$ we must solve only the scalar equation of transfer

$$\frac{\partial}{\partial\tau} V(\tau,\mu,\varphi) + V(\tau,\mu,\varphi) =$$

$$\frac{\omega}{8\pi}\ (5c-2)\int_0^{2\pi}\int_{-1}^{1}\left[\mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos(\varphi-\varphi')\right]V(\tau,\mu',\varphi')\,d\mu' d\varphi' \tag{8.9}$$

subject to the boundary conditions

$$V(0,\mu,\ ) = \pi F_V \delta(\mu-\mu_o)\delta(\varphi-\varphi_o),\ \mu,\ \mu_o > 0,\ \ ,\ \varphi_o \in [0,2\pi] \tag{8.10a}$$

and

$$V(\tau_o,-\mu,\varphi) = 0,\ \mu > 0,\ \varphi \in [0,2\pi]. \tag{8.10b}$$

If we express $V(\tau,\mu,\varphi)$ as

$$V(\tau,\mu,\varphi) = \frac{3}{8} F_v\left[\mu\mu_o V^{(o)}(\tau,\mu) + (1-\mu^2)^{\frac{1}{2}}(1-\mu_o^2)^{\frac{1}{2}}\cos(\varphi-\varphi_o)V^{(1)}(\tau,\mu)\right]$$

$$+ \pi F_v \delta(\mu-\mu_o)e^{-\tau/\mu}\left[\ \delta(\varphi-\varphi_o) - \frac{1}{2\pi} - \frac{1}{\pi}\cos(\varphi-\varphi_o)\right] \tag{8.11}$$

then it is apparent that $V^{(o)}(\tau,\mu)$ and $V^{(1)}(\tau,\mu)$ are solutions of

$$\mu\frac{\partial}{\partial\tau}V^{(o)}(\tau,\mu) + V^{(o)}(\tau,\mu) = \frac{\omega}{4}\ (5c-2)\int_{-1}^{1}\mu'^2 V^{(o)}(\tau,\mu')\,d\mu', \tag{8.12}$$

with

$$V^{(o)}(0,\mu) = \frac{4}{3\mu_0^2}\ \delta(\mu-\mu_o),\ \mu,\ \mu_o > 0, \tag{8.13a}$$

and

$$V^{(o)}(\tau_o,-\mu) = 0, \; \mu > 0, \tag{8.13b}$$

and

$$\mu \frac{\partial}{\partial\tau} V^{(1)}(\tau,\mu) + V^{(1)}(\tau,\mu) = \frac{\omega}{8}(5c-2)\int_{-1}^{1}(1-\mu'^2)V^{(1)}(\tau,\mu')d\mu', \tag{8.14}$$

with

$$V^{(1)}(0,\mu) = \frac{8}{3}(1-\mu_o^2)^{-1}\delta(\mu-\mu_o), \; \mu, \; \mu_o > 0, \tag{8.15a}$$

and

$$V^{(1)}(\tau_o,-\mu) = 0, \; \mu > 0. \tag{8.15b}$$

The radiative transfer problems defined above are of the general form we consider in Section 8.2.

Chandrasekhar [25] also deduced that the problem for the three vector $\underset{\sim}{I}(\tau,\mu,\varphi)$ can be reduced to one two-vector problem and two scalar problems. We therefore write

$$\underset{\sim}{I}(\tau,\mu,\varphi) = \underset{\sim}{\Sigma}\,\underset{\sim}{I}(\tau,\mu) + \frac{1}{4}\underset{\sim}{Q}\Big[(1-\mu^2)^{\frac{1}{2}}(1-\mu_o^2)^{\frac{1}{2}}\phi^{(1)}(\tau,\mu)\underset{\sim}{P}^{(1)}(\mu,\varphi:\mu_o,\varphi_o)$$

$$+ \phi^{(2)}(\tau,\mu)\underset{\sim}{P}^{(2)}(\mu,\varphi:\mu_o,\varphi_o)\Big]\underset{\sim}{F} + \pi\delta(\mu-\mu_o)e^{-\tau/\mu}\Big[\delta(\varphi-\varphi_o)$$

$$- \frac{1}{2\pi}\underset{\sim}{\Sigma} - \frac{2}{3\pi}(1+2\mu_o^2)^{-1}\underset{\sim}{Q}\,\underset{\sim}{P}^{(1)}(\mu,\varphi:\mu_o,\varphi_o)$$

$$- \frac{4}{3\pi} (1 + \mu_o^2)^{-2} \underset{\sim}{Q} \underset{\sim}{P}^{(2)}(\mu,\varphi:\mu_o,\varphi_o) \Big] \underset{\sim}{F} \tag{8.16}$$

where

$$\underset{\sim}{\Sigma} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{vmatrix} \tag{8.17}$$

and $\underset{\sim}{I}(\tau,\mu)$ is a two-vector that satisfies exactly the same type of problem discussed in chapter 7, i.e.

$$\mu \frac{\partial}{\partial\tau} \underset{\sim}{I}(\tau,\mu) + \underset{\sim}{I}(\tau,\mu) = \frac{\omega}{2} \underset{\sim}{Q}(\mu) \int_{-1}^{1} \underset{\sim}{Q}^T(\mu')\underset{\sim}{I}(\tau,\mu')d\mu', \tag{8.18}$$

with

$$\underset{\sim}{I}(0,\mu) = \frac{1}{2} \begin{vmatrix} F_\ell \\ F_r \end{vmatrix} \delta(\mu - \mu_o), \ \mu, \ \mu_o > 0 \tag{8.19a}$$

and

$$\underset{\sim}{I}(\tau_o,-\mu) = \lambda_o \underset{\sim}{D} \int_0^1 \underset{\sim}{I}(\tau_o,\mu')\mu' d\mu', \ \mu > 0, \tag{8.19b}$$

where $\underset{\sim}{Q}(\mu)$ and $\underset{\sim}{D}$ are those matrix defined by Eqs. (7.2) and (7.15). In addition $\phi^{(1)}(\tau,\mu)$ and $\phi^{(2)}(\tau,\mu)$ satisfy

$$\mu \frac{\partial}{\partial\tau} \phi^{(1)}(\tau,\mu) + \phi^{(1)}(\tau,\mu) = \frac{3\omega c}{8} \int_{-1}^{1} (1 - \mu'^2)(1 + 2\mu'^2)\phi^{(1)}(\tau,\mu')d\mu' \tag{8.20}$$

with

$$\phi^{(1)}(0,\mu) = \frac{8}{3}\left[(1 + 2\mu_o^2)(1 - \mu_o^2)\right]^{-1}\delta(\mu - \mu_o),\ \mu,\ \mu_o > 0, \qquad (8.21a)$$

and

$$\phi^{(1)}(\tau_o,-\mu) = 0,\ \mu > 0, \qquad (8.21b)$$

and

$$\mu \frac{\partial}{\partial\tau}\phi^{(2)}(\tau,\mu) + \phi^{(2)}(\tau,\mu) = \frac{3\omega c}{16}\int_{-1}^{1}(1 + \mu'^2)^2\phi^{(2)}(\tau,\mu')d\mu', \qquad (8.22)$$

with

$$\phi^{(2)}(0,\mu) = \frac{16}{3}(1 + \mu_o^2)^{-2}\delta(\mu - \mu_o),\ \mu,\ \mu_o > 0, \qquad (8.23a)$$

and

$$\phi^{(2)}(\tau_o,-\mu) = 0,\ \mu > 0. \qquad (8.23b)$$

We note that $\phi^{(1)}(\tau,\mu)$ and $\phi^{(2)}(\tau,\mu)$ are also defined by radiative transfer problems of the general form considered in section 8.2. We also note that the solution of the two vector was discussed in chapter 7, and therefore we assume here that this problem is already solved.

8.2. The Scalar Problems

In order to develop solutions by the F_N method for the required scalar functions $V^{(o)}(\tau,\mu)$, $V^{(1)}(\tau,\mu)$, $\phi^{(1)}(\tau,\mu)$ and $\phi^{(2)}(\tau,\mu)$ discussed in section 8.1 we now consider the general class of problems defined by

$$\mu \frac{\partial}{\partial\tau} I(\tau,\mu) + I(\tau,\mu) = \frac{\omega}{2}\int_{-1}^{1} f(\mu')I(\tau,\mu')d\mu' \tag{8.24}$$

with

$$I(0,\mu) = N\delta(\mu-\mu_o),\ \mu,\ \mu_o > 0, \tag{8.25a}$$

and

$$I(\tau_o,-\mu) = 0,\ \mu > 0. \tag{8.25b}$$

Here N is an arbitrary normalization factor, and we consider that $f(\mu) = f(-\mu)$, $f(\mu) \geq 0$ for $\mu \in [0,1]$ and that

$$\Lambda(\infty) = 1 - \omega \int_0^1 f(\mu)d\mu \neq 0. \tag{8.26}$$

By analogy with other scalar problems [58] it is clear that a solution of Eq. (8.24) can be expressed as

$$I(\tau,\mu) = \left[A(\eta_o)\phi(\eta_o,\mu)e^{-\tau/\eta_o} + A(-\eta_o)\phi(-\eta_o,\mu)e^{\tau/\eta_o}\right]\delta_{\kappa,1}$$

$$+ \int_{-1}^{1} A(\eta)\phi(\eta,\mu)e^{-\tau/\eta}d\eta, \tag{8.27}$$

where

$$\phi(\xi,\mu) = \frac{\omega\xi}{2} f(\xi)P_v\left(\frac{1}{\xi - \mu}\right) + \lambda(\xi)\delta(\xi - \mu), \tag{8.28}$$

and

$$\lambda(\eta) = 1 + \frac{\omega}{2} \eta P_v\int_{-1}^{1} f(x) \frac{dx}{x - \eta} . \tag{8.29}$$

Here $f(1) > 0 \Rightarrow \kappa = 1$, and if $f(1) = 0$ then $\kappa = 1$ if $\lambda(1) < 0$ and $\kappa = 0$ if $\lambda(1) > 0$. We do not consider here the possibility that $\lambda(t)$ and $f(t)$ can simultaneously be zero for some $t \in (0,1]$. When $\kappa = 1$ we note that η_o is a zero of

$$\Lambda(z) = 1 + \frac{\omega}{2} z\int_{-1}^{1} f(x) \frac{dx}{x - z} \tag{8.30}$$

and that η_o can be expressed [101] as

$$\eta_o = \left[\Lambda(\infty)\right]^{-\frac{1}{2}} \exp\left[-\frac{1}{\pi}\int_{0}^{1} \Theta(t)\frac{dt}{t}\right] \tag{8.31}$$

where

$$\Theta(t) = \tan^{-1}\left(\frac{\omega t\pi f(t)}{2\lambda(t)}\right) \tag{8.32}$$

is continuous for $t \in [0,1]$ and such that $\Theta(1) = \pi$.

The expansion coefficients $A(\pm\xi)$, $\xi \in P$, i.e. $\xi \in \eta_o \cup (0,1)$ if $\kappa = 1$ or $\xi \in (0,1)$ if $\kappa = 0$, in Eq. (8.27) are to be determined by the boundary conditions given as Eqs. (8.25); however the fact that the functions $\phi(\xi,\mu)$ are orthogonal, i.e.

$$\int_{-1}^{1} \mu f(\mu)\phi(\xi,\mu)\phi(\xi',\mu)d\mu = 0, \; \xi \neq \xi', \; \pm\xi, \; \pm\xi' \in P, \tag{8.33}$$

can be used, as discussed by Grandjean and Siewert [39], to deduce the following singular integral equations and constraints for the surface intensities $I(0,-\mu)$ and $I(\tau_o,\mu)$, $\mu > 0$:

$$\int_{0}^{1} \mu f(\mu)\phi(\xi,\mu)I(0,-\mu)d\mu + e^{-\tau_o/\xi}\int_{0}^{1} \mu f(\mu)\phi(-\xi,\mu)I(\tau_o,\mu)d\mu = L_1(\xi) \tag{8.34a}$$

and

$$\int_{0}^{1} \mu f(\mu)\phi(\xi,\mu)I(\tau_o,\mu)d\mu + e^{-\tau_o/\xi}\int_{0}^{1} \mu f(\mu)\phi(-\xi,\mu)I(0,-\mu)d\mu = L_2(\xi). \tag{8.34b}$$

In Eqs. (8.34) the known terms are, after we make use of Eqs. (8.25),

$$L_1(\xi) = \mu_o f(\mu_o)\phi(-\xi,\mu_o)N \tag{8.35a}$$

and

$$L_2(\xi) = \mu_o f(\mu_o)e^{-\tau_o/\xi}\phi(\xi,\mu_o)N. \tag{8.35b}$$

If we now introduce the approximations

$$I(0,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu > 0, \tag{8.36a}$$

and

$$I(\tau_o,\mu) = N\delta(\mu - \mu_o)e^{-\tau_o/\mu} + \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \quad \mu > 0, \tag{8.36b}$$

into Eqs. (8.34) we find the basic F_N equations

$$\sum_{\alpha=0}^{N} \left[a_\alpha B_\alpha(\xi) + e^{-\tau_o/\xi} b_\alpha A_\alpha(\xi) \right] = \frac{\mu_o f(\mu_o)}{\xi + \mu_o} \left[1 - e^{-\tau_o(1/\xi + 1/\mu_o)} \right] N \tag{8.37a}$$

and

$$\sum_{\alpha=0}^{N} \left[b_\alpha B_\alpha(\xi) + e^{-\tau_o/\xi} a_\alpha A_\alpha(\xi) \right] = \mu_o f(\mu_o) \left[\frac{e^{-\tau_o/\xi} - e^{-\tau_o/\mu_o}}{\xi - \mu_o} \right] N. \tag{8.37b}$$

Here

$$A_o(\xi) = \int_0^1 \mu f(\mu) \frac{d\mu}{\mu + \xi} \quad , \tag{8.38a}$$

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \int_0^1 \mu^\alpha f(\mu) d\mu, \quad \alpha \geq 1, \tag{8.38b}$$

$$B_o(\xi) = \frac{2}{\omega}\Lambda(\infty) + A_o(\xi) \tag{8.38c}$$

and

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \int_0^1 \mu^\alpha f(\mu) d\mu \ , \quad \alpha \geq 1. \tag{8.38d}$$

It is apparent that we now can readily solve the system of linear algebraic equations obtained by evaluating Eqs. (8.37) at selected values of $\xi \in P$ to find the constants a_α and b_α and thus to establish the surface quantities given by Eqs. (8.36).

8.3. Numerical Results and Conclusions

In order to evaluate the effectiveness of the F_N-method for solving the complete problem for Rayleigh and isotropic scattering in a planetary atmosphere we now would like to report some numerical results. Since the atmosphere is illuminated by a beam of natural light we have $F_\ell = F_r = \frac{1}{2}$ and $F_u = F_v = 0$, and thus $V(\tau,\mu,\varphi) = 0$.

As discussed in section 8.1, the first problem to be solved is the two-vector problem, $\underset{\sim}{I}(\tau,\mu)$, however since the F_N solution of this problem was discussed with details in chapter 7, here we will discuss only the solution of the scalar problems. To solve the two scalar problems, $\phi^{(1)}(\tau,\mu)$ and $\phi^{(2)}(\tau,\mu)$, we have used the F_N method as described in section 8.2, and thus, for the $\phi^{(1)}$-problem we have used $f(\mu) = (3c/4)(1 - \mu^2)(1 + 2\mu^2)$ and $N = (8/3)\left[(1 + 2\mu_o^2)(1 - \mu_o^2)\right]^{-1}$, and for the $\phi^{(2)}$-problem we have used $f(\mu) = (3c/8)(1 + \mu^2)^2$ and $N = (16/3)(1 + \mu_o^2)^{-2}$, to derive the F_N-equations, for selected $\xi_j \in P$,

$$\sum_{\alpha=0}^{N} \left[a_\alpha B_\alpha^{(i)}(\xi_j) + e^{-\tau_o/\xi_j} b_\alpha A_\alpha^{(i)}(\xi_j) \right] = \frac{2c\mu_o}{\xi_j+\mu_o} \left[1 - e^{-\tau_o(1/\xi_j+1/\mu_o)} \right], \tag{8.39a}$$

and

$$\sum_{\alpha=0}^{N}\left[b_\alpha B_\alpha^{(i)}(\xi_j) + e^{-\tau_o/\xi_j} a_\alpha A_\alpha^{(i)}(\xi_j)\right] = 2c\mu_o\left[\frac{e^{-\tau_o/\xi_j} - e^{-\tau_o/\mu_o}}{\xi_j - \mu_o}\right]. \tag{8.39b}$$

Here, the superscript $\underline{\underline{i}}$, in the functions $A_\alpha^{(i)}(\xi)$ and $B_\alpha^{(i)}(\xi)$, indicates the scalar problem to be solved, i.e., i = 1 indicates $\phi^{(1)}$-problem and i = 2 indicates $\phi^{(2)}$-problem. Moreover, the functions $A_\alpha^{(i)}(\xi)$ and $B_\alpha^{(i)}(\xi)$ can be found by

$$A_\alpha^{(1)}(\xi) = -\xi A_{\alpha-1}^{(1)}(\xi) + \frac{3c}{4}\left\{\frac{1}{\alpha+1} + \frac{1}{\alpha+3} - \frac{2}{\alpha+5}\right\}, \tag{8.40a}$$

with

$$A_o^{(1)}(\xi) = \frac{3c}{4}\left\{\left[-\xi-\xi^3 + 2\xi^5\right]\log(1 + \frac{1}{\xi}) + \frac{14}{15} + \frac{\xi^2}{3} + \xi^3 - 2\xi^4\right\}, \tag{8.40b}$$

and

$$A_\alpha^{(2)}(\xi) = -\xi A_{\alpha-1}^{(2)}(\xi) + \frac{3c}{8}\left\{\frac{1}{\alpha+1} + \frac{2}{\alpha+3} + \frac{1}{\alpha+5}\right\}, \tag{8.41a}$$

with

$$A_o^{(2)}(\xi) = \frac{3c}{8}\left\{\left[-\xi-2\xi^3-\xi^5\right]\log(1 + \frac{1}{\xi}) + \frac{28}{15} + \frac{7\xi^2}{3} - \frac{5\xi}{4} - \frac{\xi^3}{2} + \xi^4\right\}, \tag{8.41b}$$

and the functions $B_\alpha^{(i)}(\xi)$, i = 1, 2, can be found from Eqs. (8.38) with

$$\Lambda(\infty) = 1 - \frac{7\omega c}{10}. \tag{8.42}$$

To solve the set of linear equations, given by Eqs. (8.39), we have selected, for $\kappa = 1$, $\xi_o = \eta_o$ and the remaining $\xi_j = (2j - 1)/(2N)$, $j = 1, 2, \ldots, N$, and, for $\kappa = 0$, $\xi_{j-1} = (2j - 1)/[2(N + 1)]$, $j = 1,2,\ldots(N+1)$. Thus, for each problem, we solved a set of $2(N + 1)$ of linear algebraic equations to establish a_α and b_α and thus to establish $\phi^{(1)}$ and $\phi^{(2)}$ at the boundaries.

The quantities $\underset{\sim}{I}(0,-\mu)$, $\underset{\sim}{I}(\tau_o,\mu)$, $\phi^{(1)}(0,-\mu)$, $\phi^{(1)}(\tau_o,\mu)$, $\phi^{(2)}(0,-\mu)$ and $\phi^{(2)}(\tau_o,\mu)$ can be used with (8.16) to deduce the three-vector, I_ℓ, I_r and U, or the Stoke's parameters $I = I_\ell + I_r$, $Q = I_\ell - I_r$ and U, at the boundaries. In Tables 8.2, 8.4, 8.6, 8.8 and 8.10 we report the Stokes parameters at $(0,-\mu,\varphi)$, and in Tables 8.3, 8.5, 8.7 and 8.9 we report the Stokes parameters at (τ_o,μ,φ) for the cases considered in Table 8.1. We note that the transmitted Stokes parameters we report, are actually the diffuse component of the radiation field. We also note that our "converged" results (typically obtained from $N \sim 10 - 12$) in the tables agree, with a calculation communicated by Kawabata [44].

Table 8.1: Cases Studied for the Complete Solution of the Planetary Problem.

Case	ω	c	λ_o	τ_o	μ_o
1	0.9	1.0	0.0	1.0	0.4
2	0.9	1.0	0.1	1.0	0.4
3	0.9	0.8	0.1	5.0	0.4
4	0.9	0.8	0.2	5.0	0.4
5	0.9	0.8	-	∞	0.4

Table 8.2: Emergent Stokes Parameters for Case 1

	μ	φ=0°	φ=30°	φ=60°	φ=90°	φ=120°	φ=150°	φ=180°
I		0.4014	0.3644	0.2913	0.2575	0.2999	0.3792	0.4185
Q	0.06	-0.0117	0.0272	0.1071	0.1497	0.1156	0.0420	0.0047
U		0.0	-0.0792	-0.1314	-0.1426	-0.1155	-0.0725	0.0
I		0.3456	0.3161	0.2592	0.2369	0.2789	0.3502	0.3850
Q	0.16	-0.0205	0.0159	0.0907	0.1343	0.1103	0.0500	0.0189
U		0.0	-0.0797	-0.1248	-0.1230	-0.0883	-0.0432	0.0
I		0.2852	0.2632	0.2220	0.2105	0.2508	0.3130	0.3427
Q	0.28	-0.0331	0.0010	0.0720	0.1166	0.1007	0.0507	0.0243
U		0.0	-0.0785	-0.1161	-0.1026	-0.0617	-0.0241	0.0
I		0.2339	0.2181	0.1897	0.1864	0.2338	0.2771	0.3020
Q	0.40	-0.0452	-0.0124	0.0564	0.1016	0.0904	0.0465	0.0229
U		0.0	-0.0762	-0.1074	-0.0851	-0.0400	-0.0089	0.0
I		0.1571	0.1507	0.1414	0.1481	0.1772	0.2127	0.2287
Q	0.64	-0.0640	-0.0324	0.0342	0.0788	0.0700	0.0296	0.0076
U		0.0	-0.0700	-0.0905	-0.0559	-0.0064	0.0141	0.0
I		0.1156	0.1146	0.1155	0.1248	0.1434	0.1631	0.1715
Q	0.84	-0.0718	-0.0408	0.0240	0.0652	0.0520	0.0077	-0.0158
U		0.0	-0.0632	-0.0754	-0.0333	0.0177	0.0299	0.0
I		0.1100	0.1100	0.1100	0.1100	0.1100	0.1100	0.1100
Q	1.00	-0.0567	-0.0283	0.0283	0.0567	0.0283	-0.0283	-0.0567
U		0.0	-0.0491	-0.0491	0.0	0.0491	0.0491	0.0

Table 8.3: Transmitted Stokes Parameter for Case 1

	μ	$\varphi=0^o$	$\varphi=30^o$	$\varphi=60^o$	$\varphi=90^o$	$\varphi=120^o$	$\varphi=150^o$	$\varphi=180^o$
I		0.1043	0.0966	0.0811	0.0728	0.0794	0.0937	0.1009
Q	0.06	0.0022	0.0095	0.0239	0.0305	0.0222	0.0066	-0.0012
U		0.0	-0.0125	-0.0228	-0.0281	-0.0259	-0.0156	0.0
I		0.1338	0.1238	0.1034	0.0913	0.0977	0.1139	0.1224
Q	0.16	0.0078	0.0167	0.0339	0.0407	0.0282	0.0068	-0.0037
U		0.0	-0.0126	-0.0257	-0.0357	-0.0362	-0.0231	0.0
I		0.1612	0.1490	0.1234	0.1069	0.1115	0.1283	0.1373
Q	0.28	0.0125	0.0233	0.0437	0.0502	0.0318	0.0026	-0.0114
U		0.0	-0.0102	-0.0258	-0.0426	-0.0481	-0.0325	0.0
I		0.1711	0.1583	0.1310	0.1118	0.1113	0.1278	0.1359
Q	0.40	0.0138	0.0258	0.0482	0.0538	0.0306	-0.0046	-0.0214
U		0.0	-0.0048	-0.0209	-0.0440	-0.0553	-0.0392	0.0
I		0.1578	0.1474	0.1245	0.1057	0.1013	0.1072	0.1113
Q	0.64	0.0060	0.0201	0.0461	0.0517	0.0228	-0.0201	-0.0405
U		0.0	0.0089	-0.0043	-0.0363	-0.0586	-0.0453	0.0
I		0.1293	0.1232	0.1093	0.0959	0.0892	0.0886	0.0892
Q	0.84	-0.0108	0.0059	0.0374	0.0468	0.0174	-0.0288	-0.0509
U		0.0	0.0212	0.0125	-0.0239	-0.0538	-0.0451	0.0
I		0.0879	0.0879	0.0879	0.0879	0.0879	0.0879	0.0879
Q	1.00	-0.0426	-0.0213	0.0213	0.0426	0.0213	-0.0213	-0.0426
U		0.0	0.0369	0.0369	0.0	-0.0369	-0.0369	0.0

Table 8.4: Emergent Stokes Parameters for Case 2

	μ	φ=0°	φ=30°	φ=60°	φ=90°	φ=120°	φ=150°	φ=180°
I		0.4044	0.3674	0.2943	0.2605	0.3029	0.3823	0.4215
Q	0.06	-0.0127	0.0269	0.1068	0.1494	0.1153	0.0417	0.0044
U		0.0	-0.0792	-0.1314	-0.1426	-0.1155	-0.0633	0.0
I		0.3492	0.3198	0.2628	0.2406	0.2825	0.3539	0.3886
Q	0.16	-0.0208	0.0156	0.0904	0.1340	0.1101	0.0497	0.0186
U		0.0	-0.0797	-0.1248	-0.1230	-0.0883	-0.0432	0.0
I		0.2897	0.2677	0.2265	0.2151	0.2553	0.3175	0.3472
Q	0.28	-0.0334	0.0007	0.0718	0.1164	0.1005	0.0505	0.0241
U		0.0	-0.0785	-0.1161	-0.1026	-0.0617	-0.0241	0.0
I		0.2394	0.2235	0.1952	0.1918	0.2292	0.2825	0.3075
Q	0.40	-0.0454	-0.0126	0.0562	0.1014	0.0902	0.0463	0.0227
U		0.0	-0.0762	-0.1074	-0.0851	-0.0400	-0.0089	0.0
I		0.1643	0.1578	0.1485	0.1552	0.1843	0.2198	0.2358
Q	0.64	-0.0641	-0.0325	0.0341	0.0787	0.0699	0.0295	0.0075
U		0.0	-0.0700	-0.0905	-0.0559	-0.0064	0.0141	0.0
I		0.1239	0.1229	0.1237	0.1330	0.1517	0.1713	0.1798
Q	0.84	-0.0718	-0.0408	0.0239	0.0652	0.0519	0.0076	-0.0159
U		0.0	-0.0632	-0.0754	-0.0333	0.0177	0.0299	0.0
I		0.1189	0.1189	0.1189	0.1189	0.1189	0.1189	0.1189
Q	1.00	-0.0567	-0.0284	0.0284	0.0567	0.0284	-0.0284	-0.0567
U		0.0	-0.0491	-0.0491	0.0	0.0491	0.0491	0.0

Table 8.5: Transmitted Stokes Parameters for Case 2

	μ	φ=0°	φ=30°	φ=60°	φ=90°	φ=120°	φ=150°	φ=180°
I		0.1132	0.1055	0.0900	0.0817	0.0883	0.1026	0.1098
Q	0.06	0.0023	0.0096	0.0240	0.0306	0.0223	0.0067	-0.0011
U		0.0	-0.0125	-0.0228	-0.0281	-0.0259	-0.0156	0.0
I		0.1419	0.1320	0.1115	0.0994	0.1058	0.1220	0.1305
Q	0.16	0.0078	0.0167	0.0340	0.0408	0.0282	0.0068	-0.0036
U		0.0	-0.0126	-0.0257	-0.0357	-0.0362	-0.0231	0.0
I		0.1684	0.1562	0.1307	0.1141	0.1188	0.1356	0.1446
Q	0.28	0.0125	0.0233	0.0437	0.0501	0.0318	0.0026	-0.0114
U		0.0	-0.0102	-0.0258	-0.0426	-0.0481	-0.0325	0.0
I		0.1776	0.1648	0.1375	0.1183	0.1199	0.1343	0.1424
Q	0.40	0.0137	0.0258	0.0481	0.0537	0.0305	-0.4689	-0.0214
U		0.0	-0.0480	-0.0209	-0.0440	-0.0553	-0.0392	0.0
I		0.1630	0.1527	0.1298	0.1109	0.1065	0.1124	0.1165
Q	0.64	0.0059	0.0201	0.0460	0.0516	0.0228	-0.0202	-0.0405
U		0.0	0.0089	-0.0434	-0.0363	-0.0586	-0.0453	0.0
I		0.1339	0.1278	0.1138	0.1004	0.0937	0.0931	0.0938
Q	0.84	-0.0109	0.0059	0.0374	0.0468	0.0173	-0.0288	-0.0509
U		0.0	0.0212	0.0125	-0.0239	-0.0538	-0.0451	0.0
I		0.0919	0.0919	0.0919	0.0919	0.0919	0.0919	0.0919
Q	1.00	-0.0426	-0.0213	0.0213	0.0426	0.0213	-0.0213	-0.0426
U		0.0	0.0369	0.0369	0.0	-0.0369	-0.0369	0.0

Table 8.6: Emergent Stokes Parameters for Case 3

	μ	$\varphi=0^{\circ}$	$\varphi=30^{\circ}$	$\varphi=60^{\circ}$	$\varphi=90^{\circ}$	$\varphi=120^{\circ}$	$\varphi=150^{\circ}$	$\varphi=180^{\circ}$
I		0.3951	0.3668	0.3109	0.2849	0.3173	0.3779	0.4080
Q	0.06	-0.0129	0.0173	0.0784	0.1110	0.0848	0.0284	-0.0001
U		0.0	-0.0595	-0.0987	-0.1069	-0.0865	-0.0474	0.0
I		0.3489	0.3265	0.2831	0.2661	0.2977	0.3518	0.3781
Q	0.16	-0.0186	0.0090	0.0656	0.0985	0.0802	0.0342	0.0106
U		0.0	-0.0594	-0.0927	-0.0911	-0.0650	-0.0317	0.0
I		0.3010	0.2843	0.2528	0.2438	0.2740	0.3210	0.3434
Q	0.28	-0.0274	-0.0016	0.0520	0.0856	0.0732	0.0351	0.0150
U		0.0	-0.0585	-0.0862	-0.0757	-0.0450	-0.0172	0.0
I		0.2616	0.2494	0.2275	0.2246	0.2528	0.2931	0.3121
Q	0.40	-0.0361	-0.0112	0.0411	0.0752	0.0663	0.0325	0.0144
U		0.0	-0.0572	-0.0803	-0.0631	-0.0290	-0.0059	0.0
I		0.2019	0.1968	0.1892	0.1940	0.2162	0.2436	0.2560
Q	0.64	-0.0504	-0.0259	0.0257	0.0600	0.0527	0.0209	0.0036
U		0.0	-0.0539	-0.0693	-0.0422	-0.0037	0.0117	0.0
I		0.1679	0.1671	0.1674	0.1744	0.1889	0.2042	0.2108
Q	0.84	-0.0568	-0.0323	0.0188	0.0511	0.0402	0.0048	-0.0140
U		0.0	-0.0497	-0.0590	-0.0255	0.0148	0.0241	0.0
I		0.1615	0.1615	0.1615	0.1615	0.1615	0.1615	0.1615
Q	1.00	-0.0456	-0.0228	0.0228	0.0456	0.0228	-0.0228	-0.0456
U		0.0	-0.0395	-0.0395	0.0	0.0395	0.0395	0.0

Table 8.7. Transmitted Stokes Parameters for Case 3

	μ	$\varphi=0^{o}$	$\varphi=30^{o}$	$\varphi=60^{o}$	$\varphi=90^{o}$	$\varphi=120^{o}$	$\varphi=150^{o}$	$\varphi=180^{o}$
I		0.7221(-2)	0.7167(-2)	0.7059(-2)	0.7004(-2)	0.7054(-2)	0.7159(-2)	0.7212(-2)
Q	0.06	-0.7306(-3)	-0.6777(-3)	-0.5724(-3)	-0.5212(-3)	-0.5771(-3)	-0.6860(-3)	-0.7400(-3)
U		0.0	-0.2831(-4)	-0.5716(-4)	-0.7880(-4)	-0.7932(-4)	-0.5048(-4)	0.0
I		0.8346(-2)	0.8284(-2)	0.8158(-2)	0.8091(-2)	0.8143(-2)	0.8258(-2)	0.8316(-2)
Q	0.16	-0.7247(-3)	-0.6637(-3)	-0.5430(-3)	-0.4874(-3)	-0.5579(-3)	-0.6895(-3)	-0.7545(-3)
U		0.0	-0.1246(-4)	-0.4652(-4)	-0.9307(-4)	-0.1147(-3)	-0.8061(-4)	0.0
I		0.9718(-2)	0.9647(-2)	0.9500(-2)	0.9416(-2)	0.9468(-2)	0.9592(-2)	0.9655(-2)
Q	0.28	-0.7205(-3)	-0.6457(-3)	-0.4992(-3)	-0.4360(-3)	-0.5310(-3)	-0.7007(-3)	-0.7840(-3)
U		0.0	0.1446(-4)	-0.2699(-4)	-0.1133(-3)	-0.1692(-3)	-0.1277(-3)	0.0
I		0.1123(-1)	0.1115(-1)	0.1097(-1)	0.1087(-1)	0.1092(-1)	0.1105(-1)	0.1111(-1)
Q	0.40	-0.7069(-3)	-0.6106(-3)	-0.4235(-3)	-0.3481(-3)	-0.4805(-3)	-0.7093(-3)	-0.8209(-3)
U		0.0	0.5290(-4)	0.7134(-6)	-0.1426(-3)	-0.2477(-3)	-0.1955(-3)	0.0
I		0.1501(-1)	0.1490(-1)	0.1465(-1)	0.1447(-1)	0.1447(-1)	0.1459(-1)	0.1466(-1)
Q	0.64	-0.6375(-3)	-0.4396(-3)	-0.6115(-4)	0.7270(-4)	-0.2359(-3)	-0.7423(-3)	-0.9870(-3)
U		0.0	0.2114(-3)	0.1115(-3)	-0.2731(-3)	-0.5844(-3)	-0.4845(-3)	0.0
I		0.1884(-1)	0.1871(-1)	0.1844(-1)	0.1819(-1)	0.1811(-1)	0.1814(-1)	0.1818(-1)
Q	0.84	-0.7148(-3)	-0.3036(-3)	0.4862(-3)	0.7761(-3)	0.1549(-3)	-0.8775(-3)	-0.1377(-2)
U		0.0	0.5800(-3)	0.4356(-3)	-0.3944(-3)	-0.1119(-2)	-0.9744(-3)	0.0
I		0.2141(-1)	0.2141(-1)	0.2141(-1)	0.2141(-1)	0.2141(-1)	0.2141(-1)	0.2141(-1)
Q	1.00	-0.1540(-2)	-0.7702(-3)	0.7702(-3)	0.1540(-2)	0.7702(-3)	-0.7702(-3)	-0.1540(-2)
U		0.0	0.1334(-2)	0.1334(-2)	0.0	-0.1334(-2)	-0.1334(-2)	0.0

Table 8.8. Emergent Stokes Parameters for Case 4

	μ	$\varphi=0^{\circ}$	$\varphi=30^{\circ}$	$\varphi=60^{\circ}$	$\varphi=90^{\circ}$	$\varphi=120^{\circ}$	$\varphi=150^{\circ}$	$\varphi=180^{\circ}$
I		0.3952	0.3669	0.3109	0.2849	0.3173	0.3780	0.4080
Q	0.06	-0.0129	0.0173	0.0784	0.1110	0.0848	0.0284	-0.0001
U		0.0	-0.0595	-0.0987	-0.1069	-0.0865	-0.0474	0.0
I		0.3490	0.3266	0.2832	0.2661	0.2978	0.3518	0.3781
Q	0.16	-0.0186	0.0090	0.0656	0.0985	0.0802	0.0342	0.0106
U		0.0	-0.0594	-0.0927	-0.0911	-0.0650	-0.0317	0.0
I		0.3011	0.2843	0.2529	0.2439	0.2741	0.3210	0.3435
Q	0.28	-0.0274	-0.0016	0.0520	0.0856	0.0732	0.0351	0.0150
U		0.0	-0.0585	-0.0862	-0.0757	-0.0450	-0.0172	0.0
I		0.2617	0.2495	0.2276	0.2246	0.2528	0.2932	0.3122
Q	0.40	-0.0361	-0.0112	0.0411	0.0752	0.0663	0.0325	0.0144
U		0.0	-0.0572	-0.0803	-0.0631	-0.0290	-0.0060	0.0
I		0.2020	0.1969	0.1893	0.1941	0.2163	0.2437	0.2560
Q	0.64	-0.0504	-0.0259	0.0257	0.0600	0.0527	0.0209	0.0036
U		0.0	-0.0539	-0.0693	-0.0422	-0.0375	0.0117	0.0
I		0.1680	0.1672	0.1676	0.1745	0.1890	0.2043	0.2109
Q	0.84	-0.0568	-0.0323	0.0188	0.0511	0.0402	0.0048	-0.0140
U		0.0	-0.0497	-0.0590	-0.0255	0.0148	0.0241	0.0
I		0.1616	0.1616	0.1616	0.1616	0.1616	0.1616	0.1616
Q	1.00	-0.0456	-0.0228	0.0228	0.0456	0.0228	-0.0228	-0.0456
U		0.0	-0.0395	-0.0395	0.0	0.0395	0.0395	0.0

Table 8.9. Transmitted Stokes Parameter for Case 4

	μ	φ=0°	φ=30°	φ=60°	φ=90°	φ=120°	φ=150°	φ=180°
I		0.8329(-2)	0.8276(-2)	0.8168(-2)	0.8112(-2)	0.8163(-2)	0.8268(-2)	0.8320(-2)
Q	0.06	-0.7271(-3)	-0.6742(-3)	-0.5689(-3)	-0.5178(-3)	-0.5737(-3)	-0.6824(-3)	-0.7366(-3)
U		0.0	-0.2832(-4)	-0.5717(-4)	-0.7881(-4)	-0.7934(-4)	-0.5049(-4)	0.0
I		0.9380(-2)	0.9318(-2)	0.9193(-2)	0.9126(-2)	0.9178(-2)	0.9293(-2)	0.9351(-2)
Q	0.16	-0.7276(-3)	-0.6666(-3)	-0.5459(-3)	-0.4903(-3)	-0.5608(-3)	-0.6924(-3)	-0.7574(-3)
U		0.0	-0.1246(-4)	-0.4652(-4)	-0.9307(-4)	-0.1147(-3)	-0.8061(-4)	0.0
I		0.1068(-1)	0.1061(-1)	0.1046(-1)	0.1038(-1)	0.1043(-1)	0.1056(-1)	0.1062(-1)
Q	0.28	-0.7273(-3)	-0.6525(-3)	-0.5060(-3)	-0.4428(-3)	-0.5377(-3)	-0.7074(-3)	-0.7907(-3)
U		0.0	0.1446(-4)	-0.2700(-4)	-0.1133(-3)	-0.1692(-3)	-0.1277(-3)	0.0
I		0.1213(-1)	0.1205(-1)	0.1188(-1)	0.1178(-1)	0.1182(-1)	0.1195(-1)	0.1202(-1)
Q	0.40	-0.7152(-3)	-0.6189(-3)	-0.4318(-3)	-0.3564(-3)	-0.4889(-3)	-0.7177(-3)	-0.8293(-3)
U		0.0	0.5290(-4)	0.7131(-6)	-0.1426(-3)	-0.2477(-3)	-0.1955(-3)	0.0
I		0.1583(-1)	0.1571(-1)	0.1546(-1)	0.1528(-1)	0.1529(-1)	0.1541(-1)	0.1548(-1)
Q	0.64	-0.6449(-3)	-0.4471(-3)	-0.6856(-4)	0.6529(-4)	-0.2433(-3)	-0.7498(-3)	-0.9944(-3)
U		0.0	0.2114(-3)	0.1115(-3)	-0.2731(-3)	-0.5844(-3)	-0.4845(-3)	0.0
I		0.1959(-1)	0.1947(-1)	0.1919(-1)	0.1895(-1)	0.1886(-1)	0.1890(-1)	0.1893(-1)
Q	0.84	-0.7188(-3)	-0.3076(-3)	0.4822(-3)	0.7722(-3)	0.1509(-3)	-0.8814(-3)	-0.1381(-2)
U		0.0	0.5800(-3)	0.4356(-3)	-0.3944(-3)	-0.1119(-2)	-0.9744(-3)	0.0
I		0.2212(-1)	0.2212(-1)	0.2212(-1)	0.2212(-1)	0.2212(-1)	0.2212(-1)	0.2212(-1)
Q	1.00	-0.1540(-2)	-0.7702(-3)	0.7702(-3)	0.1540(-2)	0.7702(-3)	-0.7702(-3)	-0.1540(-2)
U		0.0	0.1334(-2)	0.1334(-2)	0.0	-0.1334(-2)	-0.1334(-2)	0.0

Table 8.10. Emergent Stokes Parameters for Case 5

	μ	φ=0°	φ=30°	φ=60°	φ=90°	φ=120°	φ=150°	φ=180°
I		0.3953	0.3670	0.3110	0.2851	0.3174	0.3781	0.4081
Q	0.06	-0.0129	0.0173	0.0784	0.1110	0.0848	0.0284	-0.0001
U		0.0	-0.0595	-0.0986	-0.1069	-0.0865	-0.0474	0.0
I		0.3491	0.3267	0.2833	0.2663	0.2979	0.3519	0.3782
Q	0.16	-0.0186	0.0090	0.0656	0.0985	0.0801	0.0342	0.0106
U		0.0	-0.0594	-0.0927	-0.0911	-0.0650	-0.0317	0.0
I		0.3013	0.2845	0.2530	0.2440	0.2742	0.3212	0.3437
Q	0.28	-0.0274	-0.0016	0.0520	0.0855	0.0732	0.0351	0.0150
U		0.0	-0.0585	-0.0862	-0.0757	-0.0450	-0.0172	0.0
I		0.2619	0.2497	0.2278	0.2248	0.2530	0.2934	0.3124
Q	0.40	-0.0361	-0.0112	0.0411	0.0752	0.0663	0.0325	0.0144
U		0.0	-0.0572	-0.0803	-0.0631	-0.0290	-0.0586	0.0
I		0.2023	0.1972	0.1896	0.1944	0.2166	0.2440	0.2563
Q	0.64	-0.0504	-0.0259	0.0257	0.0600	0.0527	0.0209	0.0036
U		0.0	-0.0539	-0.0693	-0.0422	-0.0037	0.0117	0.0
I		0.1684	0.1675	0.1679	0.1749	0.1893	0.2047	0.2113
Q	0.84	-0.0568	-0.0323	0.0188	0.0511	0.0402	0.0048	-0.0140
U		0.0	-0.0497	-0.0590	-0.0255	0.0148	0.0241	0.0
I		0.1620	0.1620	0.1620	0.1620	0.1620	0.1620	0.1620
Q	1.00	-0.0456	-0.0228	0.0228	0.0456	0.0228	-0.0228	-0.0456
U		0.0	-0.0395	-0.0395	0.0	0.0395	0.0395	0.0

9. SUMMARY AND CONCLUSIONS

This thesis has presented applications of the F_N method to basic problems in radiative transfer and neutron transport theory. In all problems considered in this thesis, the simplicity and accuracy of the F_N method was demonstrated by producing numerical results, which were compared to those obtained by "exact" techniques, when available, or by other techniques.

In chapter 3, we used the F_N method together with "exact" analysis to find the radiation field due to a point source of radiation located at the center of a finite sphere. By solving this problem we have shown that the F_N method can provide very accurate numerical results, since we have achieved excellent agreement with "exact" results using low orders of the F_N approximation. Besides, the "exact" solution of this problem requires the iterative numerical solution of a set of Fredholm integral equations, whereas the F_N method requires only the numerical solution of linear algebraic equations. We also have compared the F_N results with a Monte Carlo results, and note that to obtain the same degree of accuracy, as the F_N method, the Monte Carlo method requires a large number of histories. In short, we conclude that the F_N method can provide results so accurate as the "exact" method, using simple computational requirements.

In chapter 4, we considered the solution of the radiative transfer equation in an anisotropically plane parallel medium with specularly and diffusely reflecting boundaries, and source of the radiative heat in the walls, as well as inside the medium. This is a problem which had not been previously considered in an exact manner, and thus by reporting numerical results for the net radiative heat flux, we establish results which can be used as a "benchmark" for other approximate techniques.

In chapter 5, we considered the critical problem for multiregion reactors. The critical problem for a slab reactor with finite reflector was considered in section 5.1, and our numerical results are in agreement with those reported by Burkart, which uses an "exact" method to solve this problem. In section 5.2, we considered a critical problem for a slab reactor with a blanket and finite reflector, and since this problem had not been solved previously by any "exact" technique, again our results can be used as a "benchmark".

In chapter 6, we used the F_N method to calculate the thermal disadvantage factor in two-media slab cells with anisotropic scattering. This problem was considered previously by Eccleston and McCormick [33], and by Latelin et al. [49]. The numerical results of Eccleston and McCormick were not in agreement with those of Latelin et al., mainly for high order anisotropic scattering. In this thesis we calculated the disadvantage factor for the same cells considered by these works, and conclude that some of the results of Eccleston and McCormick were in error. The results we obtained confirms the generally accepted phsycial conclusion, that high orders terms in the scattering law have little effect on the disadvantage factor.

In chapter 7, we used the F_N method for solving the azimuth-independent equation of transfer for the polarized light in a finite plane-parallel atmosphere with a combination of Rayleigh and isotropic scattering. In particular, we solved the planetary problem, i.e., the illumination of an atmosphere of finite optical thickness with ground reflection. This problem does not have an "exact" solution, however we have compared our results with the work of Kawabata [44], who used a doubling method [40] to solve this problem, and obtained an excellent agreement.

In chapter 8, we generalize the problem discussed in chapter 7 to include the azimuth dependence. In doing so, we have to solve a three-vector equation of transfer for the Stoke's parameters, I, Q, and U, and a scalar equation of transfer for a Stoke's parameter V. To solve this problem we used an approach used by Chandrasekhar [25] to decompose the three vector problem in three different problems. The first problem is a two-vector radiative transfer problem, identical to the problem discussed in chapter 7, and the other two are scalar radiative transfer problems, which can be solved by the F_N method. Again, the numerical results we obtained were in agreement with those of Kawabata [44].

We have shown that the F_N method can be of value in solving radiation transport problems, and from the problems we solved we have concluded the following advantages of the F_N method:

i) The mathematical analysis is simple, and does not require any special technique.

ii) The computational requirements are simple, since the method needs only the computation of simple functions, and the solution of a linear algebraic system of equations. Also, with exception of the critical problems, the F_N method does not require any iterative numerical method.

iii) The method uses the actual boundary conditions, contrary to other methods, like P_N method in which is necessary to use non-physical boundary conditions. Also, the method only approximates unknown quantities at the boundaries.

iv) The method yields numerical results as accurate as "exact" techniques can provide.

On the other side, some disadvantages inherent to the F_N method are:

i) The choice of the points to satisfy the F_N equations, will affect the way the results converge. From experience, we have concluded that the best choice is the one discussed in chapter 4.

ii) The method requires a normal mode solution of the transport equation, and thus it is restricted to slab geometries, or in a geometry reducible to slab geometry.

iii) The accuracy of the method is presently limited, due to numerical reasons, to 5 or 6 significant figures, and even using higher order approximations we could not improve this accuracy.

Finally, we should mention that some additional work needs to be done if the method is going to be applied in solving practical problems. For example, the application of the method for solving the energy-dependent transport equation needs to be considered. Also, studies in other geometries, and in multidimension geometries needs to be considered, although we are not certain about this possibility. In addition, transport problems in other areas, like kinetic theory, may be capable of treatment by the F_N method.

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11. APPENDIX

11.1. On the Computation of the Eigenvalues

The computation of the discrete eigenvalues is a basic requirement in the analysis used in the F_N method, as we have shown in previous chapters, and thus here we wish to discuss how we computed these quantities.

For the isotropic scattering model, the discrete eigenvalues are the zeros of the dispersion function

$$\Lambda(z) = 1 - \frac{1}{2}\omega z \int_{-1}^{1} \frac{d\mu}{z - \mu}$$

$$= 1 - \frac{1}{2}\omega z \log\left[\frac{(z+1)}{(z-1)}\right]$$

$$= 1 - \omega z \tanh^{-1}(1/z), \; z \notin (-1,1). \tag{11.1}$$

This dispersion function is an even function and has only one pair of zeros [21]. For $\omega < 1$, the zeros are real, and for $\omega > 1$ are imaginary. To solve $\Lambda(\nu_o) = 0$, we have used the explicit expression [101],

$$\nu_o = \frac{1}{\sqrt{1-c}} \exp\left[-\frac{1}{\pi}\int_0^1 \Theta(t)\,\frac{dt}{t}\right], \tag{11.2}$$

as a first guess in a Newton-Raphson iterative scheme. Here, the function $\Theta(t)$ is given by

$$\Theta(t) = \tan^{-1}\left[\frac{\omega\pi t}{2\lambda(t)}\right], \; t \in (0,1), \tag{11.3}$$

where $\lambda(t)$ is given by

$$\lambda(t) = 1 - \omega t \tanh^{-1}(t),\ t \in (0,1). \tag{11.4}$$

In Table 11.1 we list the absolute value of the discrete eigenvalue, $|\nu_o|$, for various values of ω (single scattering albedo, or number of secondary neutrons per collision).

For the general anisotropic scattering model, the discrete eigenvalues are the zeros of the dispersion function

$$\Lambda(z) = 1 - z\int_{-1}^{1} \Psi(x)\,\frac{dx}{z - x}\ ,\ z \not\subset (-1,1), \tag{11.5}$$

where the characteristic function is

$$\Psi(x) = \frac{\omega}{2}\sum_{\ell=0}^{L} (2\ell + 1)f_\ell g_\ell(x)P_\ell(x), \tag{11.6}$$

and the polynomials $g_\ell(x)$ and $P_\ell(x)$ are defined in chapter 4. An explicit form for the dispersion function can be written as

$$\Lambda(z) = 1 + \frac{\omega z}{2}\sum_{\ell=1}^{L} (2\ell + 1)f_\ell g_\ell(z)\Gamma_\ell(z) - z\Psi(z)\log\left(\frac{z+1}{z-1}\right), \tag{11.7}$$

where the polynomials $\Gamma_\ell(z)$ can be generated from

$$(2\ell + 1)z\Gamma_\ell(z) = -2\delta_{\ell,0} + (\ell + 1)\Gamma_{\ell+1}(z) + \ell\Gamma_{\ell-1}(z)\ , \tag{11.8}$$

with

$$\Gamma_o(z) = 0 \text{ and } \Gamma_1(z) = 2.$$

Table 11.1. Discrete Eigenvalues for the Isotropic Scattering Model.

ω	$\lvert\nu_o\rvert$	ω	$\lvert\nu_o\rvert$	ω	$\lvert\nu_o\rvert$
0.10	1.0000000041	0.85	1.5885586254	1.35	0.8618848068
0.15	1.0000032393	0.90	1.9032048560	1.40	0.7937682966
0.20	1.0000908865	0.95	2.6351488343	1.45	0.7371493354
0.25	1.0006747820	0.96	2.9340205614	1.50	0.6891305030
0.30	1.0025928888	0.97	3.3740313858	1.55	0.6477488967
0.35	1.0068834333	0.98	4.1155204763	1.60	0.6116198272
0.40	1.0145858159	0.99	5.7967294513	1.65	0.5797339321
0.45	1.0267194558	0.999	18.2647257265	1.70	0.5513352116
0.50	1.0443820338	1.0	∞	1.75	0.5258443698
0.55	1.0689136569	1.05	2.5317821790	1.80	0.5028087985
0.60	1.1021320212	1.10	1.7566519663	1.85	0.4818688935
0.65	1.1467030486	1.15	1.4083093792	1.90	0.4627347408
0.70	1.2068042540	1.20	1.1982650015	1.95	0.4451695927
0.75	1.2894635253	1.25	1.0535913327	2.00	0.4289779090
0.80	1.4076343091	1.30	0.9460002249	3.00	0.2517253666

The dispersion function given by Eq. (11.6) is also an even function, and to compute the number κ of $\pm$ pairs of zeros of $\Lambda(z)$ we have used the argument principle [21], i.e., since $\Lambda(z)$ is analytic in the complex plane cut from -1 to 1, the number of zeros of $\Lambda(z)$ is given by $1/2\pi$ times the change in the argument of $\Lambda(z)$ as z varies along a contour at infinity plus a contour around the cut of $\Lambda(z)$. Since $\Lambda(\infty)$ is a constant $\Delta c_{\infty}(\arg\Lambda(z))$ is zero and we need only consider $\Delta_{-1,1}$ $(\arg\Lambda(z))$. If we denote the values of $\Lambda(z)$ as $\mathrm{Im}\Lambda(z)\to 0^{\pm}$ by $\Lambda^{\pm}(t)$, $t\in(-1,1)$, then using Plemelj-Sohotski formula [65]

$$\Lambda^{\pm}(t) = 1 + tP_{v}\int_{-1}^{1} \Psi(x)\,\frac{dx}{x - t} \pm \pi i\Psi(t)t$$

$$= \lambda(t) \pm \pi t\Psi(t)i,\ t\in(-1,1). \tag{11.9}$$

Then, using the fact that the dispersion function is even and $\overline{\Lambda^{+}} = \Lambda^{-}$,

$$\Delta_{c}\arg\Lambda(z) = 4\Delta_{01}\left[\arg\Lambda^{+}(t)\right], \tag{11.10}$$

and since

$$\arg(\Lambda^{+}(t)) = \tan^{-1}\left(\frac{\pi t\Psi(t)}{\lambda(t)}\right),$$

we can compute the number κ of $\pm$ pairs of discrete eigenvalues, by

$$\kappa = \frac{1}{\pi}\,\Delta_{0,1}\left[\tan^{-1}\left(\frac{\pi t\Psi(t)}{\lambda(t)}\right)\right]. \tag{11.11}$$

Now, we can use the formulas of Siewert [101] to find the discrete eigenvalues, thus if $\kappa = 1$ the discrete eigenvalue is given by

$$\nu_0^2 = \frac{1}{\Lambda(\infty)} \exp\left[- \frac{2}{\pi} \int_0^1 \arg\Lambda^+(t) \frac{dt}{t} \right], \quad \kappa = 1, \tag{11.12}$$

where

$$\Lambda(\infty) = \prod_{\ell=0}^{L} (1 - \omega f_\ell). \tag{11.13}$$

For $\kappa = 2$, the discrete eigenvalues are given by

$$\nu_0^2 = A + (A^2 - B)^{\frac{1}{2}}, \quad \kappa = 2, \tag{11.14a}$$

and

$$\nu_1^2 = A - (A^2 - B)^{\frac{1}{2}}, \quad \kappa = 2, \tag{11.14b}$$

where

$$A = 1 - \frac{1}{\pi} \int_0^1 t \arg\Lambda^+(t) \frac{dt}{t} + \frac{\omega}{2\Lambda(\infty)} \sum_{\ell=0}^{L} f_\ell B_\ell \tag{11.15}$$

and

$$B = \frac{1}{\Lambda(\infty)} \exp\left[- \frac{2}{\pi} \int_0^1 \arg\Lambda^+(t) \frac{dt}{t} \right] \tag{11.16}$$

with the coefficients B_ℓ being given by

$$(2\ell + 1)B_{\ell+1} = h_\ell B_\ell + \frac{(\ell+2)^2}{(2\ell+5)(2\ell+3)} h_\ell W_\ell - \frac{\ell^2}{2\ell-1} W_{\ell-1}, \ \ell \geq 0, \tag{11.17a}$$

$$B_o = \frac{1}{3} \text{ and } B_1 = \frac{3}{5} h_o \qquad \text{(11.17b and c)}$$

and h_ℓ are those quantities defined in chapter 4, and W_ℓ is given by

$$(2\ell + 1)W_{\ell+1} = \frac{h_\ell W_\ell}{(2\ell+1)} \tag{11.18a}$$

with

$$W_o = 1 \tag{11.18b}$$

Finally, if $\kappa = 3$ the discrete eigenvalues are the roots of a cubic equation and are given by

$$\nu_0^2 = (S_1 + S_2) - \frac{B^*}{3}, \ \kappa = 3, \tag{11.19a}$$

$$\nu_1^2 = -\frac{1}{2}(S_1 + S_2) - \frac{B^*}{3} + i\frac{\sqrt{3}}{2}(S_1 - S_2), \ \kappa = 3, \tag{11.19b}$$

$$\nu_2^2 = -\frac{1}{2}(S_1 + S_2) - \frac{B^*}{3} - i\frac{\sqrt{3}}{2}(S_1 - S_2), \ \kappa = 3, \tag{11.19c}$$

where

$$S_1 = \left[r + (q^3 + r^2)^{\frac{1}{2}} \right]^{1/3}, \tag{11.20a}$$

$$S_2 = \left[r - (q^3 + r^2)^{\frac{1}{2}} \right]^{1/3}, \tag{11.20b}$$

and

$$q = \frac{1}{3} C^* - \frac{1}{9} B^{*2} \tag{11.21a}$$

$$r = \frac{1}{6} (B^*C^* - 3A^*) - \frac{1}{27} B^{*3} \tag{11.21b}$$

where for real roots $(\omega < 1) \Rightarrow q^3 + r^2 < 0$. Here

$$-A^* = \nu_o^2 \nu_1^2 \nu_2^2 = \frac{1}{\Lambda(\infty)} \exp\left[-\frac{2}{\pi} \int_0^1 \arg\Lambda^+(t) \frac{dt}{t} \right], \tag{11.22a}$$

$$-B^* = \nu_o^2 + \nu_1^2 + \nu_2^2 = 3 - \frac{2}{\pi} \Theta_1 + \frac{\omega}{\Lambda(\infty)} \sum_{\ell=0}^{L} f_\ell B_\ell \tag{11.22b}$$

and

$$-C^* = \nu_o^2\nu_1^2 + \nu_o^2\nu_2^2 + \nu_1^2\nu_2^2 = 3(\Theta_1 - 1) - \Theta_3 - \frac{1}{2}\Theta_1^2$$

$$+ (\Theta_1 - 3) \frac{\omega}{\Lambda(\infty)} \sum_{\ell=0}^{L} f_\ell B_\ell + \frac{\omega}{\Lambda(\infty)} \sum_{\ell=0}^{L} f_\ell C_\ell \tag{11.22c}$$

where

$$\Theta_\alpha = \int_0^1 t^\alpha \arg\Lambda^+(t)dt, \quad \alpha = 1, 2, 3, \tag{11.23}$$

and

$$(2\ell + 1)C_{\ell+1} = h_\ell C_\ell + \frac{(\ell+2)^2}{(2\ell+5)(2\ell+3)} h_\ell T_\ell - \frac{\ell^2}{2\ell-1} T_{\ell-1}, \quad \ell \geq 0, \tag{11.24a}$$

$$T_\ell = B_\ell + \frac{1}{(2\ell+5)}\left[\frac{(\ell+3)^2}{2\ell+7} + \frac{(\ell+2)^2}{2\ell+3}\right] W_\ell \,, \tag{11.24b}$$

with

$$C_o = \frac{1}{5} \text{ and } c_1 = \frac{3}{7} h_o \,. \tag{11.24c and d}$$

In order to demonstrate the accuracy of the explicit solution given by Siewert's formulas we solved the equations above using 80 points Gaussian quadrature scheme to compute the required integrals and then refined these results by a Newton-Raphson interative scheme. In Table 11.2 we report numerical results for the discrete eigenvalues of the phase function defined by Kaper, Shultis and Veninga [45]

$$(2\ell + 1) f_\ell^L = \left(\frac{L+1}{2L}\right)\left[\ell f_{\ell-1}^{L-1} + (2\ell+1) f_{\ell-1}^{L-1} + (\ell+1) f_{\ell-1}^{L-1}\right], \tag{11.25}$$

with $f_o^L = 1$ and $f_\ell^L = 0$, if $\ell > L$. The numerical results shown in Table 11.2 are based on the L = 20 scattering law.

Table 11.2. Discrete Eigenvalues for L = 20 Scattering law

ω	ν_o		ν_1		ν_2	
	Explicit	Refined	Explicit	Refined	Explicit	Refined
0.1	1.030046	1.030042	-	-	-	-
0.5	1.536814	1.536814	1.054992	1.054987	-	-
0.95	7.480699	7.480699	1.666787	1.666787	1.019564	1.019586

For the non-conservative mixture of Rayleigh and isotropic scattering of the polarized light, the discrete eigenvalues are the zeros of $\Lambda(z) = \det\underset{\sim}{\Lambda}(z)$, where

$$\underset{\sim}{\Lambda}(z) = \underset{\sim}{I} + \frac{1}{2}\,\omega z \int_{-1}^{1} \underset{\sim}{Q}^{T}(x)\underset{\sim}{Q}(x)\,\frac{dx}{x-z}\,, \quad z \notin (-1,1). \tag{11.26}$$

An explicit form of $\Lambda(z) = \det\underset{\sim}{\Lambda}(z)$ can be written [87] as

$$\Lambda(z) = \frac{1}{8}\,c\Lambda_1(z)\Lambda_2(z) + \left[1 - c + \frac{3}{2}\,c\,(1-\omega)z^2\right]\Lambda_o(z), \tag{11.27a}$$

where

$$\Lambda_\alpha(z) = (-1)^\alpha + 3(1-z^2)\Lambda_o(z) - (-1)^\alpha 3(1-\omega)z^2, \quad \alpha = 1 \text{ and } 2, \tag{11.27b}$$

and

$$\Lambda_o(z) = 1 + \frac{1}{2}\,\omega z \log\left(\frac{z-1}{z+1}\right). \tag{11.27c}$$

To compute the discrete eigenvalue η_o (there are only two zeros $\pm\eta_o$ in the complex plane cut from -1 to 1 along the real axis), we give as first guess, in a Newton-Raphson iterative scheme, the explicit closed-form result obtained by Siewert and Burniston [87].

$$\eta_o = \left[(1-\omega)\left(1 - \frac{7}{10}\,\omega c\right)\right]^{-\frac{1}{2}} \exp\left[-\frac{1}{\pi}\int_0^1 \Theta(\mu)\,\frac{d\mu}{\mu}\right], \tag{11.28a}$$

where

$$\Theta(\mu) = \tan^{-1}\left[\frac{A(\mu)}{B(\mu)}\right] , \tag{11.28b}$$

$$A(\mu) = \frac{1}{8}\,\omega\pi\mu\left[9c(1 - \mu^2)^2\lambda_o(\mu) + 6c\mu^2(1 - \omega) + 4(1 - c)\right] , \tag{11.28c}$$

$$B(\mu) = \frac{1}{8}\,c\left\{-1 + 9(1 - \mu^2)^2\left[\lambda_o^2(\mu) - \frac{1}{4}\pi^2\omega^2\mu^2\right]\right.$$

$$\left. + 3\mu^2(1 - \omega)\left[4\lambda_o(\mu) - 3\mu^2(1 - \omega) + 2\right]\right\}$$

$$+ (1 - c)\lambda_o(\mu) \tag{11.28d}$$

and

$$\lambda_o(\mu) = 1 - \omega\mu\tanh^{-1}\mu , \tag{11.28e}$$

to obtain the results we report in Table 11.3.

Table 11.3. Discrete Eigenvalues for the non-conservative, mixture of Rayleigh and isotropic scattering, transport of polarized light

c	ω	η_o	c	ω	η_o
0.0	0.2	1.0000908865	0.2	0.2	1.0001457236
0.0	0.3	1.0025928888	0.2	0.3	1.0031921571
0.0	0.4	1.0145858159	0.2	0.4	1.0161228175
0.0	0.5	1.0443820338	0.2	0.5	1.0466830429
0.0	0.6	1.1021320212	0.2	0.6	1.1047988909
0.0	0.7	1.2068042540	0.2	0.7	1.2094560549
0.0	0.8	1.4076343091	0.2	0.8	1.4099569245
0.0	0.9	1.9032048560	0.2	0.9	1.9048963596
0.0	0.95	2.6351488343	0.2	0.95	2.6363503464
0.0	0.99	5.7967294513	0.2	0.99	5.79792667629

Table 11.3. (Continued)

c	ω	η_o	c	ω	η_o
0.4	0.2	1.0002257556	0.6	0.2	1.0003392559
0.4	0.3	1.0039224745	0.6	0.3	1.0048129561
0.4	0.4	1.0179367067	0.6	0.4	1.0200999217
0.4	0.5	1.0494444171	0.6	0.5	1.0528145207
0.4	0.6	1.1081010755	0.6	0.6	1.1122913555
0.4	0.7	1.2128509820	0.6	0.7	1.2173478316
0.4	0.8	1.4130239312	0.6	0.8	1.4172589089
0.4	0.9	1.9071915762	0.6	0.9	1.9104832965
0.4	0.95	2.6380005718	0.6	0.95	2.6404084635
0.4	0.99	5.7980114181	0.6	0.99	5.7991121257

Table 11.3. (Continued)

c	ω	η_o	c	ω	η_o
0.8	0.2	1.0004962325	1.0	0.2	1.0007086348
0.8	0.3	1.0059000901	1.0	0.3	1.0072300574
0.8	0.4	1.0227105996	1.0	0.4	1.0259044006
0.8	0.5	1.0570106741	1.0	0.5	1.0623627577
0.8	0.6	1.1177734727	1.0	0.6	1.1252305880
0.8	0.7	1.2235761273	1.0	0.7	1.2327433879
0.8	0.8	1.4234785271	1.0	0.8	1.4334777869
0.8	0.9	1.9155983217	1.0	0.9	1.9246218043
0.8	0.95	2.6442508481	1.0	0.95	2.6513510016
0.8	0.95	5.8009044752	1.0	0.99	5.8043389542

11.2. On the Inverse Problem for a Finite Rayleigh Scattering Atmosphere[14]

In a recent paper [97] the half-space solution, in terms of $\underset{\sim}{H}$ matrix [86], was used to deduce the single-scattering albedo from measurements of the polarized radiation field emerging from a Rayleigh-Scattering atmosphere. Here we consider a similar problem for a finite atmosphere with Lambert reflection at the ground. We find it sufficient to study the azimuthally symmetric component of the complete solution, and thus we consider the equation of transfer

$$\mu \frac{\partial}{\partial\tau} \underset{\sim}{I}(\tau,\mu) + \underset{\sim}{I}(\tau,\mu) = \frac{1}{2}\, \omega \underset{\sim}{Q}(\mu) \int_{-1}^{1} \underset{\sim}{Q}^{T}(\mu') \underset{\sim}{I}(\tau,\mu')\,d\mu' \qquad (11.29)$$

where $\underset{\sim}{I}(\tau,\mu)$ has components $I_{\ell}(\tau,\mu)$ and $I_{r}(\tau,\mu)$, τ is the optical variable, μ is the direction cosine as measured from the positive τ axis, ω is the albedo for single-scattering and, for Rayleigh scattering,

$$\underset{\sim}{Q}(\mu) = \frac{3^{\frac{1}{2}}}{2} \begin{vmatrix} \mu^{2} & 2^{\frac{1}{2}}(1-\mu^{2}) \\ 1 & 0 \end{vmatrix} \ . \qquad (11.30)$$

We allow boundary conditions of the form

$$\underset{\sim}{I}(0,\mu) = \underset{\sim}{F}_{1}(\mu),\ \mu > 0, \qquad (11.31a)$$

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[14] This appendix is based on a paper submitted for publication in J. Atmospheric Sci. [100].

and

$$\underset{\sim}{I}(\tau_o,-\mu) = \underset{\sim}{F}_2(\mu) = \lambda_o \underset{\sim}{D} \int_0^1 \underset{\sim}{I}(\tau_o,\mu')\mu' d\mu', \quad \mu > 0, \tag{11.31b}$$

where λ_o is the coefficient for Lambert reflection and

$$\underset{\sim}{D} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} . \tag{11.32}$$

We seek to express ω and λ_o in terms of $\underset{\sim}{I}(0,-\mu)$ and $\underset{\sim}{I}(\tau_o,\mu)$, $\mu > 0$, which we presume can be measured experimentally.

We start the analysis changing μ to $-\mu$ in Eq. (11.29) and premultiplying the resulting equation by $\underset{\sim}{I}^T(\tau,\mu)$ and integrating over μ from -1 to 1, to obtain

$$T_o(\tau) = -2 \int_0^1 \underset{\sim}{I}^T(\tau,\mu)\underset{\sim}{I}(\tau,-\mu)d\mu + \frac{1}{2}\,\omega \underset{\sim}{I}_o^T(\tau)\underset{\sim}{I}_o(\tau) \tag{11.33}$$

where

$$T_o(\tau) = \int_{-1}^1 \underset{\sim}{I}^T(\tau,\mu)\underset{\sim}{F}(\tau,-\mu)d\mu , \tag{11.34}$$

$$\underset{\sim}{F}(\tau,\mu) = \mu \frac{\partial}{\partial\tau} \underset{\sim}{I}(\tau,\mu) = \frac{1}{2}\,\omega \underset{\sim}{Q}(\mu)\underset{\sim}{I}_o(\tau) - \underset{\sim}{I}(\tau,\mu) \tag{11.35}$$

and in general

$$\underset{\sim}{I}_{\alpha}(\tau) = \int_{-1}^{1} \mu^{\alpha} \underset{\sim}{Q}^{T}(\mu)\underset{\sim}{I}(\tau,\mu)d\mu. \tag{11.36}$$

If we now differentiate Eq. (11.34) and use Eq. (11.35), we find

$$\frac{d}{d\tau} T_o(\tau) = \frac{d}{d\tau}\left[\frac{\omega}{4}\underset{\sim}{I}_o^T(\tau)\underset{\sim}{I}_o(\tau) - \int_0^1 \underset{\sim}{I}^T(\tau,\mu)\underset{\sim}{I}(\tau,-\mu)d\mu\right]. \tag{11.37}$$

Following a procedure recently used for scalar inverse problems [98], we now differentiate Eq. (11.33) to obtain

$$\frac{d}{d\tau} T_o(\tau) = 2\frac{d}{d\tau}\left[\frac{\omega}{4}\underset{\sim}{I}_o^T(\tau)\underset{\sim}{I}_o(\tau) - \int_0^1 \underset{\sim}{I}^T(\tau,\mu)\underset{\sim}{I}(\tau,-\mu)d\mu\right], \tag{11.38}$$

and thus deduce, from Eqs. (11.37) and (11.38), that $T_o(\tau)$ is a constant. We thus integrate Eq. (11.37) to obtain

$$4S_o = \omega\left[\underset{\sim}{I}_o^T(\tau_o)\underset{\sim}{I}_o(\tau_o) - \underset{\sim}{I}_o^T(0)\underset{\sim}{I}(0)\right] \tag{11.39}$$

where

$$S_o = \int_0^1 \underset{\sim}{I}^T(\tau_o,\mu)\underset{\sim}{F}_2(\mu)d\mu - \int_0^1 \underset{\sim}{F}_1^T(\mu)\underset{\sim}{I}(0,-\mu)d\mu\ . \tag{11.40}$$

Clearly if $\underset{\sim}{F}_2(\mu)$ were known we could solve Eq. (11.39) for ω. However, since $\underset{\sim}{F}_2(\mu)$ depends on λ_o we seek a second equation to relate ω and λ_o to known surfaces quantities.

To obtain the second equation, we multiply Eq. (11.29), with μ changed to $-\mu$, by $\mu^2\underset{\sim}{I}^T(\tau,\mu)$ and integrate over μ from -1 to 1 to find

$$T_2(\tau) = \frac{1}{2}\,\omega \underset{\sim}{I}_2^T(\tau)\underset{\sim}{I}_o(\tau) - 2\int_0^1 \mu^2 \underset{\sim}{I}^T(\tau,\mu)\underset{\sim}{I}(\tau,-\mu)d\mu \ , \tag{11.41}$$

where

$$T_2(\tau) = \int_{-1}^1 \mu^2 \underset{\sim}{I}^T(\tau,\mu)\underset{\sim}{F}(\tau,-\mu)d\mu \ . \tag{11.42}$$

Now, using the same technique as used before, we differentiate Eq. (11.42) to obtain

$$\frac{dT_2}{d\tau}(\tau) = \frac{\omega}{2}\frac{d}{d\tau}\left(\underset{\sim}{I}_2^T(\tau)\underset{\sim}{I}_o(\tau)\right) - \frac{\omega}{2}\left(\underset{\sim}{I}_o^T(\tau)\,\frac{d}{d\tau}\,\underset{\sim}{I}_2(\tau)\right)$$

$$- \frac{d}{d\tau}\int_0^1 \mu^2 \underset{\sim}{I}^T(\tau,\mu)\underset{\sim}{I}(\tau,-\mu)d\mu \ . \tag{11.43}$$

We thus differentiate Eq. (11.41) and solve the resulting equation simultaneously with Eq. (11.43) to deduce, after integration over τ from 0 to τ_o, the following equation

$$2S_2 = \omega\int_0^{\tau_o} \underset{\sim}{I}_o^T(\tau)\,\frac{d}{d\tau}\,\underset{\sim}{I}_2(\tau)d\tau \tag{11.44}$$

where

$$S_2 = \int_0^1 \underset{\sim}{I}^T(\tau_o,\mu)\underset{\sim}{F}_2(\mu)\mu^2 d\mu - \int_0^1 \underset{\sim}{F}_1^T(\mu)\underset{\sim}{I}(0,-\mu)\mu^2 d\mu \ . \tag{11.45}$$

If we multiply Eq. (11.29) by $\mu^\alpha \underset{\sim}{Q}^T(\mu)$, α = 0 and 1, and integrate over μ from -1 to 1, we obtain

$$\frac{d}{d\tau} \underset{\sim}{I}_1(\tau) + \underset{\sim}{\Lambda}(\infty)\underset{\sim}{I}_o(\tau) = \underset{\sim}{0} \tag{11.46}$$

where

$$\underset{\sim}{\Lambda}(\infty) = \underset{\sim}{I} - \frac{1}{2}\,\omega \int_{-1}^{1} \underset{\sim}{Q}^T(x)\underset{\sim}{Q}(x)\,dx \tag{11.47}$$

and

$$\frac{d}{d\tau} \underset{\sim}{I}_2(\tau) + \underset{\sim}{I}_1(\tau) = \underset{\sim}{0}\ . \tag{11.48}$$

Thus Eq. (11.44) can be reduced, after we use Eqs. (11.46) and (11.48), to

$$4S_2 = \omega\left[\underset{\sim}{I}_1^T(\tau_o)\underset{\sim}{\Lambda}^{-1}(\infty)\underset{\sim}{I}_1(\tau_o) - \underset{\sim}{I}_1^T(0)\underset{\sim}{\Lambda}^{-1}(\infty)\underset{\sim}{I}_1(0)\right]\ , \tag{11.49}$$

or

$$4(1-\omega)(1-\frac{7}{10}\,\omega)S_2 = \omega\Big[\underset{\sim}{I}_1^T(\tau_o)(\underset{\sim}{I} - \omega\underset{\sim}{R})\underset{\sim}{I}_1(\tau_o)$$

$$- \underset{\sim}{I}_1^T(0)(\underset{\sim}{I} - \omega\underset{\sim}{R})\underset{\sim}{I}_1(0)\Big] \tag{11.50}$$

where

$$\underset{\sim}{R} = (\det \underset{\sim}{\Delta})\underset{\sim}{\Delta}^{-1} \tag{11.51}$$

with

$$\underset{\sim}{\Delta} = \int_0^1 \underset{\sim}{Q}^T(x)\underset{\sim}{Q}(x)dx \ . \tag{11.52}$$

If we now introduce the notation

$$\underset{\sim}{\Pi}_\alpha = \int_0^1 \underset{\sim}{I}(\tau_o;\mu)\mu^\alpha d\mu \ , \tag{11.53}$$

$$S^*_\alpha = \int_0^1 \underset{\sim}{F}_1^T(\mu)\underset{\sim}{I}(0,-\mu)\mu^\alpha d\mu \ , \tag{11.54}$$

$$\underset{\sim}{\Gamma}_\alpha = \int_0^1 \underset{\sim}{Q}^T(\mu)\underset{\sim}{I}(\tau_o,\mu)\mu^\alpha d\mu \ , \tag{11.55}$$

and

$$\underset{\sim}{E}_\alpha = \int_0^1 \underset{\sim}{Q}^T(\mu)\mu^\alpha d\mu \tag{11.56}$$

then we can write Eqs. (1139) and (11.49) so that only ω and λ_o appear as unknowns:

$$4S^*_o = \omega\left[\underset{\sim}{I}_o^T(0)\underset{\sim}{I}_o(0) - (\underset{\sim}{\Gamma}_o^T + \lambda_o\underset{\sim}{\Pi}_1^T\underset{\sim}{D}\underset{\sim}{E}_o)(\underset{\sim}{\Gamma}_o + \lambda_o\underset{\sim}{E}_o\underset{\sim}{D}\underset{\sim}{\Pi}_1)\right] + 4\lambda_o\underset{\sim}{\Pi}_o^T\underset{\sim}{D}\underset{\sim}{\Pi}_1 \tag{11.57}$$

and

$$4(1-\omega)(1-\frac{7}{10}\omega)S^*_2 = \omega\left[\underset{\sim}{I}_1^T(0)(\underset{\sim}{I} - \omega\underset{\sim}{R})\underset{\sim}{I}_1(0) - (\underset{\sim}{\Gamma}_1^T - \lambda_o\underset{\sim}{\Pi}_1^T\underset{\sim}{D}\underset{\sim}{E}_1^T)(\underset{\sim}{I} - \omega\underset{\sim}{R})(\underset{\sim}{\Gamma}_1 - \lambda_o\underset{\sim}{E}_1\underset{\sim}{D}\underset{\sim}{\Pi}_1)\right] + 4\lambda_o\underset{\sim}{\Pi}_2^T\underset{\sim}{D}\underset{\sim}{\Pi}_1 . \tag{11.58}$$

It is clear that we can eliminate ω between Eqs. (11.57) and (11.58) to obtain a fifth-degree polynomial equation for λ_o. Upon solving the polynomial equation for λ_o we can readily compute ω from, say Eq. (11.57).

In order to demonstrate the effectiveness of Eqs. (11.57) and (11.58) we report some numerical results. We have used the F_N method, as described in chapter 7, to compute all the quantities required in Eqs. (11.57) and (11.58), and in Table 11.4 we show these quantities. Subsequently, we have solved the two equations, as described above, to obtain the results shown in Table 11.5. For this numerical example we use $\tau_o = 1.0$, $\omega = 0.9$, $\lambda_o = 0.2$, $\mu_o = 0.9$ and

$$\underset{\sim}{F}_1(\mu) = \frac{1}{2}\begin{vmatrix} 1 \\ \\ 1 \end{vmatrix}\delta(\mu - \mu_o) \ . \tag{11.59}$$

The columns marked 2SF, 3SF, 4SF and 5SF, in Table 11.5, are based on using results for the surfaces quantities (that would be measured in an experiment) that have been rounded to yield 2, 3, 4 and 5 correct significant figures.

Table 11.4. Moments for the Inverse Calculation

$\underset{\sim}{\Pi}_o$	0.36844		0.46712
$\underset{\sim}{\Pi}_1$	0.25612		0.29341
$\underset{\sim}{\Pi}_2$	0.20732		0.22685
S_0^*		0.30591	
S_2^*		0.24779	
$\underset{\sim}{I}_o(0)$	1.2404		0.29809
$\underset{\sim}{I}_1(0)$	0.48804		0.030509
$\underset{\sim}{\Gamma}_o$	0.58408		0.19734
$\underset{\sim}{\Gamma}_1$	0.40690		0.097591

Table 11.5. The Computed Values of ω and λ_o

Quantity	2SF	3SF	4SF	5SF	Exact
ω	0.88	0.891	0.8993	0.90006	0.9
λ_o	0.24	0.224	0.2018	0.19986	0.2

11.3. On the Modification of Computation of $g_\ell(\nu)$ Polynomials and the Functions $A_\alpha(\nu)$, $B_\alpha(\nu)$ for Large Values of ν.

The principal numerical difficulty encountered in problems with a general anisotropic scattering law, such as those discussed in chapters 4 and 6, is when the single scattering albedo (ω) is near unity. This difficulty arises in the calculation of the polynomials $g_\ell(\nu_{\beta,1})$ (note that $\nu_{\beta,1}$ denotes the biggest discrete eigenvalue), defined by Eq. (4.5) and calculated by Eqs. (4.6). As has been shown by Kuščer [46], the value of $g_\ell(\nu_{\beta,1})$, $\ell \geq 1$, tends to zero as $\nu_{\beta,1}$ tends to infinity (note that $\nu_{\beta,1} \to \infty$ as $\omega \to 1$). However, the use of recursion relation, Eq. (4.6), to calculate $g_\ell(\nu_{\beta,1})$ does not give the correct behavior of this polynomials for large values of $\nu_{\beta,1}$, especially for larger values of ℓ. To overcome this computational difficulty we defined, for larger values of ℓ, say $\ell = 40$, $g^*_\ell(\nu_{\beta,1}) = 0$, $g^*_{\ell-1}(\nu_{\beta,1}) = 1$, and then used the recursion relation, as given by Eq. (4.6), backwards to compute $g^*_\ell(\nu_{\beta,1})$, i.e.

$$\ell g^*_{\ell-1}(\nu) = \nu h_\ell g^*_\ell(\nu) - \ell g^*_{\ell-1}(\nu), \tag{11.60}$$

and since we know that $g_o(\nu_{\beta,1}) = 1$, we can find the $g_\ell(\nu)$ polynomial by

$$g\ (\nu) = \frac{g^*_\ell(\nu)}{g^*_o(\nu)}\ , \ \ell = 0,\ 1,\ 2,\ .\ .\ .\ . \tag{11.61}$$

A second difficulty encountered when ω is near unity arises in the computation of the functions $A_\alpha(\nu_{\beta,1})$ and $B_\alpha(\nu_{\beta,1})$ using the recursion formulas given by Eqs. (4.13) and (4.14). To overcome this numerical difficulty, we computed these functions using a nested series derived from the original equations of definition of $A_\alpha(\nu_{\beta,1})$ and $B_\alpha(\nu_{\beta,1})$. To derive the nested series, we first write the original definition of $A_\alpha(\nu_{\beta,1})$ and $B_\alpha(\nu_{\beta,1})$

$$A_\alpha(\nu_{\beta,1}) = \int_0^1 \mu^{\alpha+1} g(\nu_{\beta,1},-\mu) \frac{d\mu}{\mu + \nu_{\beta,1}} \quad , \tag{11.62a}$$

and

$$B_\alpha(\nu_{\beta,1}) = \int_0^1 \mu^{\alpha+1} g(\nu_{\beta,1},\mu) \frac{d\mu}{\mu - \nu_{\beta,1}} \quad . \tag{11.62b}$$

Then, we expand $1/(\mu \pm \nu_{\beta,1})$ in series to obtain, after integration,

$$\nu_{\beta,1} A_\alpha(\nu_{\beta,1}) = \sum_{\ell=0}^{L} (2\ell + 1) f_\ell g_\ell(\nu_{\beta,1}) (-1)^\ell \left[\Delta_{\alpha,\ell} - \frac{1}{\nu_{\beta,1}} \Delta_{\alpha+1,\ell} + \frac{1}{\nu_{\beta,1}^2} \Delta_{\alpha+2,\ell} - \frac{1}{\nu_{\beta,1}^3} \Delta_{\alpha+3,\ell} \cdots \right] , \tag{11.63a}$$

and

$$\nu_{\beta,1} B_\alpha(\nu_{\beta,1}) = \sum_{\ell=0}^{L} (2\ell + 1) f_\ell g_\ell(\nu_{\beta,1}) \left[\Delta_{\alpha,\ell} + \frac{1}{\nu_{\beta,1}} \Delta_{\alpha+3,\ell} + \frac{1}{\nu_{\beta,1}^3} \Delta_{\alpha+3,\ell} \cdots \right] , \tag{11.63b}$$

where $\Delta_{\alpha,\ell}$ are those quantities defined in chapter 4. Now, if we define

$$X_\alpha = \sum_{\ell=0}^{L} (2\ell + 1) f_\ell g_\ell(\nu_{\beta,1})(-1)^\alpha \Delta_{\alpha,\ell} \ , \tag{11.64a}$$

and

$$Y_\alpha = \sum_{\alpha=0}^{L} (2\ell + 1) f_\ell g_\ell(\nu_{\beta,1}) \Delta_{\alpha,\ell} \ , \tag{11.64b}$$

then we can compute the functions $A_\alpha(\nu_{\beta,1})$ and $B_\alpha(\nu_{\beta,1})$ by using the following nested series:

$$\nu_{\beta,1} A_\alpha(\nu_{\beta,1}) = X_\alpha \left[1 - \frac{1}{\nu_{\beta,1}} \frac{X_{\alpha+1}}{X_\alpha} \left(1 - \frac{X_{\alpha+2}}{\nu_{\beta,1} X_{\alpha+1}} \left(1 - \frac{X_{\alpha+3}}{\nu_{\beta,1} X_{\alpha+2}} \left(\ldots , \right.\right.\right.\right. \tag{11.65a}$$

and

$$\nu_{\beta,1} B_\alpha(\nu_{\beta,1}) = Y_\alpha \left[1 + \frac{1}{\nu_{\beta,1}} \frac{Y_{\alpha+1}}{Y_\alpha} \left(1 + \frac{Y_{\alpha+2}}{\nu_{\beta,1} Y_{\alpha+1}} \left(1 + \frac{Y_{\alpha+3}}{\nu_{\beta,1} Y_{\alpha+2}} \left(\ldots , \right.\right.\right.\right. \tag{11.65b}$$